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CONVERGENCE THEOREMS OF AN IMPLICIT ITERATES WITH ERRORS FOR NON-LIPSCHITZIAN ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS

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Abstract. The aim of this paper is to study an implicit iterative process with errors for two finite families of non-Lipschitzian asymptotically quasi-nonexpansive type mappings in the framework of real Banach spaces. In this paper, we have obtained a necessary and sufficient condition to converge to common fixed points for proposed scheme and mappings and also obtained strong convergence theorems by using semi-compactness and *Condition* (B').

1. **Introduction**

Let E be a real Banach space and $\mathcal{U}_E = \{x \in E : \|x\| = 1\}$. E is said to be *uniformly convex* if for any $\varepsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in \mathcal{U}_E$,

$$\|x - y\| \geq \varepsilon \text{ implies } \left\|\frac{x - y}{2}\right\| \leq 1 - \delta.$$

In 1973, Petryshyn and Williamson [13] established a necessary and sufficient condition for a Mann [12] iterative sequence to converge strongly to a fixed point of quasi-nonexpansive mapping. Subsequently, Ghosh and Debnath [5] extended the results of [13] and obtained some necessary and sufficient conditions for an Ishikawa-type iterative sequence to converge to a fixed point of quasi-nonexpansive mapping. In 2001, Liu in [10, 11] extended the results of Ghosh and Debnath [5] to a more general asymptotically quasi-nonexpansive mappings. In 2003, Sahu and Jung [15] studied Ishikawa and Mann iteration process in Banach spaces and they proved some weak and strong convergence theorems for asymptotically quasi-nonexpansive type mapping. In 2006, Shahzad and Udomene [17] gave the necessary and sufficient condition for convergence of common fixed point of two-step modified Ishikawa iterative sequence for two asymptotically quasi-nonexpansive mappings in real Banach space.

Recently, Qin et al. [14] studied a general implicit iterative process for a finite family of generalized asymptotically quasi-nonexpansive mappings and established

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strong convergence theorem of the proposed iterative process in the framework of real Banach space.

The main goal of this paper is to establish the strong convergence of general implicit iterative process studied by Qin et al. [14] which includes Schu's explicit iterative processes and Sun's implicit iterative processes as special cases for two finite families of non-Lipschitzian asymptotically quasi-nonexpansive type mappings on a closed convex unbounded set in a real uniformly convex Banach spaces. Our results unify, improve and generalize many known results given in the existing current literature.

2. **Preliminaries and lemmas**

Let C be a nonempty subset of a normed space E and $T\colon C \to C$ be a given mapping. The set of fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : T(x) = x\}$. The mapping T is said to be

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{2.1}$$

for all $x, y \in C$.

(2) quasi-nonexpansive [2] if

$$\|Tx - p\| \leq \|x - p\| \tag{2.2}$$

for all $x \in C$, $p \in F(T)$.

(3) asymptotically nonexpansive [6] if there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + u_n)\,\|x - y\| \tag{2.3}$$

for all $x, y \in C$ and $n \geq 1$.

(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that

$$\|T^n x - p\| \leq (1 + u_n)\,\|x - p\| \tag{2.4}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

(5) uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\,\|x - y\| \tag{2.5}$$

for all $x, y \in C$ and $n \geq 1$.

(6) asymptotically nonexpansive type [8], if

$$\limsup_{n\to\infty} \left\{ \sup_{x,y\in C} \Big(\|T^n x - T^n y\| - \|x - y\| \Big) \right\} \leq 0. \tag{2.6}$$

(7) asymptotically quasi-nonexpansive type [15], if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \left\{ \sup_{x\in C,\ p\in F(T)} \Big(\|T^n x - p\| - \|x - p\| \Big) \right\} \leq 0. \tag{2.7}$$

Remark 2.1. It is easy to see that if $F(T)$ is nonempty, then asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping and asymptotically nonexpansive type mapping are the special cases of asymptotically quasi-nonexpansive type mappings.

The Mann and Ishikawa iteration processes have been used by a number of authors to approximate the fixed points of nonexpansive, asymptotically nonexpansive mappings, and quasi-nonexpansive mappings on Banach spaces (see, e.g., [5, 7, 9, 10, 11, 13, 18, 23]).

Recall that the modified Mann iteration which was introduced by Schu [16] generates a sequence $\{x_n\}$ in the following manner:

$$x_1 \in C,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \ \ n \geq 1, \tag{2.8}$$

where $\{\alpha_n\}$ is a real sequence in the interval $(0,1)$ and $T\colon C \to C$ is an asymptotically nonexpansive mapping.

In 2001, Xu and Ori [24] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space H. Let C be a nonempty subset of H. Let $T_1, T_2, \ldots, T_N$ be self-mappings of C and suppose that $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$, the set of common fixed points of $T_i, i = 1, 2, \ldots, N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows, with $\{t_n\}$ a real sequence in $(0,1)$, $x_0 \in C$:

$$\begin{aligned}
x_1 &= t_1 x_0 + (1-t_1)T_1 x_1,\\
x_2 &= t_2 x_1 + (1-t_2)T_2 x_2,\\
&\ \vdots\\
x_N &= t_N x_{N-1} + (1-t_N)T_N x_N,\\
x_{N+1} &= t_{N+1} x_N + (1-t_{N+1})T_1 x_{N+1},\\
&\ \vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1-t_n)T_n x_n, \quad n \geq 1 \tag{2.9}$$

where $T_k = T_{k \bmod N}$. (Here the mod N function takes values in $\mathcal{N}$). And they proved the weak convergence of the process (2.4).

