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On Some Properties of a New Truncated Model With Applications to Lifetime Data

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Abstract. This research explored the exponentiated left truncated power distribution which is a bounded model. Various statistical properties which include the moments and their associated measures, Bonferroni and Lorenz curves, reliability measures, shapes, quantile function, entropy, and order statistics were discussed in detail. A simulation study was provided and applications to two real-world data were considered. The performance of the exponentiated left truncated power distribution over other bounded models like Topp-Leone distribution, Beta distribution, Kumaraswamy distribution, Lehmann type–I distribution, Lehmann type–II distribution, generalized power function, Weibull power function, and Mustapha type–II distribution is quite commendable.

1. Introduction

Probability models play important roles in describing real-life events. They have been discussed in the past to model several real-time events so proficiently. The rainfall event was addressed by [1]. Pollution events were addressed by ([2], [3], [4]). Manifold dynamics of COVID-19 were addressed by ([5], [6]). Engineering issues were addressed by ([7], [8]), and several others. Some

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probability models are bounded while some are unbounded. Unbounded distributions extend from negative infinity to positive infinity while bounded ones are confined to lie between two determined values. According to [9], probability models with unit intervals are useful in the area of biology, economics, engineering, and psychology among others. Examples of bounded models include the continuous uniform distribution, beta distribution, Kumaraswamy distribution by [10], [11], [12], [13], and several other notable ones.

It is also worthy of note that some of these bounded probability models have been used to develop generalized families of distributions, examples include the Beta-G family of distributions by [14], Kumaraswamy-G family of distributions by [15], Topp-Leone G family of distributions by ([16], [17]), and so on. A quest to develop models that can adequately fit real-life events has led to the extension of the existing probability models.

1.1. Definition

A random variable X is said to follow the ELTr-PF distribution if the associated cumulative distribution function (CDF) and corresponding probability density function (PDF) begin at k, and are given respectively by;

$$F_{ELTr-PF}(x;a,b) = \left(\frac{x^a - k^a}{1 - k^a}\right)^b, k < x < 1, a, b > 0, \tag{1}$$

$$f_{ELTr-PF}(x;a,b) = \frac{ab}{(1-k^a)^b} x^{a-1}(x^a - k^a)^{b-1}, \tag{2}$$

where $k < x$ is a possible minimum assured life, and it can be defined as an unknown starting point at which age of some certain component/device initiates, and $(a, b > 0)$ are two shape parameters. However, if parameters b=1 and k=0, the model reduces to the baseline model (x^a). This research is aimed at extending the power function and introducing a new bounded probability model; the exponentiated left truncated power (ELTr-PF) function which can be used as an alternative to the existing ones because of its superior modeling capabilities. Its properties are identified, a simulation and real-life applications are provided.

The rest of the paper is structured as follows; general mathematical properties of the ELTr-PF distribution including reliability measures are derived in Section 2, its miscellaneous measures are established in Section 3. The model parameters are estimated in Section 4 while a simulation experiment is performed in Section 5. Applications to real-world data sets are discussed in Section 6, and finally, the conclusion is reported in Section 7.

2. Mathematical Properties

This section covers several mathematical properties of the exponentiated left truncated power distribution.

2.1. Useful representation

Linear combination provides a much informal approach to discuss the CDF and PDF than the conventional integral computation when determining the mathematical properties. For this, the following binomial expansion is considered:

$$(1-y)^{\beta}=\sum_{i=0}^{\infty}\binom{\beta}{i}(-1)^{i}y^{i}, |y|<1.$$

Owing to Equations (1) and (2), infinite linear combinations (LC) of the ELTr-PF CDF becomes:

$$F_{LC-ELTr-PF}(x;a,b)=\frac{1}{(1-k^{a})^{b}}\sum\nolimits_{i=0}^{\infty}\binom{b}{i}(-1)^{i}k^{ai}x^{a(b-i)}, \tag{3}$$

and the corresponding PDF is given as follows:

$$f_{LC-ELTr-PF}(x;a,b)=\frac{ab}{(1-k^{a})^{b}}\sum\nolimits_{j=0}^{\infty}\binom{b-1}{j}(-1)^{j}k^{aj}x^{a(b-j)-1}. \tag{4}$$

2.2. Moments with associated measures

Moments play remarkable roles in the discussion of distribution theory in studying the significant characteristics of a probability distribution like the mean, variance, skewness, and kurtosis.

Theorem 1. If $X\sim ELTr-PF(x;k,a,b)$, with $a,b>0$, and $k<x$, then the r-th ordinary moment (μ'_r) of X is given by:

$$\mu'_{r-ELTr-PF}=\frac{ab}{\left(1-k^{a}\right)^{b}}\sum\nolimits_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^{j}k^{aj}}{r+a(b-j)}\left(1-k^{r+a(b-j)}\right).$$

Proof μ'_r can be written directly following Equation (4) as follows:

$$\mu'_{r-ELTr-PF}=\frac{ab}{\left(1-k^{a}\right)^{b}}\int_{k}^{1}x^{r}\,x^{a-1}\left(x^{a}-k^{a}\right)^{b-1}dx,$$

$$\mu'_{r-ELTr-PF}=\frac{ab}{\left(1-k^{a}\right)^{b}}\sum\nolimits_{j=0}^{\infty}\binom{b-1}{j}(-1)^{j}k^{aj}\int_{k}^{1}x^{r+a(b-j)-1}dx.$$

Further, by solving the simple integral computation, it leads to the final form of the *r*-th ordinary moment, and it is given by:

$$\mu'_{r-ELTr-PF} = \frac{ab}{(1-k^a)^b}\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{r+a(b-j)}\left(1-k^{r+a(b-j)}\right). \tag{5}$$

The expression in Equation (5) is quite impressive and useful in the development of several statistical measures. For instance, to obtain the mean of *X*, substitute *r* = 1 in Equation (5) as follows:

$$\mu'_{1-ELTr-PF} = \frac{ab}{(1-k^a)^b}\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{1+a(b-j)}\left(1-k^{1+a(b-j)}\right). \tag{6}$$

One may perhaps further determine the well-established statistics such as skewness ($\beta_1 = \mu_3^2/\mu_2^3$), and kurtosis ($\beta_2 = \mu_4/\mu_2^2$), of X by integrating Equation (6). A well-established relationship between the central moments (μ_s) and cumulants (K_s) of X may easily be defined by ordinary moments $\mu_s = \sum_{k=0}^{s}\binom{s}{k}(-1)^k\left(\mu'_1\right)^s\mu'_{s-k}$. Hence, the first four cumulants can be calculated by $K_1 = \mu'_1$, $K_2 = \mu'_2 - \mu_1'^2$, $K_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$, and $K_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu_2'^2 + 12\mu'_2\mu_1'^2 - 6\mu_1'^4$, etc., respectively.

