

## Semi Analytical Solution for Fuzzy Autonomous Differential Equations

Mazin H. Suhhiem<sup>1,\*</sup>, Raad I. Khwayyit<sup>2</sup>

<sup>1</sup>University of Sumer, Iraq

<sup>2</sup>Ministry of Education, Iraq

\*Corresponding author: mazin.suhhiem@yahoo.com

**ABSTRACT.** In this work, we have used fuzzy Adomian decomposition method to find the fuzzy semi analytical solution of the fuzzy autonomous differential equations with fuzzy initial conditions. This method allows for the solution of the fuzzy initial value problems to be calculated in the form of an infinite fuzzy series in which the fuzzy components can be easily calculated. The fuzzy series solutions that we have obtained are accurate solutions and very close to the fuzzy exact analytical solutions. Some numerical results have been given to illustrate the efficiency of the used method.

### 1. Introduction

The topic of fuzzy semi analytical methods (fuzzy series method) for solving fuzzy initial value problems (FIVPs) has been rapidly growing in recent years. Several fuzzy semi analytical methods have been proposed to obtain the fuzzy series solution of the linear and non-linear FIVB. Fuzzy Adomian decomposition method is one of the fuzzy semi analytical methods used to obtain the fuzzy series solution of the FIVBs.

Researchers and scientists are continuing to develop this method for solving various types of the FIVBs because it represents an efficient and effective technique (For more details, see [1, 2, 3, 4, 6, 7, 8, 10]).

---

Received: Sep. 13, 2022.

2010 Mathematics Subject Classification. 34A07.

Key words and phrases. fuzzy Adomian decomposition method; fuzzy autonomous differential equation; fuzzy series solution.

In this work, we will need many basic concepts in the fuzzy theory. These concepts can be found in detail in [5, 8, 11].

## 2. Fuzzy Autonomous Differential Equations

A fuzzy ordinary differential equation is called autonomous if it is independent of its independent crisp variable  $x$ . This implies that the  $n$ th order fuzzy autonomous differential equation is of the form [11]:

$$u^{(n)}(x) = f(u(x), u'(x), u''(x), \dots, u^{(n-1)}(x)), x \in [x_0, h] \quad (2.1)$$

with the fuzzy initial conditions:

$$u(x_0) = u_0$$

$$u'(x_0) = u'_0$$

$$u''(x_0) = u''_0$$

⋮

$$u^{(n-1)}(x_0) = u_0^{(n-1)}$$

where:  $u$  is a fuzzy function of the crisp variable  $x$ ,  $f(u(x), u'(x), u''(x), \dots, u^{(n-1)}(x))$  is a fuzzy function of the crisp variable  $x$  and the fuzzy variable  $u$ ,  $u^{(n)}(x)$  is the fuzzy derivative of  $u(x)$ ,  $u'(x)$ ,  $u''(x)$ , ...,  $u^{(n-1)}(x)$ , and  $u(x_0)$ ,  $u'(x_0)$ ,  $u''(x_0)$ , ...,  $u^{(n-1)}(x_0)$  are fuzzy numbers.

The general idea of solving the fuzzy differential equation is based on transforming this equation into a system of non-fuzzy (crisp) differential equations.

Thus, problem (2.1) can be written as:

$$\begin{aligned} \underline{u}^{(n)}(x) &= \underline{f}(u, u', u'', \dots, u^{(n-1)}) \\ &= H(\underline{u}, \underline{u}', \underline{u}'', \dots, \underline{u}^{(n-1)}, \bar{u}, \bar{u}', \bar{u}'', \dots, \bar{u}^{(n-1)}) \end{aligned} \quad (2.2)$$

With the initial conditions:

$$\underline{u}(x_0) = \underline{u}_0$$

$$\underline{u}'(x_0) = \underline{u}'_0$$

$$\underline{u}''(x_0) = \underline{u}''_0$$

$$\begin{aligned}
 & \cdot \\
 & \cdot \\
 & \cdot \\
 \underline{u}^{(n-1)}(x_0) &= \underline{u}_0^{(n-1)} \\
 \overline{u}^{(n)}(x) &= \overline{f}(u, u', u'', \dots, u^{(n-1)}) \\
 &= G(\underline{u}, \underline{u}', \underline{u}'', \dots, \underline{u}^{(n-1)}, \overline{u}, \overline{u}', \overline{u}'', \dots, \overline{u}^{(n-1)})
 \end{aligned} \tag{2.3}$$

With the initial conditions:

$$\begin{aligned}
 \overline{u}(x_0) &= \overline{u}_0 \\
 \overline{u}'(x_0) &= \overline{u}'_0 \\
 \overline{u}''(x_0) &= \overline{u}''_0 \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 \overline{u}^{(n-1)}(x_0) &= \overline{u}_0^{(n-1)}
 \end{aligned}$$