In 2003, Sun [19] extend the process (2.9) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with $\{\alpha_n\}$ a real sequence in $(0,1)$

and an initial point $x_0 \in C$, which is defined as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1,\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2 x_{N+1},\\
&\vdots\\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2 x_{2N},\\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3 x_{2N+1},\\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^k x_n, \quad n \geq 1 \tag{2.10}$$

where $n = (k-1)N + i$, $i \in \mathcal{N}$.

Sun [19] proved the strong convergence of the process (2.10) to a common fixed point, requiring only one member T in the family $\{T_i : i \in \mathcal{N}\}$ to be semi-compact. The result of Sun [19] generalized and extended the corresponding main results of Wittmann [21] and Xu and Ori [24].

In 2010, Qin et al. [14] studied the following general implicit iteration process for two finite families of generalized asymptotically quasi-nonexpansive mappings $\{S_1, S_2, \dots, S_N\}$ and $\{T_1, T_2, \dots, T_N\}$:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 S_1 x_0 + \gamma_1 T_1 x_1 + \delta_1 u_1,\\
x_2 &= \alpha_2 x_1 + \beta_2 S_2 x_1 + \gamma_2 T_2 x_2 + \delta_2 u_2,\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + \beta_N S_N x_{N-1} + \gamma_N T_N x_N + \delta_N u_N,\\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} S_1^2 x_N + \gamma_{N+1} T_1^2 x_{N+1} + \delta_{N+1} u_{N+1}, \qquad (2.11)\\
&\vdots\\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} S_N^2 x_{2N-1} + \gamma_{2N} T_N^2 x_{2N} + \delta_{2N} u_{2N},\\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} S_1^3 x_{2N} + \gamma_{2N+1} T_1^3 x_{2N+1} + \delta_{2N+1} u_{2N+1},\\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x_{n-1} + \gamma_n T_{i(n)}^{h(n)} x_n + \delta_n u_n, \quad n \geq 1 \tag{2.12}$$

where x_0 is the initial value, $\{u_n\}$ is a bounded sequence in C, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Since for each $n \geq 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, \dots, N\} = \mathcal{N}$, $h = h(n) \geq 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$.

In this paper, motivated by [14], we study general implicit iterative process (2.12) for two finite families of non-Lipschitzian asymptotically quasi-nonexpansive type mappings $\{S_1, S_2, \ldots, S_N\}$ and $\{T_1, T_2, \ldots, T_N\}$ in Banach spaces. Also we establish some strong convergence theorems for said scheme and mappings.

We remark that implicit iterative process (2.12) which includes the explicit iterative process (2.8) and the implicit iterative process (2.10) as a special case in general.

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then the implicit iterative process (2.12) is reduced to the following implicit iterative process:

$$x_n = (\alpha_n + \beta_n)x_{n-1} + \gamma_n T_{i(n)}^{h(n)} x_n + \delta_n u_n, \quad n \geq 1. \tag{2.13}$$

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\} = \mathcal{N}$, then the implicit iterative process (2.12) is reduced to the following explicit iterative process:

$$x_n = \frac{\alpha_n}{1-\gamma_n} x_{n-1} + \frac{\beta_n}{1-\gamma_n} S_{i(n)}^{h(n)} x_n + \frac{\delta_n}{1-\gamma_n} u_n, \quad n \geq 1. \tag{2.14}$$

Denote the indexing set $\{1, 2, \ldots, N\}$ by $\mathcal{N}$. Let $\{T_i : i \in \mathcal{N}\}$ be N uniformly $L_{t,i}$-Lipschitzian asymptotically quasi-nonexpansive type self-mappings of C and $\{S_i : i \in \mathcal{N}\}$ be N uniformly $L_{s,i}$-Lipschitzian asymptotically quasi-nonexpansive type self-mappings of C. We show that (2.12) exists. Let $x_0 \in C$ and $x_1 = \alpha_1 x_0 + \beta_1 S_1 x_0 + \gamma_1 T_1 x_1 + \delta_1 u_1$. Define $W \colon C \to C$ by: $Wx = \alpha_1 x_0 + \beta_1 S_1 x_0 + \gamma_1 T_1 x_1 + \delta_1 u_1$ for all $x \in C$. The existence of x_1 is guaranteed if W has a fixed point. For any $x, y \in C$, we have

$$\begin{aligned} \|Wx - Wy\| &\leq \gamma_1 \|T_1 x - T_1 y\| \leq \gamma_1 L_{t,1} \|x - y\| \\ &\leq \gamma_1 L_t \|x - y\|, \end{aligned} \tag{2.15}$$

where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$. Now, W is a contraction if $\gamma_1 L_t < 1$ or $L_t < 1/\gamma_1$. As $\gamma_1 \in (0,1)$, therefore W is a contraction even if $L_t > 1$. By the Banach contraction principle, W has a unique fixed point. Thus, the existence of x_1 is established. Similarly, we can establish the existence of $x_2, x_3, x_4, \ldots$. Thus, the implicit algorithm (2.12) is well defined.

The distance between a point x and a set C and closed ball with center zero and radius r in E are, respectively, defined by

$$d(x, C) = \inf_{y \in C} \|x - y\|, \quad B_r(0) = \{x \in E : \|x\| \leq r\}. \tag{2.16}$$

In order to prove our main results, we need the following definition and lemmas.

Definition 2.1.(see [19]) Let C be a closed subset of a normed space E and let $T \colon C \to C$ be a mapping. Then T is said to be semi-compact if for any bounded sequence $\{x_n\}$ in C with $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^* \in C$ as $n_k \to \infty$.