Table 1 presents some numerical results of the first four ordinary moments ($\mu'_1, \mu'_2, \mu'_3, \mu'_4$), σ^2 = variance, β_1 = skewness, and β_2 = kurtosis for some choices of model parameters (k = 0.1) for the ELTr-PF distribution.

Table 1. Some numerical results of moments, variance, skewness, and kurtosis.

Statistics	$a = 0.1, k = 0.1$					Remarks
μ'_r	$b = 1.2$	$b = 1.3$	$b = 1.4$	$b = 1.5$	$b = 1.6$	
μ'_1	0.4425	0.4586	0.4737	0.4880	0.5015	Decreasing
μ'_2	0.2607	0.2752	0.2890	0.3023	0.3151	
μ'_3	0.1814	0.1931	0.2045	0.2155	0.2262	
μ'_4	0.1386	0.1482	0.1576	0.1668	0.1758	
σ^2	0.0373	0.0314	0.0244	0.0165	0.0078	
β_1	0.0016	0.0029	0.0031	0.0021	0.0008	Decreasing
β_2	0.1329	0.1266	0.1150	0.0971	0.0714	Decreasing

Table 2. Some numerical results of moments, variance, skewness, and kurtosis.

Statistics	$b=1.5, k=0.1$		$b=1.7, k=0.1$			Remarks
μ'_r	$a=0.1$	$a=0.2$	$a=0.01$	$a=0.03$	$a=0.05$	
μ'_1	0.4880	0.5063	0.4972	0.5009	0.5047	Decreasing
μ'_2	0.3023	0.3208	0.3101	0.3138	0.3177	
μ'_3	0.2155	0.2317	0.2214	0.2247	0.2281	
μ'_4	0.1668	0.1807	0.1714	0.1743	0.1772	
σ^2	0.0166	0.0046	0.0103	0.0079	0.0053	
β_1	0.0022	0.0006	0.0008	0.0005	0.0004	Decreasing
β_2	0.0971	0.0645	0.0730	0.0671	0.0602	Decreasing

Tables 1 and 2 illustrate decreasing behavior of the first four moments, variance, skewness, and kurtosis with some choices of model parameters.

Moment generating function $M_X(t)$ can be defined as:

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'_r.$$

Therefore, the moment generating function (mgf) of X is given by:

$$M_{X-ELTr-PF}(t)=\frac{ab}{(1-k^a)^b}\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum\nolimits_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{r+a(b-j)}\left(1-k^{r+a(b-j)}\right).$$

Characteristic function is defined as:

$$\emptyset_X(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\mu'_r.$$

By following Equation (5), it is obtained as:

$$\emptyset_{X-ELTr-PF}(t)=\frac{ab}{(1-k^a)^b}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\sum\nolimits_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{r+a(b-j)}\left(1-k^{r+a(b-j)}\right).$$

The factorial generating function of X is defined as:

$$F_x(t)=E(1+t)^x=E\left(e^{x ln(1+t)}\right)=\sum_{r=0}^{\infty}\frac{\left(ln(1+t)\right)^r}{r!}\mu'_r.$$

By using Equation (5), it is obtained as:

$$F_{x-ELTr-PF}(t)=\left(\frac{ab}{(1-k^a)^b}\sum_{r=0}^{\infty}\frac{\left(ln(1+t)\right)^r}{r!}\sum\nolimits_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{r+a(b-j)}\left(1-k^{r+a(b-j)}\right)\right).$$

Negative moments of X, substitute r with – w in Equation (5) and it is given by:

$$\mu'_{-w-ELTr-PF} = \frac{ab}{(1-k^a)^b} \sum_{j=0}^{\infty} \binom{b-1}{j} \frac{(-1)^j k^{aj}}{a(b-j)-w} \left(1-k^{a(b-j)-w}\right).$$

Furthermore, for fractional positive and fractional negative moments of X, substitute r with $\left(\frac{m}{n}\right)$ and $\left(-\frac{m}{n}\right)$ in Equation (6) respectively. In the theory of statistics, the Mellin transformation is famous as a distribution of the product as well as a quotient for independent random variables. The Mellin transformation is represented by

$$M_x(m) = E(x^{m-1}) = \int_1^k x^{m-1} f(x)dx.$$

Mellin transformation of X is given by:

$$M_{x-ELTr-PF}(m) = \frac{ab}{(1-k^a)^b} \sum_{j=0}^{\infty} \binom{b-1}{j} \frac{(-1)^j k^{aj}}{a(b-j)+m-1} \left(1-k^{a(b-j)+m-1}\right).$$

2.3. Incomplete moments

The r – th lower incomplete moments of X is defined as:

$\Phi_r(t) = \int_k^t x^r f(x)dx$,

and it is given by:

$$\Phi_{r-ELTr-PF}(t) = \frac{ab}{(1-k^a)^b} \sum_{j=0}^{\infty} \binom{b-1}{j} \frac{(-1)^j k^{aj}}{r+a(b-j)} \left(t^{r+a(b-j)} - k^{r+a(b-j)}\right). \quad (7)$$

The first incomplete moment can be obtained by substituting r = 1 in Equation (7) as follows:

$$\Phi_{1-ELTr-PF}(t) = \frac{ab}{(1-k^a)^b} \sum_{j=0}^{\infty} \binom{b-1}{j} \frac{(-1)^j k^{aj}}{1+a(b-j)} \left(t^{1+a(b-j)} - k^{1+a(b-j)}\right). \quad (8)$$

The residual life function is the probability that a component whose life says x, will survive in an additional interval at t. It is given by:

$$R\left({}^t/_x\right) = \frac{S(x+t)}{S(t)}.$$

Therefore, the residual life function of X is:

$$S_{R(t)-ELTr-PF}\left({}^t/_x\right) = \frac{(1-k^a)^b - ((x+t)^a - k^a)^b}{(1-k^a)^b - (t^a - k^a)^b}, \qquad x > 0.$$

The reverse residual life is obtained by $S_{\bar{R}(t)-ELTr-Pow}\left({}^t/_x\right) = \frac{S(x-t)}{S(t)}$. The reverse residual life function of X is therefore given by:

$$S_{\bar{R}(t)-ELTr-PF}\left({}^t/_x\right) = \frac{(1-k^a)^b - ((x-t)^a - k^a)^b}{(1-k^a)^b - (t^a - k^a)^b}, \qquad x > 0.$$