Where

$$\begin{aligned}
 H(\underline{u}, \underline{u}', \underline{u}'', \dots, \underline{u}^{(n-1)}, \overline{u}, \overline{u}', \overline{u}'', \dots, \overline{u}^{(n-1)}) \\
 = \text{Min} \left\{ f(x, z) : z \in [\underline{u}, \underline{u}', \underline{u}'', \dots, \underline{u}^{(n-1)}, \overline{u}, \overline{u}', \overline{u}'', \dots, \overline{u}^{(n-1)}] \right\},
 \end{aligned} \tag{2.4}$$

$$\begin{aligned}
 G(\underline{x}, \underline{x}', \underline{x}'', \dots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \dots, \overline{x}^{(n-1)}) \\
 = \text{Max} \left\{ f(x, z) : z \in [\underline{u}, \underline{u}', \underline{u}'', \dots, \underline{u}^{(n-1)}, \overline{u}, \overline{u}', \overline{u}'', \dots, \overline{u}^{(n-1)}] \right\},
 \end{aligned} \tag{2.5}$$

The parametric form of system (2.4 - 2.5) is given by:

$$\begin{aligned}
 \underline{u}^{(n)}(x, r) \\
 = H(\underline{u}(x, r), \underline{u}'(x, r), \dots, \underline{u}^{(n-1)}(x, r), \overline{u}(x, r), \overline{u}'(x, r), \dots, \overline{u}^{(n-1)}(x, r))
 \end{aligned} \tag{2.6}$$

With the initial conditions:

$$\begin{aligned}
 \underline{u}(x_0, r) &= \underline{u}_0(r), \\
 \underline{u}'(x_0, r) &= \underline{u}'_0(r) \\
 \underline{u}''(x_0, r) &= \underline{u}''_0(r) \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 \underline{u}^{(n-1)}(x_0, r) &= \underline{u}_0^{(n-1)}(r)
 \end{aligned}$$

$$\begin{aligned} & \bar{u}^{(n)}(x, r) \\ & = G(\underline{u}(x, r), \underline{u}'(x, r), \dots, \underline{u}^{(n-1)}(x, r), \bar{u}(x, r), \bar{u}'(x, r), \dots, \bar{u}^{(n-1)}(x, r)) \end{aligned} \quad (2.7)$$

With the initial conditions:

$$\bar{u}(x_0, r) = \bar{u}_0(r)$$

$$\bar{u}'(x_0, r) = \bar{u}'_0(r)$$

$$\bar{u}''(x_0, r) = \bar{u}''_0(r)$$

⋮

$$\bar{u}^{(n-1)}(x_0, r) = \bar{u}_0^{(n-1)}(r)$$

Both equation (2.6) and equation (2.7) have only one solution on the interval  $[x_0, h]$ . Therefore, equation (2.1) has a unique fuzzy solution on  $[x_0, h]$ , where  $r \in [0, 1]$  (For more details, see [11]).

In order to illustrate the above, we give the following example:

If we consider the second order fuzzy autonomous differential equation

$$u''(x) = 4u'(x) - 4u(x), \quad x \in [0, 1] \quad (2.8)$$

With the fuzzy initial conditions:

$$u(0) = [2 + r, 4 - r],$$

$$u'(0) = [5 + r, 7 - r] \text{ and } r \in [0, 1].$$

To convert problem (2.8) into a system of the second order crisp ordinary differential equations, we apply the following steps:

$$[u''(x)]_r = [4u'(x) - 4u(x)]_r, \text{ for all } r \in [0, 1] \quad (2.9)$$

With the fuzzy initial conditions:

$$[u(0)]_r = [2 + r, 4 - r],$$

$$[u'(0)]_r = [5 + r, 7 - r]$$

Then we get

$$[u''(x)]_r = 4[u'(x)]_r - 4[u(x)]_r, \text{ for all } r \in [0, 1] \quad (2.10)$$

With the fuzzy initial conditions:

$$[u(0)]_r = [2 + r, 4 - r],$$

$$[u'(0)]_r = [5 + r, 7 - r]$$

Then we have

$$[ [u''(x)]_r^L, [u''(x)]_r^U ] = [ 4[u'(x)]_r^L - 4[u(x)]_r^L, 4[u'(x)]_r^U - 4[u(x)]_r^U ] \quad (2.11)$$

With the fuzzy initial conditions:

$$[ [u(0)]_r^L, [u(0)]_r^U ] = [ 2 + r, 4 - r ]$$

$$[ [u'(0)]_r^L, [u'(0)]_r^U ] = [ 5 + r, 7 - r ]$$

Then we get the following system of second order crisp ordinary differential equations:

$$[u''(x)]_r^L = 4[u'(x)]_r^L - 4[u(x)]_r^L; \quad (2.12)$$

With the initial conditions:

$$[u(0)]_r^L = 2 + r$$

$$[u'(0)]_r^L = 5 + r$$

$$[u''(x)]_r^U = 4[u'(x)]_r^U - 4[u(x)]_r^U; \quad (2.13)$$

With the initial conditions:

$$[u(0)]_r^U = 4 - r$$

$$[u'(0)]_r^U = 7 - r$$

Which gives the unique crisp solutions

$$[u(x)]_r^L = (2 + r) e^{2x} + (1 - r) x e^{2x} \quad (2.14)$$

$$[u(x)]_r^U = (4 - r) e^{2x} + (r - 1) x e^{2x} \quad (2.15)$$

Then the unique fuzzy solution of problem (8) is

$$[u(x)]_r = [ (2 + r) e^{2x} + (1 - r) x e^{2x}, (4 - r) e^{2x} + (r - 1) x e^{2x} ] \quad (2.16)$$