Lemma 2.1.(see [20]) Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq a_n + b_n, \; n \geq 1. \tag{2.17}$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2.(see [14]) Let E be a uniformly convex Banach space, $s > 0$ a positive number, and $B_s(0)$ a closed ball of E. Then there exists a continuous, strictly increasing and convex function $g\colon [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned} \|ax + by + cz + dw\|^2 &\leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 \\ &\quad -abg(\|x-y\|) \end{aligned} \tag{2.18}$$

for all $x, y, z, w \in B_s(0) = \{x \in E : \|x\| \leq s\}$ and $a, b, c, d \in [0,1]$ such that $a + b + c + d = 1$.

3. Main Results

We begin with a necessary and sufficient condition for convergence of $\{x_n\}$ generated by the general implicit iterative process (2.12) to prove the following result.

Theorem 3.1. Let C be a nonempty closed convex subset of a Banach space E. Let $T_i\colon C \to C$ be a uniformly $L_{t,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping and let $S_i\colon C \to C$ be a uniformly $L_{s,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping for each $1 \leq i \leq N$. Assume that $F = \cap_{i=1}^{N} F(T_i) \bigcap \cap_{i=1}^{N} F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n+\delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a iterative sequence generated in (2.12). Put

$$A_n = \max\Big\{0, \sup_{p\in F,\; n\geq 1} \Big(\Big\|T_{i(n)}^{h(n)} x_n - p\Big\| - \|x_n - p\| \Big) : i \in \mathcal{N}\Big\} \tag{3.1}$$

and

$$B_n = \max\Big\{0, \sup_{p\in F,\; n\geq 1} \Big(\Big\|S_{i(n)}^{h(n)} x_{n-1} - p\Big\| - \|x_{n-1} - p\| \Big) : i \in \mathcal{N}\Big\} \tag{3.2}$$

such that $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$. Assume that the following restrictions are satisfied:

(a) there exist constants $a, b, c, d \in (0,1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$, and $c \leq \gamma_n \leq d < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\{x_n\}$ converges strongly to some point in F if and only if

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

Proof. The necessity is obvious and so it is omitted. Now, we prove the sufficiency. For any $p \in F$, from (2.12), (3.1) and (3.2), we have

$$\begin{aligned}
\|x_n - p\| &= \left\|\alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x_{n-1} + \gamma_n T_{i(n)}^{h(n)} x_n + \delta_n u_n - p\right\| \\
&= \left\|\alpha_n (x_{n-1} - p) + \beta_n (S_{i(n)}^{h(n)} x_{n-1} - p) + \gamma_n (T_{i(n)}^{h(n)} x_n - p) + \delta_n (u_n - p)\right\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n \left\|S_{i(n)}^{h(n)} x_{n-1} - p\right\| + \gamma_n \left\|T_{i(n)}^{h(n)} x_n - p\right\| \\
&\quad + \delta_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n (\|x_{n-1} - p\| + B_n) + \gamma_n (\|x_n - p\| + A_n) \\
&\quad + \delta_n \|u_n - p\| \\
&= (\alpha_n + \beta_n) \|x_{n-1} - p\| + \gamma_n \|x_n - p\| + \beta_n B_n + \gamma_n A_n \\
&\quad + \delta_n \|u_n - p\| \\
&= (1 - \gamma_n - \delta_n) \|x_{n-1} - p\| + \gamma_n \|x_n - p\| + \beta_n B_n + \gamma_n A_n \\
&\quad + \delta_n \|u_n - p\| \\
&\leq (1 - \gamma_n) \|x_{n-1} - p\| + \gamma_n \|x_n - p\| + A_n + B_n \\
&\quad + \delta_n \|u_n - p\| . \qquad (3.3)
\end{aligned}$$

Since from restriction (a) $\gamma_n \leq d$ it follows from (3.3) that

$$\begin{aligned}
\|x_n - p\| &\leq \|x_{n-1} - p\| + \frac{1}{1-\gamma_n}(A_n + B_n) + \frac{\delta_n}{1-\gamma_n} \|u_n - p\| \\
&\leq \|x_{n-1} - p\| + \frac{1}{1-d}(A_n + B_n) + \frac{\delta_n}{1-d} \|u_n - p\| \\
&\leq \|x_{n-1} - p\| + \frac{1}{1-d}(A_n + B_n) + \frac{M}{1-d}\delta_n, \qquad (3.4)
\end{aligned}$$

where $M = \sup_{n \geq 1}\{\|u_n - p\|\}$, since $\{u_n\}$ is a bounded sequence in C. This implies that

$$d(x_n, F) \leq d(x_{n-1}, F) + m_n, \qquad (3.5)$$

where $m_n = \frac{1}{1-d}(A_n + B_n) + \frac{M}{1-d}\delta_n$. Since by assumptions of the theorem, $\sum_{n=1}^{\infty} A_n < \infty$, $\sum_{n=1}^{\infty} B_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, it follows that $\sum_{n=1}^{\infty} m_n < \infty$. Therefore, from Lemma 2.1, we know that $\lim_{n\to\infty} d(x_n, F)$ exists. Since by hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$, so by Lemma 2.1, we have

$$\lim_{n\to\infty} d(x_n, F) = 0. \qquad (3.6)$$

Next we prove that $\{x_n\}$ is a Cauchy sequence in C. It follows from (3.4) that for any $m \geq 1$, for all $n \geq n_0$ and for any $p \in F$, we have