Mean residual life (MRL) function is defined as $\frac{1-\Phi_1(t)}{S(t)-t}$. It is obtained for X as

$$\mathrm{MRL}=\frac{1-\frac{ab}{(1-k^a)^b}\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{1+a(b-j)}\left(t^{1+a(b-j)}-k^{1+a(b-j)}\right)}{S(t)-t}.$$

Mean inactivity time (MIT) is defined as $t-\frac{\Phi_1(t)}{P(t)}$. It is obtained for X as

$$\mathrm{MRL}=t-\frac{1}{P(t)}\left(\frac{ab}{(1-k^a)^b}\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{1+a(b-j)}\left(t^{1+a(b-j)}-k^{1+a(b-j)}\right)\right)$$

Vitality function is defined as $V(x)=\frac{1}{S(x)}\int_x^1 xf(x)dx$. It is obtained for X as

$$V(x)=\frac{1}{1-F(x)}\left(\frac{ab}{(1-k^a)^b}\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{1+a(b-j)}\left(1-x^{1+a(b-j)}\right)\right).$$

The conditional moments are defined as$E(x^r|x>t)=\frac{1}{\bar{P}(t)}\int_t^1 x^r f(x)dx$. It is obtained for X as

$$E(x^r|x>t)=\frac{1}{1-P(t)}\left(\frac{ab}{(1-k^a)^b}\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{r+a(b-j)}\left(1-t^{r+a(b-j)}\right)\right)$$

2.4. Bonferroni and Lorenz curves

The Bonferroni $B(x)$ and Lorenz $L(x)$ curves are important not only in the study of economics, the distribution of income, poverty, or wealth, but they play a vital role in the fields of insurance, demography, medicine, reliability, and others. These curves are defined respectively by:

$$B(x)=\frac{\int_0^t xf(x)dx}{\mu_1^{/}}, \qquad L(x)=\frac{B(x)}{F(x)},$$

Lorenz curve $L(x)$

$$L_{ELTr-PF}(t)=\frac{\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{1+a(b-j)}\left(t^{1+a(b-j)}-k^{1+a(b-j)}\right)}{\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{1+a(b-j)}\left(1-k^{1+a(b-j)}\right)}, \tag{9}$$

and Bonferroni curve $B(x)$ are given by:

$$B_{ELTr-PF}(t)=\frac{\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{1+a(b-j)}\left(t^{1+a(b-j)}-k^{1+a(b-j)}\right)}{\left(\left(\frac{x^a-k^a}{1-k^a}\right)^b\right)\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j k^{aj}}{1+a(b-j)}\left(1-k^{1+a(b-j)}\right)}.$$

2.5.Reliability measures

The survival function is defined as the probability that a component will survive till time x. Analytically, it is defined as:

$S(x)=1-F(x)$.

The survival function of X is therefore given by:

$$S_{ELTr-PF}(x;a,b) = 1 - \left(\frac{x^a - k^a}{1 - k^a}\right)^b.$$

The hazard rate function (HRF) is defined as measuring the failure rate of a component in a particular time *x.* Mathematically, it is defined as:

$h(x) = f(x)/S(x)$.

Hence, the hazard rate function of X is given by:

$$h_{ELTr-PF}(x;a,b) = \frac{abx^{a-1}(x^a - k^a)^{b-1}}{(1 - k^a)^b - (x^a - k^a)^b}.$$

Further, several notable reliability measures may be derived for X such as the reversed hazard rate function. It is defined as:

$h_r(x) = f(x)/F(x)$.

The reversed hazard rate function of X is given by:

$$h_{r-ELTr-PF}(x;a,b) = \frac{abx^{a-1}}{(x^a - k^a)}.$$

The Mills ratio is defined as $M(x) = S(x)/f(x)$. Hence, the Mills ratio of X is given by:

$$M_{ELTr-PF}(x;a,b) = \frac{(1 - k^a)^b - (x^a - k^a)^b}{abx^{a-1}(x^a - k^a)^{b-1}}.$$

The Odd function is defined as $O(x) = F(x)/S(x)$. Therefore, the Odd function of X is given by:

$$O_{ELTr-PF}(x;a,b) = \left(\frac{x^a - k^a}{1 - k^a}\right)^{-b} - 1.$$

3. Miscellaneous measures

This section covers several measures including limiting behavior, shapes of density and hazard rate functions, quantile function, entropy measures, and distribution of order statistics, bivariate, and multivariate extensions for ELTr-PF distribution.

3.1.Limiting behavior

The limiting behavior of the CDF, PDF, and HRF of *X* for $x \to k$ and $x \to 1$ is discussed in propositions 1 and 2.

Proposition 1. Limiting behaviors of the CDF, PDF, and HRF of *X* for $x \to k$ are given respectively by:

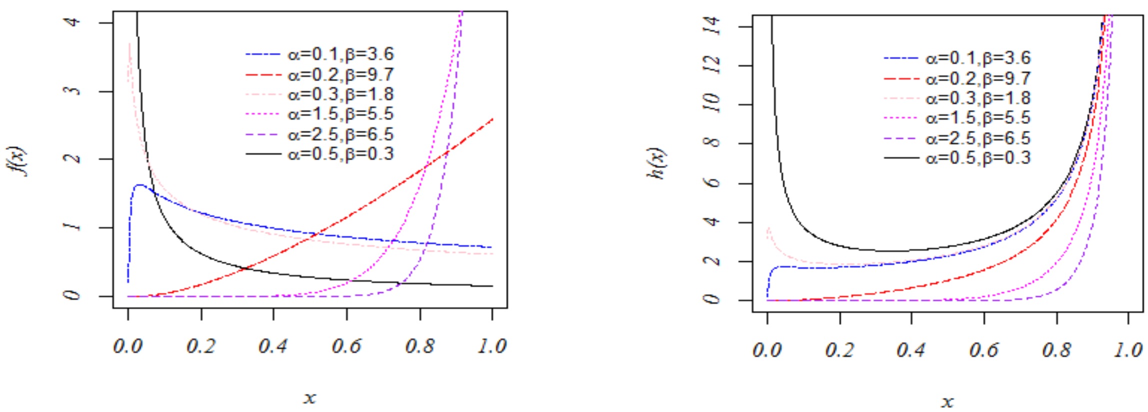
$$F_{ELTr-PF}\,(k) \sim 0,$$
$$f_{ELTr-PF}(k) \sim 0,$$
$$h_{ELTr-PF}\,(k) \sim 0.$$

Proposition 2. Limiting behaviors of the CDF, PDF, and HRF of X for $x \to 1$ are given respectively by:

$$F_{ELTr-PF}\,(1) \sim 1,$$
$$f_{ELTr-PF}\,(1) \sim \frac{ab}{(1-k^{a})},$$
$$h_{ELTr-PF}\,(1) \sim Indeterminate.$$

3.2. Shapes of density and hazard rate functions

Different curves of PDF and HRF of X are presented in Figures 1 and 2, for different choices of model parameters. Note that in Figure 1, curves of the PDF present some possible shapes including increasing, upside-down increasing, and decreasing. However, possible shapes of the HRF in Figure 2 present increasing and bathtub-shaped.