### 3. Fuzzy Adomian Decomposition Method

To understand the fuzzy Adomian decomposition method, we consider the  $n$ th order fuzzy differential equation [2,3,7]:

$$[F(u(x))]_r = [g(x)]_r \quad (3.1)$$

Where  $F$  represents a general nonlinear fuzzy ordinary (or fuzzy partial) differential operator including both linear and nonlinear terms,  $x$  denotes the independent crisp variable,  $u(x)$  and  $g(x)$  are unknown fuzzy functions.

From section (2), We can conclude that:

$$[F(u(x))]_r = [F(u(x))]_r^L, [F(u(x))]_r^U \quad (3.2)$$

$$[g(x)]_r = [g(x)]_r^L, [g(x)]_r^U \quad (3.3)$$

where

$$[F(u(x))]_r^L = [g(x)]_r^L \quad (3.4)$$

$$[F(u(x))]_r^U = [g(x)]_r^U \quad (3.5)$$

The fuzzy linear terms of (3.1) are decomposed into  $[L]_r + [R]_r$ , where:  $[L]_r$  is invertible and is taken as the highest order fuzzy derivative.

This implies that:

$$L(*) = \frac{d^m}{dx^m} (*), \quad m = 1, 2, 3, \dots \quad (3.6)$$

$$L^{-1}(*) = \int_0^x \int_0^x \dots \int_0^x (*) dx dx \dots dx \quad (3.7)$$

and  $[R]_r$  is the remainder of the fuzzy linear operator.

Thus the equation (3.1) may be written as:

$$[L(u)]_r + [R(u)]_r + [N(u)]_r = [g]_r \quad (3.8)$$

Where  $[N(u)]_r$  represents the fuzzy nonlinear terms.

By the concepts of section (2), We get:

$$[L(u)]_r^L + [R(u)]_r^L + [N(u)]_r^L = [g]_r^L \quad (3.9)$$

$$[L(u)]_r^U + [R(u)]_r^U + [N(u)]_r^U = [g]_r^U \quad (3.10)$$

The fuzzy Adomian decomposition method represents the fuzzy solution  $[u(x)]_r$  of problem (3.1) as a fuzzy series of the form:

$$[u(x)]_r = [u(x)]_r^L, [u(x)]_r^U \quad (3.11)$$

where

$$[u(x)]_r^L = \sum_{n=0}^{\infty} [u_n(x)]_r^L = [u_0(x)]_r^L + [u_1(x)]_r^L + [u_2(x)]_r^L + [u_3(x)]_r^L + \dots \quad (3.12)$$

$$[u(x)]_r^U = \sum_{n=0}^{\infty} [u_n(x)]_r^U = [u_0(x)]_r^U + [u_1(x)]_r^U + [u_2(x)]_r^U + [u_3(x)]_r^U + \dots \quad (3.13)$$

Such that:

$$\bullet [u_0(x)]_r = [u_0(x)]_r^L, [u_0(x)]_r^U \quad (3.14)$$

where

$$[u_0(x)]_r^L = [\theta_0]_r^L + L^{-1}([g(x)]_r^L) \quad (3.15)$$

$$[u_0(x)]_r^U = [\theta_0]_r^U + L^{-1}([g(x)]_r^U) \quad (3.16)$$

$$\bullet) [u_1(x)]_r = [u_1(x)]_r^L, [u_1(x)]_r^U \quad (3.17)$$

where

$$[u_1(x)]_r^L = -L^{-1}([R(u_0)]_r^L) - L^{-1}([A_0]_r^L) \quad (3.18)$$

$$[u_1(x)]_r^U = -L^{-1}([R(u_0)]_r^U) - L^{-1}([A_0]_r^U) \quad (3.19)$$

$$\bullet) [u_2(x)]_r = [u_2(x)]_r^L, [u_2(x)]_r^U \quad (3.20)$$

where

$$[u_2(x)]_r^L = -L^{-1}([R(u_1)]_r^L) - L^{-1}([A_1]_r^L) \quad (3.21)$$

$$[u_2(x)]_r^U = -L^{-1}([R(u_1)]_r^U) - L^{-1}([A_1]_r^U) \quad (3.22)$$

⋮  
⋮  
⋮

$$\bullet) [u_{n+1}(x)]_r = [u_{n+1}(x)]_r^L, [u_{n+1}(x)]_r^U, n \geq 0 \quad (3.23)$$

where

$$[u_{n+1}(x)]_r^L = -L^{-1}([R(u_n)]_r^L) - L^{-1}([A_n]_r^L) \quad (3.24)$$

$$[u_{n+1}(x)]_r^U = -L^{-1}([R(u_n)]_r^U) - L^{-1}([A_n]_r^U) \quad (3.25)$$

Note that:

$$[\theta_0]_r = [\theta_0]_r^L, [\theta_0]_r^U \quad (3.26)$$

and it can be calculated as follows:

•) If  $L = \frac{d}{dx}$ , then we have:

$$[\theta_0]_r^L = [u(0)]_r^L \quad (3.27)$$

$$[\theta_0]_r^U = [u(0)]_r^U \quad (3.28)$$

•) If  $L = \frac{d^2}{dx^2}$ , then we have:

$$[\theta_0]_r^L = [u(0)]_r^L + x[u'(0)]_r^L \quad (3.29)$$

$$[\theta_0]_r^U = [u(0)]_r^U + x[u'(0)]_r^U \quad (3.30)$$

•) If  $L = \frac{d^3}{dx^3}$ , then we have:

$$[\theta_0]_r^L = [u(0)]_r^L + x[u'(0)]_r^L + \frac{x^2}{2!} [u''(0)]_r^L \quad (3.31)$$

$$[\theta_0]_r^U = [u(0)]_r^U + x[u'(0)]_r^U + \frac{x^2}{2!} [u''(0)]_r^U \quad (3.32)$$

·  
·  
·

•) If  $L = \frac{d^{n+1}}{dx^{n+1}}$ , then we have:

$$[\theta_0]_r^L = [u(0)]_r^L + x[u'(0)]_r^L + \frac{x^2}{2!} [u''(0)]_r^L + \cdots + \frac{x^n}{n!} [u^{(n)}(0)]_r^L \quad (3.33)$$

$$[\theta_0]_r^U = [u(0)]_r^U + x[u'(0)]_r^U + \frac{x^2}{2!} [u''(0)]_r^U + \cdots + \frac{x^n}{n!} [u^{(n)}(0)]_r^U \quad (3.34)$$

Also, Note that:

$[A_0]_r, [A_1]_r, [A_2]_r, \dots, [A_n]_r$  are the fuzzy Adomian polynomials,

which can be found as follows:

$$\bullet) [A_0]_r = [A_0]_r^L, [A_0]_r^U \quad (3.35)$$

where

$$[A_0]_r^L = [N(u_0)]_r^L \quad (3.36)$$

$$[A_0]_r^U = [N(u_0)]_r^U \quad (3.37)$$

$$\bullet) [A_1]_r = [A_1]_r^L, [A_1]_r^U \quad (3.38)$$

where

$$[A_1]_r^L = [u_1]_r^L [N'(u_0)]_r^L \quad (3.39)$$

$$[A_1]_r^U = [u_1]_r^U [N'(u_0)]_r^U \quad (3.40)$$

$$\bullet) [A_2]_r = [A_2]_r^L, [A_2]_r^U \quad (3.41)$$

where

$$[A_2]_r^L = [u_2]_r^L [N'(u_0)]_r^L + \frac{1}{2!} ([u_1]_r^L)^2 [N''(u_0)]_r^L \quad (3.42)$$

$$[A_2]_r^U = [u_2]_r^U [N'(u_0)]_r^U + \frac{1}{2!} ([u_1]_r^U)^2 [N''(u_0)]_r^U \quad (3.43)$$

·  
·  
·

$$\bullet) [A_n]_r = [A_n]_r^L, [A_n]_r^U, n = 0, 1, 2, \dots \quad (3.44)$$

where

$$[A_n]_r^L = \frac{1}{n!} \frac{d^n}{d\beta^n} ( [N(\sum_{n=0}^{\infty} \beta^n u_n)]_r^L )|_{\beta=0} \quad (3.45)$$

$$[A_n]_r^U = \frac{1}{n!} \frac{d^n}{d\beta^n} ( [N(\sum_{n=0}^{\infty} \beta^n u_n)]_r^U )|_{\beta=0} \quad (3.46)$$



#### 4. Applied Examples

In this section, we will solve three fuzzy problems to illustrate the accuracy of the fuzzy Adomian decomposition method.

Example 1: Consider the first order fuzzy autonomous differential equation

$$u'(x) = u(x) + u^2(x), \quad x \in [0, 0.1],$$

With the fuzzy initial condition:

$$[u(0)]_r = [0.96 + 0.04r, 1.01 - 0.01r], \quad r \in [0, 1].$$

Solution:

We define:

$$L(u) = \frac{d}{dx}(u)$$

$$R(u) = [R(u)]_r^L = [R(u)]_r^U = -u$$

$$N(u) = [N(u)]_r^L = [N(u)]_r^U = -u^2$$

$$g(x) = [g(x)]_r^L = [g(x)]_r^U = 0$$

From section (3), We can find:

$$[\theta_0]_r^L = 0.96 + 0.04r$$

$$[u_0]_r^L = (0.96 + 0.04r)$$

$$[A_0]_r^L = -(0.96 + 0.04r)^2$$

$$[u_1]_r^L = ((0.96 + 0.04r) + (0.96 + 0.04r)^2)x$$

$$[A_1]_r^L = (-2(0.96 + 0.04r)^2 - 2(0.96 + 0.04r)^3)x$$

$$[u_2]_r^L = \left(\frac{1}{2}(0.96 + 0.04r) + \frac{3}{2}(0.96 + 0.04r)^2 + (0.96 + 0.04r)^3\right)x^2$$

$$[A_2]_r^L = (-2(0.96 + 0.04r)^2 - 5(0.96 + 0.04r)^3 - 3(0.96 + 0.04r)^4)x^2$$

$$[u_3]_r^L = \left(\frac{1}{6}(0.96 + 0.04r) + \frac{7}{6}(0.96 + 0.04r)^2 + 2(0.96 + 0.04r)^3 + (0.96 + 0.04r)^4\right)x^3$$

⋮  
⋮  
⋮

Also, we find:

$$[\theta_0]_r^U = 1.01 - 0.01r$$

$$[u_0]_r^U = (1.01 - 0.01r)$$

$$[A_0]_r^U = -(1.01 - 0.01r)^2$$

$$[u_1]_r^U = ((1.01 - 0.01r) + (1.01 - 0.01r)^2)x$$

$$[A_1]_r^U = (-2(1.01 - 0.01r)^2 - 2(1.01 - 0.01r)^3)x$$

$$[u_2]_r^U = \left(\frac{1}{2}(1.01 - 0.01r) + \frac{3}{2}(1.01 - 0.01r)^2 + (1.01 - 0.01r)^3\right)x^2$$

$$[A_2]_r^U = (-2(1.01 - 0.01r)^2 - 5(1.01 - 0.01r)^3 - 3(1.01 - 0.01r)^4)x^2$$

$$[u_3]_r^U = \left(\frac{1}{6}(1.01 - 0.01r) + \frac{7}{6}(1.01 - 0.01r)^2 + 2(1.01 - 0.01r)^3 + (1.01 - 0.01r)^4\right)x^3$$

⋮  
⋮  
⋮

Therefore, the fuzzy semi analytical solution is:

$$[u(x)]_r = [u(x)]_r^L, [u(x)]_r^U$$

Where

$$[u(x)]_r^L = (0.96 + 0.04r) + ((0.96 + 0.04r) + (0.96 + 0.04r)^2)x + \left(\frac{1}{2}(0.96 + 0.04r) + \frac{3}{2}(0.96 + 0.04r)^2 + (0.96 + 0.04r)^3\right)x^2 + \left(\frac{1}{6}(0.96 + 0.04r) + \frac{7}{6}(0.96 + 0.04r)^2 + 2(0.96 + 0.04r)^3 + (0.96 + 0.04r)^4\right)x^3 + \dots$$

$$[u(x)]_r^U = (1.01 - 0.01r) + ((1.01 - 0.01r) + (1.01 - 0.01r)^2)x + \left(\frac{1}{2}(1.01 - 0.01r) + \frac{3}{2}(1.01 - 0.01r)^2 + (1.01 - 0.01r)^3\right)x^2 + \left(\frac{1}{6}(1.01 - 0.01r) + \frac{7}{6}(1.01 - 0.01r)^2 + 2(1.01 - 0.01r)^3 + (1.01 - 0.01r)^4\right)x^3 + \dots$$

Example 2: Consider the second order fuzzy autonomous differential equation

$$u''(x) + u(x) = 5, \quad x \in [0, 1],$$

With the fuzzy initial conditions:

$$[u(0)]_r = [r, 2 - r]$$

$$[u'(0)]_r = [1 + r, 3 - r], \quad r \in [0, 1].$$

Solution:

We define:

$$L(u) = \frac{d^2}{dx^2}(u)$$

$$R(u) = [R(u)]_r^L = [R(u)]_r^U = u$$

$$N(u) = [N(u)]_r^L = [N(u)]_r^U = 0$$

$$g(x) = [g(x)]_r^L = [g(x)]_r^U = 5$$

From section (3), We can find:

$$[\theta_0]_r^L = r + (1+r)x$$

$$[u_0]_r^L = r + (1+r)x + \frac{5}{2}x^2$$

$$[A_0]_r^L = 0$$

$$[u_1]_r^L = -\frac{r}{2}x^2 - \frac{(r+1)}{6}x^3 - \frac{5}{24}x^4$$

$$[A_1]_r^L = 0$$

$$[u_2]_r^L = \frac{r}{24}x^4 + \frac{(r+1)}{120}x^5 + \frac{5}{720}x^6$$

$$[A_2]_r^L = 0$$

$$[u_3]_r^L = -\frac{r}{720}x^6 - \frac{(r+1)}{5040}x^7 - \frac{5}{40320}x^8$$

⋮  
⋮  
⋮

Also, we find:

$$[\theta_0]_r^U = (2-r) + (3-r)x$$

$$[u_0]_r^U = (2-r) + (3-r)x + \frac{5}{2}x^2$$

$$[A_0]_r^U = 0$$

$$[u_1]_r^U = -\frac{(2-r)}{2}x^2 - \frac{(3-r)}{6}x^3 - \frac{5}{24}x^4$$

$$[A_1]_r^U = 0$$

$$[u_2]_r^U = \frac{(2-r)}{24}x^4 + \frac{(3-r)}{120}x^5 + \frac{5}{720}x^6$$

$$[A_2]_r^U = 0$$

$$[u_3]_r^U = -\frac{(2-r)}{720}x^6 - \frac{(3-r)}{5040}x^7 - \frac{5}{40320}x^8$$