$$\begin{aligned}
\|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + \frac{1}{1-d}(A_{n+m} + B_{n+m}) + \frac{M}{1-d}\delta_{n+m} \\
&\leq \|x_{n+m-2} - p\| + \frac{1}{1-d}(A_{n+m-1} + B_{n+m-1}) + \frac{M}{1-d}\delta_{n+m-1} \\
&\quad + \frac{1}{1-d}(A_{n+m} + B_{n+m}) + \frac{M}{1-d}\delta_{n+m} \\
&\leq \|x_{n+m-2} - p\| + \frac{1}{1-d}[(A_{n+m} + A_{n+m-1}) + (B_{n+m} + B_{n+m-1})] \\
&\quad + \frac{M}{1-d}[\delta_{n+m} + \delta_{n+m-1}] \\
&\leq \ldots \\
&\leq \ldots \\
&\leq \|x_n - p\| + \frac{1}{1-d}\sum_{k=n+1}^{n+m}(A_k + B_k) + \frac{M}{1-d}\sum_{k=n+1}^{n+m}\delta_k. \qquad (3.7)
\end{aligned}$$

So, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\
&\leq \|x_n - p\| + \frac{1}{1-d}\sum_{k=n+1}^{n+m}(A_k + B_k) + \frac{M}{1-d}\sum_{k=n+1}^{n+m}\delta_k \\
&\quad + \|x_n - p\| \\
&= 2\|x_n - p\| + \frac{1}{1-d}\sum_{k=n+1}^{n+m}(A_k + B_k) + \frac{M}{1-d}\sum_{k=n+1}^{n+m}\delta_k.
\end{aligned}$$

(3.8)

Then, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq 2d(x_n, F) + \frac{1}{1-d}\sum_{k=n+1}^{n+m}(A_k + B_k) \\
&\quad + \frac{M}{1-d}\sum_{k=n+1}^{n+m}\delta_k, \quad \forall n \geq n_0. \qquad (3.9)
\end{aligned}$$

For any given $\varepsilon > 0$, there exists a positive integer $n_1 \geq n_0$ such that for any $n \geq n_1$,

$$d(x_n, F) < \frac{\varepsilon}{6}, \qquad \sum_{k=n+1}^{n+m}(A_k + B_k) < \frac{(1-d)\varepsilon}{3}, \qquad (3.10)$$

and

$$\sum_{k=n+1}^{n+m}\delta_k < \frac{(1-d)\varepsilon}{3M}. \qquad (3.11)$$

Thus, from (3.9)-(3.11) and $n \geq n_1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &< 2.\frac{\varepsilon}{6} + \frac{1}{1-d}.\frac{(1-d)\varepsilon}{3} \\ &\quad + \frac{M}{(1-d)}.\frac{(1-d)\varepsilon}{3M} \\ &= \varepsilon. \end{aligned} \tag{3.12}$$

This implies that $\{x_n\}$ is a Cauchy sequence in C. Thus, the completeness of E implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = p$. Now, we have to show that $\{x_n\}$ converges to some common fixed point in F. Indeed, we know that the set $F = \cap_{i=1}^N F(T_i) \bigcap \cap_{i=1}^N F(S_i)$ is closed. From the continuity of $d(x, F) = 0$ with $\lim_{n\to\infty} d(x_n, F) = 0$ and $\lim_{n\to\infty} x_n = p$, we get

$$d(p, F) = 0, \tag{3.13}$$

and so $p \in F$, that is, $\{x_n\}$ converges to some common fixed point in F. This completes the proof.

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then Theorem 3.1 is reduced to the following result:

Corollary 3.1. Let C be a nonempty closed convex subset of a Banach space E. Let $T_i \colon C \to C$ be a uniformly $L_{t,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping for each $1 \leq i \leq N$. Assume that $F = \cap_{i=1}^N F(T_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a iterative sequence generated in (2.13). Put

$$A_n = \max\Big\{0, \sup_{p\in F,\ n\geq 1} \Big(\Big\|T_{i(n)}^{h(n)} x_n - p\Big\| - \|x_n - p\| \Big) : i \in \mathcal{N}\Big\},$$

such that $\sum_{n=1}^{\infty} A_n < \infty$. Assume that the following restrictions are satisfied:

(a) there exist constants $a, b, c \in (0, 1)$ such that $a \leq \alpha_n + \beta_n$ and $b \leq \gamma_n \leq c < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\{x_n\}$ converges strongly to some point in F if and only if

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then Theorem 3.1 is reduced to the following result:

Corollary 3.2. Let C be a nonempty closed convex subset of a Banach space E. Let $S_i \colon C \to C$ be a uniformly $L_{s,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping for each $1 \leq i \leq N$. Assume that $F = \cap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$

be sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n+\delta_n=1$ for each $n\geq 1$. Let $\{x_n\}$ be a iterative sequence generated in (2.14). Put

$$B_n \quad = \quad \max\Big\{0, \sup_{p\in F,\ n\geq 1}\Big(\Big\|S_{i(n)}^{h(n)}x_{n-1}-p\Big\|-\|x_{n-1}-p\|\Big):i\in\mathcal{N}\Big\},$$

such that $\sum_{n=1}^{\infty}B_n<\infty$. Assume that the following restrictions are satisfied:

(a) there exist constants $a,b,c,d\in(0,1)$ such that $a\leq\alpha_n$, $b\leq\beta_n$ and $c\leq\gamma_n$, for all $n\geq 1$;

(b) $\sum_{n=1}^{\infty}\delta_n<\infty$.