(a) (b)

Figure 1. Plots of PDF (a) and HRF (b) for ELTr-PF distribution.

3.3. Quantile function

The concept of quantile function was introduced by [18]. The q^{th} quantile function of the ELTr-PF distribution is obtained by inverting the CDF in Equation (1). It is defined by:

$$q = F(x_q) = P(X \le x_q), \qquad q \in (0,1).$$

Then, the quantile function of X is given by:

$$x_{q-ELTr-PF} = \left(k^{a} + (1-k^{a})q^{1/\beta}\right)^{1/\alpha}. \tag{10}$$

To derive the 1st quartile, median and 3rd quartile of X, one may place q = 0.25, 0.5, and 0.75 respectively in Equation (10). Henceforth, to generate random numbers, one may assume that the expression in Equation (10) follows to uniform distribution $u = U(0, 1)$.

3.4.Entropy measures

This subsection covers several well-known entropy measures addressed by ([19], [20], [21], [22], [23], [24]).

The entropy of *r.v.* X is a measure of uncertainty. The Rényi entropy of X is defined by:

$$I_\delta(X) = \frac{1}{1-\delta} log \int_k^1 f^\delta(x)dx \ , \qquad \delta > 0 \ and \delta \neq 1. \tag{11}$$

First, $f_{ELTr-Pow}(x)$ is simplified in terms of $f^\delta{}_{ELTr-Pow}(x)$ by considering Equation (2) as:

$$f^\delta{}_{ELTr-PF}(x) = \left(\frac{ab}{(1-k^a)^b}\right)^\delta x^{\delta(a-1)}(x^a - k^a)^{\delta(b-1)},$$

by applying the binomial expansion,

$$f^\delta{}_{ELTr-PF}(x) = \left(\frac{ab}{(1-k^a)^b}\right)^\delta \sum_{i=0}^{\infty} \binom{\delta(b-1)}{i} (-1)^i k^{ai} x^{\delta(ab-1)-ai},$$

and substituting this into Equation (11) gives the Rényi entropy of X as:

$$I_{\delta-ELTr-PF}(X) = \left(\frac{ab}{(1-k^a)^b}\right)^\delta \sum_{i=0}^{\infty} \binom{\delta(b-1)}{i} (-1)^i k^{ai} \int_k^1 x^{\delta(ab-1)-ai} dx \ ,$$

hence, by integrating the last expression the reduced form of the Rényi entropy for X is obtained and it is given by:

$$I_{\delta-ELTr-PF}(X) = \left(\frac{ab}{(1-k^a)^b}\right)^\delta log \sum_{i=0}^{\infty} A_{i,\delta} \frac{1}{\psi_{i,\delta}} \left(1 - k^{\psi_{i,\delta}}\right), \tag{12}$$

where $\psi_{i,\delta} = \delta(ab-1) - ai$, $A_{i,\delta} = \binom{\delta(b-1)}{i} (-1)^i k^{ai}$.

A generalization of the Boltzmann-Gibbs entropy is the $\eta -$ entropy. Although in physics, it is referred to as the Tsallis entropy. Tsallis entropy / $\eta -$ entropy is defined by

$$H_\eta(X) = \frac{1}{\eta - 1}\left(1 - \int_k^1 f^\eta(x)dx\right), \qquad \eta > 0 \ and \ \eta \neq 1.$$

The Tsallis entropy of X is given by

$$H_{\eta-ELTr-PF}(X) = \frac{1}{\eta-1}\left(1 - \left(\frac{ab}{(1-k^a)^b}\right)^\eta log \sum_{i=0}^{\infty} A_{i,\eta} \frac{1}{\psi_{i,\eta}} \left(1 - k^{\psi_{i,\eta}}\right)\right),$$

where $\psi_{i,\eta} = \eta(ab-1) - ai$, $A_{i,\eta} = \binom{\eta(b-1)}{i} (-1)^i k^{ai}$.

The Havrda and Charvat introduced $\omega -$ entropy measure. It is defined by

$$H_\omega(X) = \frac{1}{2^{1-\omega} - 1}\left(\int_k^1 f^\omega(x)dx - 1\right), \qquad \omega > 0 \ and \ \omega \neq 1.$$

Havrda and Charvat entropy of X is given by

$$H_{\omega-ELTr-PF}(X)=\frac{1}{2^{1-\omega}-1}\left(\left(\left(\frac{ab}{(1-k^{a})^{b}}\right)^{\omega}log\sum_{i=0}^{\infty}A_{i,\omega}\frac{1}{\psi_{i,\omega}}\left(1-k^{\psi_{i,\omega}}\right)\right)-1\right),$$

where $\psi_{i,\omega}=\omega(ab-1)-ai$, $A_{i,\omega}=\binom{\omega(b-1)}{i}(-1)^{i}k^{ai}$.

Arimoto generalized the work of Havrda and Charvat by introducing $\varepsilon-$ entropy measure. It is defined by

$$H_{\varepsilon}(X)=\frac{\varepsilon}{2^{1-\varepsilon}-1}\left(\left(\int_{k}^{1}f^{\varepsilon}(x)dx\right)^{\frac{1}{\varepsilon}}-1\right),\qquad \varepsilon>0\ and\ \varepsilon\neq 1.$$

Arimoto entropy of X is given by

$$H_{\varepsilon-ELTr-PF}(X)=\frac{\varepsilon}{2^{1-\varepsilon}-1}\left(\left(\left(\left(\frac{ab}{(1-k^{a})^{b}}\right)^{\frac{1}{\varepsilon}}log\sum_{i=0}^{\infty}A_{i,\frac{1}{\varepsilon}}\frac{1}{\psi_{i,\frac{1}{\varepsilon}}}\left(1-k^{\psi_{i,\frac{1}{\varepsilon}}}\right)\right)\right)^{\varepsilon}-1\right),$$

where $\psi_{i,\frac{1}{\varepsilon}}=\frac{1}{\varepsilon}(ab-1)-ai$, $A_{i,\frac{1}{\varepsilon}}=\binom{\frac{1}{\varepsilon}(b-1)}{i}(-1)^{i}k^{ai}$.