⋮  
⋮  
⋮

Therefore, the fuzzy semi analytical solution is:

$$[u(x)]_r = [u(x)]_r^L, [u(x)]_r^U$$

Where

$$\begin{aligned}
[u(x)]_r^L &= r + (r+1)x + \left(\frac{5-r}{2}\right)x^2 - \left(\frac{r+1}{6}\right)x^3 + \left(\frac{r-5}{24}\right)x^4 + \left(\frac{r+1}{120}\right)x^5 + \left(\frac{5-r}{720}\right)x^6 - \\
&\left(\frac{r+1}{5040}\right)x^7 - \left(\frac{5}{40320}\right)x^8 + \dots \\
&= 5 + (r-5)\cos x + (r+1)\sin x \\
[u(x)]_r^U &= (2-r) + (3-r)x + \left(\frac{3+r}{2}\right)x^2 - \left(\frac{3-r}{6}\right)x^3 - \left(\frac{3+r}{24}\right)x^4 + \left(\frac{3-r}{120}\right)x^5 + \left(\frac{3+r}{720}\right)x^6 - \\
&\left(\frac{3-r}{5040}\right)x^7 - \left(\frac{5}{40320}\right)x^8 + \dots \\
&= 5 - (3+r)\cos x + (3-r)\sin x
\end{aligned}$$

Which is the fuzzy exact analytical solution.

Example 3: Consider the second order fuzzy autonomous differential equation

$$u''(x) + [2r+1, -2r+5]u(x) + \sqrt{u(x)} = 0, \quad x \in [0, 1],$$

With the fuzzy initial conditions:

$$[u(0)]_r = [0, 0]$$

$$[u'(0)]_r = [r+4, -r+6], \quad r \in [0, 1].$$

Solution:

We define:

$$L(u) = \frac{d^2}{dx^2}(u)$$

$$[R(u)]_r^L = (2r+1)u$$

$$[R(u)]_r^U = (-2r+5)u$$

$$N(u) = [N(u)]_r^L = [N(u)]_r^U = \sqrt{u}$$

$$g(x) = [g(x)]_r^L = [g(x)]_r^U = 0$$

From section (3), We can find:

$$[\theta_0]_r^L = (r+4)x$$

$$[u_0]_r^L = (r+4)x$$

$$[A_0]_r^L = ((r+4)x)^{\frac{1}{2}}$$

$$[u_1]_r^L = -\frac{(2r+1)}{6(r+4)^2}((r+4)x)^3 - \frac{4}{15(r+4)^2}((r+4)x)^{\frac{5}{2}}$$

$$[A_1]_r^L = -\frac{(2r+1)}{12(r+4)^2}((r+4)x)^{\frac{5}{2}} - \frac{2}{15(r+4)^2}((r+4)x)^2$$

$$\begin{aligned}
[u_2]_r^L &= \frac{1}{90(r+4)^4} ((r+4)x)^4 + \frac{(2r+1)^2}{120(r+4)^4} ((r+4)x)^5 + \frac{(2r+1)}{90(r+4)^4} ((r+4)x)^{\frac{9}{2}} \\
[A_2]_r^L &= \frac{(2r+1)}{60(r+4)^4} ((r+4)x)^4 - \frac{1}{300(r+4)^4} ((r+4)x)^{\frac{7}{2}} + \frac{(2r+1)^2}{1440(r+4)^4} ((r+4)x)^{\frac{9}{2}} \\
[u_3]_r^L &= -\frac{(2r+1)}{1080(r+4)^6} ((r+4)x)^6 - \frac{(2r+1)^3}{5040(r+4)^6} ((r+4)x)^7 + \frac{1}{7425(r+4)^6} ((r+4)x)^{\frac{11}{2}} - \\
&\frac{17(2r+1)^2}{51480(r+4)^6} ((r+4)x)^{\frac{13}{2}} \\
&\vdots \\
&\vdots \\
&\vdots
\end{aligned}$$