Then $\{x_n\}$ converges strongly to some point in F if and only if

$$\liminf_{n\to\infty} d(x_n,F)=0.$$

We prove a lemma which plays an important role in establishing strong convergence of the general implicit iterative process (2.12) in a uniformly convex Banach space.

Lemma 3.1. Let C be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_i\colon C\to C$ be a uniformly $L_{t,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping and let $S_i\colon C\to C$ be a uniformly $L_{s,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping for each $1\leq i\leq N$. Assume that $F=\cap_{i=1}^{N}F(T_i)\bigcap\cap_{i=1}^{N}F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n+\delta_n=1$ for each $n\geq 1$. Let $\{x_n\}$ be a iterative sequence generated in (2.12). Put

$$A_n \quad = \quad \max\Big\{0, \sup_{p\in F,\ n\geq 1}\Big(\Big\|T_{i(n)}^{h(n)}x_n-p\Big\|-\|x_n-p\|\Big):i\in\mathcal{N}\Big\}$$

and

$$B_n \quad = \quad \max\Big\{0, \sup_{p\in F,\ n\geq 1}\Big(\Big\|S_{i(n)}^{h(n)}x_{n-1}-p\Big\|-\|x_{n-1}-p\|\Big):i\in\mathcal{N}\Big\},$$

such that $\sum_{n=1}^{\infty}A_n<\infty$ and $\sum_{n=1}^{\infty}B_n<\infty$. Assume that the following restrictions are satisfied:

(a) there exist constants $a,b,c,d\in(0,1)$ such that $a\leq\alpha_n$, $b\leq\beta_n$, and $c\leq\gamma_n\leq d<1/L_t$, where $L_t=\max\{L_{t,i}:1\leq i\leq N\}$, for all $n\geq 1$;

(b) $\sum_{n=1}^{\infty}\delta_n<\infty$.

Then

$$\lim_{n\to\infty}\|x_n-T_rx_n\|=\lim_{n\to\infty}\|x_n-S_rx_n\|=0,\quad \forall\ r\in\{1,2,\ldots,N\}.$$

Proof. As in the proof of Theorem 3.1, $\lim_{n\to\infty}\|x_n-q\|$ exists for all $q\in F$. It follows that the sequence $\{x_n\}$ is bounded. In view of Lemma 2.2, we see that

$$\begin{aligned}
\|x_n-q\|^2 &\le \alpha_n\|x_{n-1}-q\|^2+\beta_n\left\|S_{i(n)}^{h(n)}x_{n-1}-q\right\|^2+\gamma_n\left\|T_{i(n)}^{h(n)}x_n-q\right\|^2\\
&\quad +\delta_n\|u_n-q\|^2-\alpha_n\beta_n g\Big(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\Big)\\
&\le \alpha_n\|x_{n-1}-q\|^2+\beta_n[\|x_{n-1}-q\|+B_n]^2+\gamma_n[\|x_n-q\|+A_n]^2\\
&\quad +\delta_n\|u_n-q\|^2-\alpha_n\beta_n g\Big(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\Big)\\
&\le \alpha_n\|x_{n-1}-q\|^2+\beta_n[\|x_{n-1}-q\|^2+K'_n]+\gamma_n[\|x_n-q\|^2+K''_n]\\
&\quad +M_1\delta_n-\alpha_n\beta_n g\Big(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\Big)\\
&= (\alpha_n+\beta_n)\|x_{n-1}-q\|^2+\gamma_n\|x_n-q\|^2+\beta_nK'_n+\gamma_nK''_n\\
&\quad +M_1\delta_n-\alpha_n\beta_n g\Big(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\Big)\\
&= (1-\gamma_n-\delta_n)\|x_{n-1}-q\|^2+\gamma_n\|x_n-q\|^2+\beta_nK'_n+\gamma_nK''_n\\
&\quad +M_1\delta_n-\alpha_n\beta_n g\Big(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\Big)\\
&\le (1-\gamma_n)\|x_{n-1}-q\|^2+\gamma_n\|x_n-q\|^2+(K'_n+K''_n)\\
&\quad +M_1\delta_n-\alpha_n\beta_n g\Big(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\Big), \qquad (3.14)
\end{aligned}$$

where M_1 is a appropriate constant such that $M_1=\sup_{n\ge1}\{\|u_n-q\|^2\}$ and $K'_n=B_n^2+2\|x_n-q\|B_n$ and $K''_n=A_n^2+2\|x_n-q\|A_n$, since $\sum_{n=1}^\infty A_n<\infty$ and $\sum_{n=1}^\infty B_n<\infty$, it follows that $\sum_{n=1}^\infty K'_n<\infty$ and $\sum_{n=1}^\infty K''_n<\infty$. This implies that

$$\begin{aligned}
\alpha_n\beta_n g\Big(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\Big) &\le (1-\gamma_n)\big[\,\|x_{n-1}-q\|^2-\|x_n-q\|^2\,\big]\\
&\quad +(K'_n+K''_n)+M_1\delta_n. \qquad (3.15)
\end{aligned}$$

In view of restrictions (a), (b), $\sum_{n=1}^\infty K'_n<\infty$ and $\sum_{n=1}^\infty K''_n<\infty$, we obtain that

$$\lim_{n\to\infty} g\Big(\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|\Big)=0. \qquad (3.16)$$

Since $g\colon[0,\infty)\to[0,\infty)$ is a continuous, strictly increasing, and convex function with $g(0)=0$, we obtain that

$$\lim_{n\to\infty}\left\|S_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\right\|=0. \qquad (3.17)$$