Booker and Lubba developed the $\tau-$ entropy measure. It is defined by

$$H_{\tau}(X)=\frac{\tau}{\tau-1}\left(1-\left(\int_{k}^{1}f^{\tau}(x)dx\right)^{\frac{1}{\tau}}\right),\qquad \tau>0\ and\ \tau\neq 1.$$

Boekee and Lubba entropy of X is given by

$$H_{\tau-ELTr-PF}(X)=\frac{\tau}{\tau-1}\left(1-\left(\left(\frac{ab}{(1-k^{a})^{b}}\right)^{\tau}log\sum_{i=0}^{\infty}A_{i,\tau}\frac{1}{\psi_{i,\tau}}\left(1-k^{\psi_{i,\tau}}\right)\right)^{\frac{1}{\tau}}\right).$$

where $\psi_{i,\tau}=\tau(ab-1)-ai$, $A_{i,\tau}=\binom{\tau(b-1)}{i}(-1)^{i}k^{ai}$.

Mathai and Haubold generalized the classical Shannon entropy is known as $\zeta-$ entropy. It is defined by

$$H_{\zeta}(X)=\frac{1}{\zeta-1}\left(\int_{k}^{1}f^{2-\zeta}(x)dx-1\right),\qquad \zeta>0\ and\ \zeta\neq 1.$$

Mathai and Haubold entropy of X is given by

$$H_{\zeta-ELTr-PF}(X)=\frac{1}{\zeta-1}\left(\left(\left(\frac{ab}{(1-k^{a})^{b}}\right)^{2-\zeta}log\sum_{i=0}^{\infty}A_{i,2-\zeta}\frac{1}{\psi_{i,2-\zeta}}\left(1-k^{\psi_{i,2-\zeta}}\right)\right)-1\right),$$

where $\psi_{i,2-\zeta} = (2-\zeta)(ab-1) - ai$, $A_{i,2-\zeta} = \binom{(2-\zeta)(b-1)}{i}(-1)^i k^{ai}$.

Table 3 presents the flexible behavior of the entropy measures for some choices of model parameters for S-I $(\boldsymbol{a = 1.1, b = 2.1, k = 0.01})$, S-II$(\boldsymbol{a = 1.1, b = 1.5, k = 0.015})$, and S-III$(\boldsymbol{a = 1.5, b = 1.1, k = 0.05})$.

Table 3. Some numerical results of Rényi, Tsallis, Havrda and Charvat, Arimoto, Boekee and Lubba, Mathai and Haubold entropy measures.

Entropy	Int.	S-I	S-II	S-III
Rényi	$\delta = 1.1$	4.8902	1.3591	0.4750
	$\delta = 1.5$	1.3337	0.3706	0.1295
	$\delta = 1.7$	1.0796	0.3000	0.1048
	$\delta = 1.9$	0.9385	0.2608	0.0911
Tsallis	$\eta = 1.1$	0.3634	0.1047	0.0323
	$\eta = 1.5$	0.1500	0.0431	-0.0070
	$\eta = 1.7$	0.0780	0.0181	-0.0248
	$\eta = 1.9$	0.0174	-0.0044	-0.0417
Havrda and Charvat	$\omega = 1.1$	0.0028	0.0019	0.0045
	$\omega = 1.5$	0.0820	0.0389	0.0554
	$\omega = 1.7$	0.1887	0.0851	0.1109
	$\omega = 1.9$	0.3680	0.1601	0.1969
Arimoto	$\varepsilon = 1.1$	0.0039	0.0010	0.0004
	$\varepsilon = 1.5$	0.3128	0.0529	0.0213
	$\varepsilon = 1.7$	1.5767	0.1481	0.0596
	$\varepsilon = 1.9$	85.315	0.3558	0.1333
Boekee and Lubba	$\tau = 1.1$	0.3701	0.1063	0.0333
	$\tau = 1.5$	0.2238	0.0665	0.0087
	$\tau = 1.7$	0.1871	0.0552	0.0012
	$\tau = 1.9$	0.1606	0.0466	-0.0045
Mathai and Haubold	$\zeta = 1.1$	-0.1569	0.0401	0.3761
	$\zeta = 1.5$	-0.1500	-0.0431	0.0070
	$\zeta = 1.7$	-0.1031	-0.0306	-0.0051
	$\zeta = 1.9$	-0.0403	-0.0116	-0.0035

Table 3 presents' versatile behavior of entropy measures for different parametric values. Note that the Rényi entropy is decreasing, Tsallis entropy is decreasing, Havrda and Charvat entropy is increasing, Arimoto entropy is increasing, Boekee and Lubba entropy is decreasing, and Mathai and Haubold's entropy is decreasing.

3.5.Distribution of order statistics

This subsection covers *i*-th order statistics PDF, minimum order statistics PDF, maximum order statistics PDF, order statistics CDF, median order statistics PDF, and Joint order statistics PDF.

In reliability analysis and life testing of a component in quality control, OS has a noteworthy contribution. Let X_1, X_2, X_3, …, X_n be a random sample of size n which follows the ELTr-PF distribution and $\{X_{(1)} < X_{(2)} < X_{(3)} < \ldots < X_{(n)}\}$ be the corresponding order statistics. The PDF of the *i*-th OS is given by:

$$f_{(i:n)}(x) = \frac{1}{B(i,n-i+1)!}\left(F(x)\right)^{i-1}\left(1-F(x)\right)^{n-i} f(x), \quad i=1, 2, 3, \ldots, n.$$

By incorporating Equations (3) and (4), the PDF of *i*-th OS is given by:

$$f_{(i:n)-ELTr-PF}(x) = \left(\begin{array}{c}\frac{1}{B(i,n-i+1)!}\left(\left(\frac{x^a-k^a}{1-k^a}\right)^b\right)^{i-1}\left(1-\left(\frac{x^a-k^a}{1-k^a}\right)^b\right)^{n-i} \times \\ \left(\frac{ab}{(1-k^a)^b}x^{a-1}(x^a-k^a)^{b-1}\right)\end{array}\right). \tag{13}$$

For minimum OS, substitute ($i = 1$) into Equation (13) as;

$$f_{(1:n)-ELTr-PF}(x) = \left(\begin{array}{c}\frac{1}{B(i,n-i+1)!}\left(1-\left(\frac{x^a-k^a}{1-k^a}\right)^b\right)^{n-1} \times \\ \left(\frac{ab}{(1-k^a)^b}x^{a-1}(x^a-k^a)^{b-1}\right)\end{array}\right).$$

While the maximum OS is obtained by substituting ($i = n$) into Equation (13) as:

$$f_{(n:n)-ELTr-pF}(x) = \frac{1}{B(i,n-i+1)!}\left(\left(\frac{x^a-k^a}{1-k^a}\right)^b\right)^{n-1}\left(\frac{ab}{(1-k^a)^b}x^{a-1}(x^a-k^a)^{b-1}\right).$$