Also, we find:

$$\begin{aligned}
[\theta_0]_r^U &= (6-r)x \\
[u_0]_r^U &= (6-r)x \\
[A_0]_r^U &= ((6-r)x)^{\frac{1}{2}} \\
[u_1]_r^U &= -\frac{(5-2r)}{6(6-r)^2} ((6-r)x)^3 - \frac{4}{15(6-r)^2} ((6-r)x)^{\frac{5}{2}} \\
[A_1]_r^U &= -\frac{(5-2r)}{12(6-r)^2} ((6-r)x)^{\frac{5}{2}} - \frac{2}{15(6-r)^2} ((6-r)x)^2 \\
[u_2]_r^U &= \frac{1}{90(6-r)^4} ((6-r)x)^4 + \frac{(5-2r)^2}{120(6-r)^4} ((6-r)x)^5 + \frac{(5-2r)}{90(6-r)^4} ((6-r)x)^{\frac{9}{2}} \\
[A_2]_r^U &= \frac{(2r+1)}{60(6-r)^4} ((6-r)x)^4 - \frac{1}{300(6-r)^4} ((6-r)x)^{\frac{7}{2}} + \frac{(2r+1)^2}{1440(6-r)^4} ((6-r)x)^{\frac{9}{2}} \\
[u_3]_r^U &= -\frac{(5-2r)}{1080(6-r)^6} ((6-r)x)^6 - \frac{(5-2r)^3}{5040(6-r)^6} ((6-r)x)^7 + \frac{1}{7425(6-r)^6} ((6-r)x)^{\frac{11}{2}} - \\
&\frac{17(5-2r)^2}{51480(6-r)^6} ((6-r)x)^{\frac{13}{2}} \\
&\vdots \\
&\vdots \\
&\vdots
\end{aligned}$$

Therefore, the fuzzy semi analytical solution is:

$$[u(x)]_r = [u(x)]_r^L, [u(x)]_r^U$$

$$\begin{aligned}
\text{Where } [u(x)]_r^L &= (r+4) \left[ x - (2r+1) \frac{x^3}{3!} + (2r+1)^2 \frac{x^5}{5!} - (2r+1)^3 \frac{x^7}{7!} \right] + (r+ \\
&4)^{\frac{1}{2}} \left[ -\frac{4}{15} x^{\frac{5}{2}} + \frac{1}{90} (2r+1) x^{\frac{9}{2}} - \frac{17}{51480} (2r+1)^2 x^{\frac{13}{2}} \right] + \left[ \frac{1}{90} x^4 - \frac{1}{1080} (2r+1) x^6 + \right. \\
&\left. \frac{1}{7425} (r+4)^{-\frac{1}{2}} x^{\frac{11}{2}} \right] + \dots
\end{aligned}$$

$$[u(x)]_r^U = (6-r) \left[ x - (5-2r) \frac{x^3}{3!} + (5-2r)^2 \frac{x^5}{5!} - (5-2r)^3 \frac{x^7}{7!} \right] + (6-r)^{\frac{1}{2}} \left[ -\frac{4}{15} x^{\frac{5}{2}} + \frac{1}{90} (5-2r) x^{\frac{9}{2}} - \frac{17}{51480} (5-2r)^2 x^{\frac{13}{2}} \right] + \left[ \frac{1}{90} x^4 - \frac{1}{1080} (5-2r) x^6 + \frac{1}{7425} (6-r)^{-\frac{1}{2}} x^{\frac{11}{2}} \right] + \dots$$

## 5. Discussion

To show the accuracy of the results, we will give a numerical comparison between the exact analytical solution and the semi analytical solution.

If we go back to example 1:

$$u'(x) = u(x) + u^2(x), \quad x \in [0, 0.1],$$

The fuzzy exact analytical solution for this problem is:

$$[u(x)]_r = [u(x)]_r^L, [u(x)]_r^U$$

where

$$[u(x)]_r^L = \frac{(1.96+0.04r)}{(1.96+0.04r)-(0.96+0.04r)e^x} - 1$$

$$[u(x)]_r^U = \frac{(2.01-0.01r)}{(2.01-0.01r)-(1.01-0.01r)e^x} - 1$$

While the fuzzy semi analytical solution that we got (at  $r = 0.5$ ) is:

$$[u(x)]_r = [u(x)]_r^L, [u(x)]_r^U$$

where

$$[u(x)]_r^L = 0.98 + 1.9404x + 2.871792 x^2 + 4.08855216x^3 + \dots$$

$$[u(x)]_r^U = 1.005 + 2.015025x + 3.032612625 x^2 + 4.396163251x^3 + \dots$$

We test the accuracy by computing the absolute errors:

$$[error]_r^L = | [u_{exact}(x)]_r^L - [u_{series}(x)]_r^L |$$

$$[error]_r^U = | [u_{exact}(x)]_r^U - [u_{series}(x)]_r^U |$$

Table 1: Numerical results for example 1.

$x$	$[u_{series}(x)]_r^L$	$[error]_r^L$	$[u_{series}(x)]_r^U$	$[error]_r^U$
0	0.9800000000000000	0	1.0050000000000000	0
0.01	0.999695267752160	5.90 e-8	1.025457907425751	6.46 e-8
0.02	1.019989425217280	9.57 e-7	1.046548714356008	1.05 e-6
0.03	1.040907003708320	4.92 e-6	1.068298797770277	5.39 e-6
0.04	1.062472534538240	1.58 e-5	1.090734534648064	1.73 e-5
0.05	1.084710549020000	3.91 e-5	1.113882301968875	4.29 e-5
0.06	1.107645578466560	8.23 e-5	1.137768476712216	9.03 e-5
0.07	1.131302154190880	1.55 e-4	1.162419435857593	1.70 e-4
0.08	1.155704807505920	2.69 e-4	1.187861556384512	2.95 e-4
0.09	1.180878069724640	4.37 e-4	1.214121215272479	4.80 e-4
0.10	1.206846472160000	6.78 e-4	1.241224789501000	7.44 e-4

Table 2: Numerical results for example 1.