Next, we show that

$$\lim_{n\to\infty}\left\|T_{i(n)}^{h(n)}x_n-x_{n-1}\right\|=0. \qquad (3.18)$$

From Lemma 2.2, we also see that

$$\begin{aligned}
\|x_n - q\|^2 &\leq \alpha_n \|x_{n-1} - q\|^2 + \beta_n \left\|S_{i(n)}^{h(n)} x_{n-1} - q\right\|^2 + \gamma_n \left\|T_{i(n)}^{h(n)} x_n - q\right\|^2 \\
&\quad + \delta_n \|u_n - q\|^2 - \alpha_n \gamma_n g\Big(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\| \Big) \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + \beta_n [\|x_{n-1} - q\| + B_n]^2 + \gamma_n [\|x_n - q\| + A_n]^2 \\
&\quad + \delta_n \|u_n - q\|^2 - \alpha_n \gamma_n g\Big(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\| \Big) \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + \beta_n [\|x_{n-1} - q\|^2 + K'_n] + \gamma_n [\|x_n - q\|^2 + K''_n] \\
&\quad + M_1 \delta_n - \alpha_n \gamma_n g\Big(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\| \Big) \\
&= (\alpha_n + \beta_n) \|x_{n-1} - q\|^2 + \gamma_n \|x_n - q\|^2 + \beta_n K'_n + \gamma_n K''_n \\
&\quad + M_1 \delta_n - \alpha_n \gamma_n g\Big(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\| \Big) \\
&= (1 - \gamma_n - \delta_n) \|x_{n-1} - q\|^2 + \gamma_n \|x_n - q\|^2 + \beta_n K'_n + \gamma_n K''_n \\
&\quad + M_1 \delta_n - \alpha_n \gamma_n g\Big(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\| \Big) \\
&\leq (1 - \gamma_n) \|x_{n-1} - q\|^2 + \gamma_n \|x_n - q\|^2 + (K'_n + K''_n) \\
&\quad + M_1 \delta_n - \alpha_n \gamma_n g\Big(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\| \Big), \qquad (3.19)
\end{aligned}$$

This implies that

$$\begin{aligned}
\alpha_n \gamma_n g\Big(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\| \Big) &\leq (1 - \gamma_n) \big[\, \|x_{n-1} - q\|^2 - \|x_n - q\|^2 \,\big] \\
&\quad + (K'_n + K''_n) + M_1 \delta_n. \qquad (3.20)
\end{aligned}$$

In view of restrictions (a), (b), $\sum_{n=1}^{\infty} K'_n < \infty$ and $\sum_{n=1}^{\infty} K''_n < \infty$, we obtain that

$$\lim_{n\to\infty} g\Big(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\| \Big) = 0. \qquad (3.21)$$

Since $g\colon [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$, we obtain that (3.18) holds. Notice that

$$\begin{aligned}
\|x_n - x_{n-1}\| &\leq \beta_n \left\|S_{i(n)}^{h(n)} x_{n-1} - x_{n-1}\right\| + \gamma_n \left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\| \\
&\quad + \delta_n \|u_n - x_{n-1}\|. \qquad (3.22)
\end{aligned}$$

In view of (3.17) and (3.18), we see from the restriction (b) that

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0, \qquad (3.23)$$

which implies that

$$\lim_{n\to\infty} \|x_n - x_{n+j}\| = 0, \quad \forall\ j \in \{1, 2, \ldots, N\}. \qquad (3.24)$$

Since for any positive integer $n > N$, it can be written as $n = (h(n)-1)N + i(n)$, where $i(n) \in \{1, 2, \ldots, N\} = I$, observe that

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\leq \left\|x_{n-1} - T_{i(n)}^{h(n)} x_n\right\| + \left\|T_{i(n)}^{h(n)} x_n - T_n x_n\right\| \\ &\leq \left\|x_{n-1} - T_{i(n)}^{h(n)} x_n\right\| + L_t \left\|T_{i(n)}^{h(n)-1} x_n - x_n\right\| \\ &\leq \left\|x_{n-1} - T_{i(n)}^{h(n)} x_n\right\| \\ &\quad + L_t\Big(\left\|T_{i(n)}^{h(n)-1} x_n - T_{i(n-N)}^{h(n)-1} x_{n-N}\right\| \\ &\quad + \left\|T_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\right\| \\ &\quad + \left\|x_{(n-N)-1} - x_n\right\| \Big). \end{aligned} \tag{3.25}$$

Since for each $n > N$, $n = (n-N)(mod\ N)$, on the other hand, we obtain from $n = (h(n)-1)N+i(n)$ that $n-N = ((h(n)-1)-1)N+i(n) = (h(n-N)-1)N+i(n-N)$. That is,

$$h(n-N) = h(n) - 1, \qquad i(n-N) = i(n). \tag{3.26}$$

Notice that

$$\begin{aligned} \left\|T_{i(n)}^{h(n)-1} x_n - T_{i(n-N)}^{h(n)-1} x_{n-N}\right\| &= \left\|T_{i(n)}^{h(n)-1} x_n - T_{i(n)}^{h(n)-1} x_{n-N}\right\| \\ &\leq L_t \left\|x_n - x_{n-N}\right\|, \end{aligned} \tag{3.27}$$

and

$$\left\|T_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\right\| = \left\|T_{i(n-N)}^{h(n-N)} x_{n-N} - x_{(n-N)-1}\right\|. \tag{3.28}$$