Correspondingly, the CDF of the *i*-th OS is defined by:

$$F_{(i:n)}(x) = \sum_{r=i}^{n}\binom{n}{r}\left(F(x)\right)^r\left(1-F(x)\right)^{n-r}.$$

By incorporating Equation (1), the CDF of the *i*-th OS is obtained and it is given by:

$$F_{(i:n)-ELTr-PF}(x) = \sum_{r=1}^{n} \binom{n}{r} \left(\left(\left(\frac{x^a - k^a}{1 - k^a} \right)^b \right)^{n-1} \right)^r \left(1 - \left(\left(\frac{x^a - k^a}{1 - k^a} \right)^b \right)^{n-1} \right)^{n-r}.$$

The Median of the i-th OS is defined by:

$$f_{(m+1:n)}(x) = \frac{(2m+1)!}{(m!)^2} f(x) \big(F(x)\big)^m \big(1 - F(x)\big)^m.$$

By incorporating Equations (1) and (2), the PDF of the median X_{m+1} OS is obtained as:

$$f_{(m+1:n)-ELTr-PF}(x) = \frac{(2m+1)!}{(m!)^2} \left(\left(\left(\frac{x^a - k^a}{1 - k^a} \right)^b \right)^{n-1} \right)^m \left(\left(1 - \left(\frac{x^a - k^a}{1 - k^a} \right)^b \right)^{n-1} \right)^m.$$

The Joint distribution of the i-th and j-th OS is defined by:

$$f_{(i:j:n)}(x_i, x_j) = C\big(F(x_i)\big)^{i-1} \Big(F(x_j) - F(x_i)\Big)^{j-i-1} \Big(1 - F(x_j)\Big)^{n-j} f(x_i) f(x_j). \tag{14}$$

To obtain the joint distributions, Equations (1) and (2) are further written as follows:

$$F_{ELTr-PF}(x_i) = \left(\frac{x_i{}^a - k^a}{1 - k^a} \right)^b, F_{ELTr-Pow}(x_j) = \left(\frac{x_j{}^a - k^a}{1 - k^a} \right)^b,$$

$$f_{ELTr-PF}(x_i) = \frac{ab}{(1 - k^a)^b} x_i{}^{a-1} (x_i{}^a - k^a)^{b-1},$$

$$f_{ELTr-PF}(x_j) = \frac{ab}{(1 - k^a)^b} x_j{}^{a-1} (x_j{}^a - k^a)^{b-1}.$$

By substituting these expressions into Equation (14), the joint distribution of the i-th and j-th OS is obtained as:

$$f_{(i:j:n)-ELTr-PF}(x_i, x_j) = \left(\begin{array}{c} C \left(\left(\frac{x_i{}^a - k^a}{1 - k^a} \right)^b \right)^{i-1} \left(\left(\frac{x_j{}^a - k^a}{1 - k^a} \right)^b - \left(\frac{x_i{}^a - k^a}{1 - k^a} \right)^b \right)^{j-i-1} \times \\ \left(1 - \left(\frac{x_j{}^a - k^a}{1 - k^a} \right)^b \right)^{n-j} \times \\ \left(\left(\frac{ab}{(1 - k^a)^b} \right)^2 x_i{}^{a-1} (x_i{}^a - k^a)^{b-1} \right) \times \\ \left(x_j{}^{a-1} (x_j{}^a - k^a)^{b-1} \right) \end{array} \right).$$

3.6.Bivariate and multivariate extensions

In this subsection, we discuss the bivariate and multivariate extensions for the ELTr–PF distribution by following the Morgenstern and the Clayton families.

The CDF of the Bi–ELTr–PF distribution followed by the Morgenstern family for the random vector (V_1, V_2) is

$$F_{\phi-ELTr-PF}\,(V_1, V_2) = \Big(1 + \phi\big(1 - F_1(v_1)\big)\big(1 - F_2(v_2)\big)\Big) F_1(v_1) F_2(v_2),$$

where $|\phi| \le 1$, $F_1(v_1) = \left(\frac{{v_1}^a - k^a}{1 - k^a}\right)^b$, and $F_2(v_2) = \left(\frac{{v_2}^a - k^a}{1 - k^a}\right)^b$.

The CDF of the Bi– ELTr–PF distribution followed by the Clayton family for the random vector (X, Y) is

$$C(x, y) = \left(x^{-(\zeta_1+\zeta_2)} + y^{-(\zeta_1+\zeta_2)} - 1\right)^{-\frac{1}{(\zeta_1+\zeta_2)}};\ \zeta_1 + \zeta_2 \ge 0.$$

Let $v_1 \sim$ ELTr–PF (α_1, β_1), and $v_2 \sim$ ELTr–PF (α_2, β_2). Then setting $x = F_1(v_1) = \left(\frac{{v_1}^a - k^a}{1 - k^a}\right)^b$ and $y = F_2(v_2) = \left(\frac{{v_2}^a - k^a}{1 - k^a}\right)^b$.

The CDF of the Bi– ELTr–PF distribution followed by the Clayton family for the random vector (V_1, V_2) is

$$G_{Bi-ELTr-PF}\,(v_1, v_2) = \left(\left(\frac{{v_1}^a - k^a}{1 - k^a}\right)^{-b(\zeta_1+\zeta_2)} + \left(\frac{{v_2}^a - k^a}{1 - k^a}\right)^{-b(\zeta_1+\zeta_2)} - 1\right)^{-\frac{1}{(\zeta_1+\zeta_2)}}.$$

A simple *n*-dimensional extension of the last version will be

$$H_{Multi-ELTr-PF}\,(x_1, x_2, x_3, \dots, x_n) = \left(\sum_{i=1}^{n}\left(\left(\frac{{x_i}^{a_i} - {k_i}^{a_i}}{1 - {k_i}^{a_i}}\right)^{-b(\zeta_1+\zeta_2)}\right) + 1 - n\right)^{-\frac{1}{(\zeta_1+\zeta_2)}}.$$

4. Inference

In this section, the X's parameters are estimated by following the method of maximum likelihood estimation (MLE), as this method provides full information about the unknown model parameter.