$x$	$[u_{series}(x)]_r^L$	$[error]_r^L$	$[u_{series}(x)]_r^U$	$[error]_r^U$
0.001	0.981943275880552	5.82 e-12	1.007018062008788	6.37 e-12
0.003	0.985847156518908	4.73 e-10	1.011072487210033	5.18 e-10
0.005	0.989774305869020	3.66 e-9	1.015151489836031	4.01 e-9
0.007	0.993724920181391	1.41 e-8	1.019255280902620	1.54 e-8
0.009	0.997699195706525	3.86 e-8	1.023384071425635	4.23 e-8
0.011	1.001697328694925	8.64 e-8	1.027538072420912	9.47 e-8
0.013	1.005719515397095	1.69 e-7	1.031717494904287	1.85 e-7
0.015	1.009765952063540	3.01 e-7	1.035922549891597	3.29 e-7
0.017	1.013836834944762	4.97 e-7	1.040153448398677	5.45 e-7
0.019	1.017932360291266	7.78 e-7	1.044410401441363	8.53 e-7
0.021	1.022052724353554	1.17 e-6	1.048693620035492	1.28 e-6



## 6. Conclusion

In this work, we have studied the fuzzy semi analytical solutions of the fuzzy autonomous differential equations. We have used the fuzzy Adomian decomposition method to find these solutions. Based on the numerical results we obtained, the fuzzy Adomian decomposition method is a highly efficient method in solving and gives accurate results, and in some cases, this method gives us the exact analytical solution. The accuracy of this method varies from one fuzzy differential equation to another, and this depends on the type of fuzzy differential equation, whether it is of the first order or the highest order, and also depends on the nature of the fuzzy differential equation, whether it is linear or non-linear.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References

- [1] T. Allahviranloo, L. Jamshidi, Solution of Fuzzy Differential Equations Under Generalized Differentiability by Adomian Decomposition Method, *Iran. J. Optim.* 1 (2009), 57-75.
- [2] L. Wang, S. Guo, Adomian Method for Second-Order Fuzzy Differential Equation, *Int. J. Math. Comput. Sci.* 5 (2011), 613-616.
- [3] S. Narayanamoorthy, T. Yookesh, An Adomian Decomposition Method to Solve Linear Fuzzy Differential Equations, in: *Proceeding of the International Conference on Mathematical Methods and Computation*, India, 13-14 February 2014.
- [4] M. Paripour, E. Hajilou, A. Hajilou, H. Heidari, Application of Adomian Decomposition Method to Solve Hybrid Fuzzy Differential Equations, *J. Taibah Univ. Sci.* 9 (2015), 95–103.  
<https://doi.org/10.1016/j.jtusci.2014.06.002>.
- [5] A. Jameel, Numerical and Approximate – Analytical Solutions of Fuzzy Initial Value Problems, Ph.D. Thesis, School of Quantitative Sciences, University Utara Malaysia, Malaysia, 2015.
- [6] S. Biswas, S. Banerjee, T. Roy, Solving Intuitionistic Fuzzy Differential Equations with Linear Differential Operator by Adomian Decomposition Method, *Notes IFS*, 22 (2016), 25-41.  
<http://ifigenia.org/wiki/issue:nifs/22/4/25-41>.
- [7] S. Biswas, T. Roy, Adomian Decomposition Method for Fuzzy Differential Equations with Linear Differential Operator, *J. Inform. Comput. Sci.* 11 4 (2016), 243-250.

- [8] M. Suhhiem, Fuzzy Artificial Neural Network for Solving Fuzzy and Non-Fuzzy Differential Equations, Ph.D. Thesis, College of Sciences, University of Al-Mustansiriyah, Iraq, 2016.
- [9] A.K. Ateeah, Approximate Solution for Fuzzy Differential Algebraic Equations of Fractional Order Using Adomian Decomposition Method, *Ibn Al-Haitham J. Pure Appl. Sci.* 30 (2017), 202-213.
- [10] S. Askari, T. Allahviranloo, S. Abbasbandy, Solving Fuzzy Fractional Differential Equations by Adomian Decomposition Method Used in Optimal Control Theory, *Int. Trans. J. Eng. Manage. Appl. Sci. Technol.* 10 (2019), 10A12P.
- [11] H.A. Sabr, B.N. Abood, M. Suhhiem, Fuzzy Homotopy Analysis Method for Solving Fuzzy Autonomous Differential Equation, *Ratio Math.* 40 (2021), 191-212. <https://doi.org/10.23755/rm.v40i1.589>.