Substituting (3.27) and (3.28) into (3.25), we obtain that

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\leq \left\|x_{n-1} - T_{i(n)}^{h(n)} x_n\right\| + L_t\Big(L_t \left\|x_n - x_{n-N}\right\| \\ &\quad + \left\|T_{i(n-N)}^{h(n-N)} x_{n-N} - x_{(n-N)-1}\right\| + \left\|x_{(n-N)-1} - x_n\right\| \Big). \end{aligned} \tag{3.29}$$

In view of (3.18) and (3.24), we obtain that

$$\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0. \tag{3.30}$$

Notice that

$$\|x_n - T_n x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|. \tag{3.31}$$

It follows from (3.23) and (3.30) that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.32}$$

Notice that

$$\begin{aligned} \|x_n - T_{n+j} x_n\| &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| \\ &\quad + \|T_{n+j} x_{n+j} - T_{n+j} x_n\| \\ &\leq (1 + L_t) \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| \end{aligned} \tag{3.33}$$

for all $j \in \{1, 2, \ldots, N\}$.

From (3.24) and (3.32), we obtain that

$$\lim_{n\to\infty} \|x_n - T_{n+j}x_n\| = 0, \quad \forall\ j \in \{1, 2, \dots, N\}. \tag{3.34}$$

Note that any subsequence of a convergent sequence converges to the same limit, it follows that

$$\lim_{n\to\infty} \|x_n - T_r x_n\| = 0, \quad \forall\ r \in \{1, 2, \dots, N\}. \tag{3.35}$$

Letting $L_s = \max\{L_{s,i} : 1 \le i \le N\}$, we have

$$\begin{aligned} \left\|S_{i(n)}^{h(n)} x_n - x_{n-1}\right\| &\le \left\|S_{i(n)}^{h(n)} x_n - S_{i(n)}^{h(n)} x_{n-1}\right\| + \left\|S_{i(n)}^{h(n)} x_{n-1} - x_{n-1}\right\| \\ &\le L_s \|x_n - x_{n-1}\| + \left\|S_{i(n)}^{h(n)} x_{n-1} - x_{n-1}\right\|. \end{aligned} \tag{3.36}$$

In view of (3.17) and (3.23), we see that

$$\left\|S_{i(n)}^{h(n)} x_n - x_{n-1}\right\| = 0. \tag{3.37}$$

Observe that

$$\begin{aligned} \|x_{n-1} - S_n x_{n-1}\| &\le \left\|x_{n-1} - S_{i(n)}^{h(n)} x_{n-1}\right\| + \left\|S_{i(n)}^{h(n)} x_{n-1} - S_n x_{n-1}\right\| \\ &\le \left\|x_{n-1} - S_{i(n)}^{h(n)} x_{n-1}\right\| + L_s \left\|S_{i(n)}^{h(n)-1} x_{n-1} - x_{n-1}\right\| \\ &\le \left\|x_{n-1} - S_{i(n)}^{h(n)} x_{n-1}\right\| + L_s\Big(\left\|S_{i(n)}^{h(n)-1} x_{n-1} - S_{i(n-N)}^{h(n)-1} x_{n-N}\right\| \\ &\quad + \left\|S_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\right\| + \left\|x_{(n-N)-1} - x_{n-1}\right\| \Big). \end{aligned} \tag{3.38}$$

In view of

$$\begin{aligned} \left\|S_{i(n)}^{h(n)-1} x_{n-1} - S_{i(n-N)}^{h(n)-1} x_{n-N}\right\| &= \left\|S_{i(n)}^{h(n)-1} x_{n-1} - S_{i(n)}^{h(n)-1} x_{n-N}\right\| \\ &\le L_s \|x_{n-1} - x_{n-N}\|, \end{aligned} \tag{3.39}$$

and

$$\left\|S_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\right\| = \left\|S_{i(n-N)}^{h(n-N)} x_{n-N} - x_{(n-N)-1}\right\|, \tag{3.40}$$

we obtain that

$$\begin{aligned} \|x_{n-1} - S_n x_{n-1}\| &\le \left\|x_{n-1} - S_{i(n)}^{h(n)} x_{n-1}\right\| + L_s\Big(L_s \|x_{n-1} - x_{n-N}\| \\ &\quad + \left\|S_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\right\| + \left\|x_{(n-N)-1} - x_{n-1}\right\| \Big). \end{aligned} \tag{3.41}$$

In view of (3.17), (3.24) and (3.37), we obtain that

$$\lim_{n\to\infty} \|x_{n-1} - S_n x_{n-1}\| = 0. \tag{3.42}$$

Notice that

$$\begin{aligned} \|x_n - S_n x_n\| &\le \|x_n - x_{n-1}\| + \|x_{n-1} - S_n x_{n-1}\| + \|S_n x_{n-1} - S_n x_n\| \\ &\le (1 + L_s) \|x_n - x_{n-1}\| + \|x_{n-1} - S_n x_{n-1}\|. \end{aligned} \tag{3.43}$$

From (3.23) and (3.42), we see that

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{3.44}$$

On the other hand, we have

$$
\begin{aligned}
\|x_n - S_{n+j}x_n\| &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - S_{n+j}x_{n+j}\| + \|S_{n+j}x_{n+j} - S_{n+j}x_n\| \\
&\leq (1+L_s)\|x_n - x_{n+j}\| + \|x_{n+j} - S_{n+j}x_{n+j}\|, \quad \forall\ j \in \{1,2,\ldots,N\}.
\end{aligned} \tag{3.45}
$$

It follows from (3.24) and (3.45) that

$$\lim_{n\to\infty} \|x_n - S_{n+j}x_n\| = 0, \quad \forall\ j \in \{1,2,\ldots,N\}. \tag{3.46}$$

Note that any subsequence of a convergent sequence converges to the same limit, it follows that

$$\lim_{n\to\infty} \|x_n - S_r x_n\| = 0, \quad \forall\ r \in \{1,2,\ldots,N\}. \tag{3.47}$$

This completes the proof.