Let $X_1, X_2, \dots, X_n$ be a random sample of size *n* from *X*, then the likelihood function $L(\phi) = \prod_{i=1}^{n} f(x_i; a, b)$ of *X* is given by:

$$L_{ELTr-PF}(\phi) = \left(\frac{ab}{(1 - k^a)^b}\right)^n \prod_{i=1}^{n} {x_i}^{a-1}({x_i}^a - k^a)^{b-1}.$$

The log-likelihood function, $l(\phi)$ is thus given by:

$$l_{ELTr-PF}(\phi) = \begin{pmatrix} nloga + nlogb - nlog(1-k^a) + (a-1)\sum_{i=1}^{n} logx_i \\ +(b-1)\sum_{i=1}^{n} log(x_i{}^a - k^a) \end{pmatrix}. \quad (15)$$

Partial derivatives for parameters a and b of Equation (15) yield the following:

$$\frac{\partial l_{ELTr-PF}(\phi)}{\partial a} = \frac{n}{a} + \frac{nk^a loga}{1-k^a} + \sum_{i=1}^{n} logx_i + (b-1)\sum_{i=1}^{n} \frac{x_i{}^a logx_i - k^a logk}{(x_i{}^a - k^a)},$$

$$\frac{\partial l_{ELTr-PF}(\phi)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} log(x_i{}^a - k^a).$$

The maximum likelihood estimates $(\hat{\phi}_i = \hat{a}, \hat{b})$ of the ELTr-PF distribution can be obtained by maximizing Equation (15) or by solving the above non-linear equations simultaneously. These non-linear equations although do not provide an analytical solution for the MLEs and the optimum value of a, and b. Consequently, the Newton-Raphson type algorithm is an appropriate choice in the support of MLEs.

5. Simulation

In this subsection, we discuss the following algorithm (step 1 to 5) to observe the asymptotic performance of MLE's $\hat{\phi} = (\hat{\alpha}, \hat{\beta})$.

Step -1: A random sample x1, x2, x3, …, xn of sizes n = 100, 200, 300, 400, and 1000 from Equation (14).

Step -2: Results of root mean square error (RMSE), coverage probability (CP), and average width (AW) are calculated with the assist of statistical software R. These results are presented in Tables 4 and 5.

Step -3: Each sample is replicated 1000 times.

Step -4: Gradual decrease with the increase in sample sizes is observed for RMSEs.

Step -5: CPs of all the parameters $\phi = (\alpha, \beta)$ is approximately 0.975 approaches to the nominal value and AW decreases as sample sizes increase.

Furthermore, the following measures are defined in the development of average estimate (AE), RMSE, CP, and AW, where I(.) is an indicator function and $se_{\hat{\phi}_i} = \sqrt{var\hat{\phi}_i}$ is the standard error of estimate ϕ_i.and it is given as follows:

$$AE(\hat{\phi}) = \frac{1}{N}\sum_{i=1}^{N}\hat{\phi}, RMSE(\hat{\phi}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}_i - \phi)^2},$$

$$CP(\hat{\phi}) = \sum_{i=1}^{N} I(\hat{\phi}_i - 0.95se_{\hat{\phi}_i}, \hat{\phi}_i + 0.95se_{\hat{\phi}_i}), \text{and}$$

$$AW(\hat{\varphi}) = \frac{1}{N}\sum_{i=1}^{N}\left|(\hat{\phi}_i + 0.95) - (\hat{\phi}_i - 0.95)\right|$$

Table 4. Root mean square error, coverage probability, and average width for (a =1.1, b =1.1, k=0.01).

Sample	$\mathrm{RMSE}_{(a)}$	$\mathrm{CP}_{(a)}$	$\mathrm{AW}_{(a)}$
100	0.5848	1	3.4874
200	0.5359	1	2.1784
300	0.5078	1	2.2354
400	0.4991	1	2.0208
1000	0.3330	1	1.1878

Table 5. Root mean square error, coverage probability, and average width for (a =1.1, b =1.1, k=0.01).

Sample	$\mathrm{RMSE}_{(b)}$	$\mathrm{CP}_{(b)}$	$\mathrm{AW}_{(b)}$
100	0.2320	0.98	1.9249
200	0.2339	0.98	1.4396
300	0.2259	0.98	1.1951
400	0.2235	0.98	1.0603
1000	0.2020	0.98	0.6999

In both Tables 4 and 5, the RMSE and AW values reduced as the sample size increases. This indicates that the parameters of the ELTr-PF distribution are good and stable.

6. Analysis of Real-Life Data

In this section, the application of the ELTr-PF distribution is discussed. For this, two engineering datasets are explored. The ELTr-PF distribution is compared with its competing models (presented in Table 6) based on some criteria called, -Log-likelihood (-LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), along with the good-of-fit statistics Cramer-Von Mises (CM), Anderson-Darling (AD), and Kolmogorov Smirnov (K-S) with its p-value. All the numerical results are calculated with the assistance of statistical software R with its exclusive package AdequacyModel (https://www.r-project.org/).

Table 6. List of Some Competitive models CDFs.

Model	Model's CDFs	Parameter / variable Range	Author(s)
L-I	$G_I(x) = x^a$	$a > 0$, $0 < x < 1$	[25]
L-II	$G_{II}(x) = 1 - (1-x)^a$		
Beta	$G_{III}(x) = I_x(a,b)$	$a, b > 0$ $0 < x < 1$	
Topp-Leone	$G_{IV}(x) = (2x - x^2)^a$	$a > 0$, $0 < x < 1$	[11]
Kum	$G_V(x) = 1 - (1 - x^a)^b$	$a, b > 0$, $0 < x < 1$	[10]
GPF	$G_{VI}(x) = 1 - (\mathrm{g} - x)^a (\mathrm{g} - k)^{-a}$	$a > 0$ $k < x < \mathrm{g}$	[26]
WPF	$G_{VII}(x) = 1 - e^{-a\left(\frac{x^b}{\mathrm{g}^b - x^b}\right)^c}$	$a, b, c > 0$ $0 < x < \mathrm{g}$	[27]
MT-II	$G_{VIII}(x) = e^{x^a \log 2} - 1$	$a > 0$, $0 < x < 1$	[28]

Lehmann Type–I =L–I, Lehmann Type–II =L–II, Kumaraswamy=Kum, Generalized Power Function=GPF, Weibull Power Function=WPF, Mustapha Type–II = MT-II.

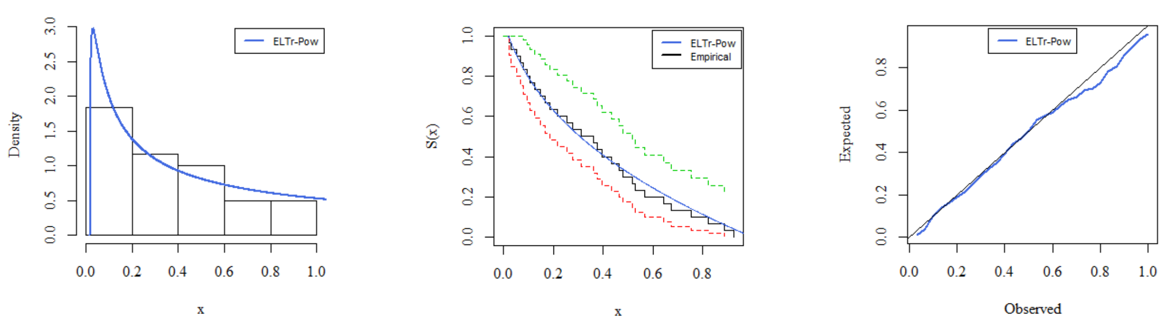
6.1. Application 1

The first data set relates to 30 measurements of tensile strength of polyester fibers discussed by [29]. The parameter estimates with standard errors (in parenthesis) and goodness of fit statistics are obtained and illustrated in Table 7.

Table 7. Parameter estimates, standard errors (in parenthesis), and goodness of fit statistics for tensile strength of polyester fibers data.

Model	Parameter estimates (Standard errors)			Statistics					
	$\hat{a}$	$\hat{b}$	$\hat{c}$	AIC	BIC	CM	AD	KS	P-value
ELTr-PF	0.2169 (0.4352)	1.4096 (0.7666)	-	-5.3369	-2.5345	0.0097	0.0791	0.0474	1.0000
WPF	3.0299 (2.2330)	1.3464 (0.9412)	0.7957 (0.373)	0.2444	4.4480	0.0174	0.1382	0.0611	0.9995
Kum	0.9627 (0.2017)	1.6081 (0.4135)	-	-2.6221	0.1803	0.0183	0.1550	0.0650	0.9987
Top-Leon	1.1091 (0.2024)	-	-	-3.8078	-2.4066	0.0189	0.1600	0.0665	0.9981
Beta	0.9666 (0.2237)	1.6204 (0.4106)	-	-2.6101	0.1923	0.0184	0.1559	0.0669	0.9979
L-II	1.6624 (0.3035)	-	-	-4.5885	-3.1873	0.0184	0.1558	0.0740	0.9924
L-I	0.7254 (0.1324)	-	-	-1.4495	-0.0483	0.0168	0.1425	0.1374	0.5754
MT-II	0.5847 (0.1176)	-	-	0.4176	1.8188	0.0212	0.1788	0.1555	0.4201

The minimum goodness of fit statistics is the criteria of a better fit model which the ELTr-PF distribution eventually satisfies. Hence, this research supports that the ELTr-PF distribution provides a better fit than its competitors. Furthermore, the curves of fitted density (a) Kaplan-Meier survival (b), and Probability-Probability (PP) (c) plots are presented in Figure 2.



(a) (b) (c)

Figure 2. Fitted plots for 30 measurements of tensile strength of polyester fibers data.

6.2. Application 2

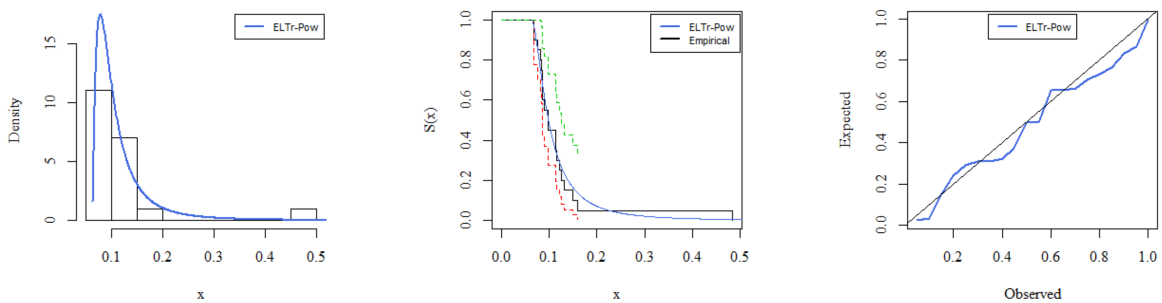
The second data set represents the failure times of 20 mechanical components studied by [30]. The parameter estimates with standard errors (in parenthesis) and goodness of fit statistics are obtained and illustrated in Table 8.

Table 8. Parameter estimates, standard errors (in parenthesis), and goodness of fit statistics for the mechanical components data.

Model	Parameter estimates (Standard errors)			statistics					
	$\hat{a}$	$\hat{b}$	$\hat{c}$	AIC	BIC	CM	AD	KS	P-value
ELTr-PF	-2.9668 (0.7335)	2.3598 (0.8327)	-	-74.6350	-72.6435	0.0488	0.3594	0.1054	0.9794
Beta	3.1119 (0.9365)	21.8184 (7.0402)	-	-51.7626	-49.7711	0.3700	2.3155	0.2538	0.1520
Kum	1.5877 (0.2444)	21.8682 (10.210)	-	-47.2969	-45.3054	0.4370	2.6508	0.2626	0.1267
WPF	25.3216 (10.981)	8.6983 (30.616)	0.1887 (0.6640)	-46.8444	-43.8572	0.3972	2.4524	0.2642	0.1226
L-II	7.3406 (1.6414)	-	-	-43.1863	-42.1906	0.3698	2.3142	0.3989	0.0034
GPF	3.1354 (0.7011)	-	-	-50.4166	-49.4209	0.4156	2.5011	0.4263	0.0014
Top-Leon	0.6247 (0.1397)	-	-	-25.4857	-24.4900	0.3391	2.1565	0.4842	0.0002
L-I	0.4484 (0.1002)	-	-	-15.1164	-14.1207	0.3211	2.0627	0.5104	0.0001
MT-II	0.3402 (0.0843)	-	-	-12.1937	-11.1979	0.3386	2.1538	0.5000	0.0001

In Table 8, it is also clear that the ELTr-PF distribution has the lowest values for all the goodness of fit statistics. Therefore, the ELTr-PF distribution is recommended over its competing distributions.

The corresponding curves of fitted density (a) Kaplan-Meier survival (b), and Probability-Probability (PP) (c) plots are presented in Figure 3.



(a) (b) (c)

Figure 3. Fitted plots for failure times of 20 mechanical components data.

7. Conclusion

The Exponentiated Left Truncated Power (ELTr-PF) distribution has been successfully explored in this research. Its various statistical properties were investigated and established. The simulation study showed that the parameters of the ELTr-PF distribution are good and stable, as the root mean square error reduces as the sample size increases. The two datasets provided in this research support that the ELTr-PF distribution is a better fit compared to the Beta distribution, Kumaraswamy distribution, Lehmann Type I and Type II distributions, Generalized Power Function, Weibull Power Function, and Mustapha Type–II distribution. The density, Kaplan-Meier, and PP curves/plots also provide sufficient information about the closest fit to subject datasets.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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