Now, we are in a position to prove our strong convergence theorems.

Theorem 3.2. Let C be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_i\colon C \to C$ be a uniformly $L_{t,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping and let $S_i\colon C \to C$ be a uniformly $L_{s,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping for each $1 \leq i \leq N$. Assume that $F = \cap_{i=1}^N F(T_i) \bigcap \cap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a iterative sequence generated in (2.12). Put

$$A_n = \max\Big\{0, \sup_{p\in F,\ n\geq 1} \Big(\Big\|T_{i(n)}^{h(n)}x_n - p\Big\| - \|x_n - p\| \Big) : i \in \mathcal{N}\Big\}$$

and

$$B_n = \max\Big\{0, \sup_{p\in F,\ n\geq 1} \Big(\Big\|S_{i(n)}^{h(n)}x_{n-1} - p\Big\| - \|x_{n-1} - p\| \Big) : i \in \mathcal{N}\Big\},$$

such that $\sum_{n=1}^\infty A_n < \infty$ and $\sum_{n=1}^\infty B_n < \infty$. Assume that the following restrictions are satisfied:

(a) there exist constants $a, b, c, d \in (0,1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$, and $c \leq \gamma_n \leq d < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;

(b) $\sum_{n=1}^\infty \delta_n < \infty$.

If one of $\{S_1, S_2, \ldots, S_N\}$ or one of $\{T_1, T_2, \ldots, T_N\}$ is semicompact, then the sequence $\{x_n\}$ converges strongly to some point in F.

Proof. By Lemma 3.1, it follows that

$$\lim_{n\to\infty} \|x_n - T_r x_n\| = \lim_{n\to\infty} \|x_n - S_r x_n\| = 0, \quad \forall\ r \in \{1,2,\ldots,N\}. \tag{3.48}$$

Without any loss of generality, we may assume that S_1 is semi-compact. Therefore, by (3.48), it follows that $\lim_{n\to\infty} \|x_n - S_1 x_n\| = 0$. Since S_1 is semi-compact, therefore there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in C$. For each $r \in \{1, 2, \dots, N\}$, we get that

$$\|x^* - S_r x^*\| \leq \left\|x^* - x_{n_j}\right\| + \left\|x_{n_j} - S_r x_{n_j}\right\| + \left\|S_r x_{n_j} - S_r x^*\right\|. \tag{3.49}$$

Since S_r is Lipschitz continuous, we obtain from (3.48) that $x^* \in \cap_{r=1}^N F(S_r)$. Notice that

$$\|x^* - T_r x^*\| \leq \left\|x^* - x_{n_j}\right\| + \left\|x_{n_j} - T_r x_{n_j}\right\| + \left\|T_r x_{n_j} - T_r x^*\right\|. \tag{3.50}$$

Since T_r is Lipschitz continuous, we obtain from (3.48) that $x^* \in \cap_{r=1}^N F(T_r)$. This means that $x^* \in F$. In view of Theorem 3.1, we obtain that $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$, therefore $\{x_n\}$ converges to $x^* \in F$, and hence the result. This completes the proof.

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \dots, N\}$, then Theorem 3.2 is reduced to the following result:

Corollary 3.3. Let C be a nonempty closed convex subset of a Banach space E. Let $T_i : C \to C$ be a uniformly $L_{t,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping for each $1 \leq i \leq N$. Assume that $F = \cap_{i=1}^N F(T_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a iterative sequence generated in (2.13). Put

$$A_n = \max\Big\{0, \sup_{p\in F,\ n\geq 1} \Big(\Big\|T_{i(n)}^{h(n)} x_n - p\Big\| - \|x_n - p\|\Big) : i \in \mathcal{N}\Big\},$$

such that $\sum_{n=1}^{\infty} A_n < \infty$. Assume that the following restrictions are satisfied:

(a) there exist constants $a, b, c \in (0, 1)$ such that $a \leq \alpha_n + \beta_n$ and $b \leq \gamma_n \leq c < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If one of $\{T_1, T_2, \dots, T_N\}$ is semicompact, then the sequence $\{x_n\}$ converges strongly to some point in F.

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \dots, N\}$, then Theorem 3.2 is reduced to the following result:

Corollary 3.4. Let C be a nonempty closed convex subset of a Banach space E. Let $S_i : C \to C$ be a uniformly $L_{s,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping for each $1 \leq i \leq N$. Assume that $F = \cap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a iterative sequence generated in (2.14). Put

$$B_n = \max\Big\{0, \sup_{p\in F,\ n\geq 1} \Big(\Big\|S_{i(n)}^{h(n)} x_{n-1} - p\Big\| - \|x_{n-1} - p\|\Big) : i \in \mathcal{N}\Big\},$$

such that $\sum_{n=1}^{\infty} B_n < \infty$. Assume that the following restrictions are satisfied:

(a) there exist constants $a, b, c, d \in (0, 1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$ and $c \leq \gamma_n$, for all $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If one of $\{S_1, S_2, \ldots, S_N\}$ is semicompact, then the sequence $\{x_n\}$ converges strongly to some point in F.

Remark 3.1. Theorem 3.2 extends and improves Theorem 3.3 due to Sun [19] to the case of more general class of asymptotically quasi-nonexpansive mapping and general implicit iterative process and without the boundedness of C which in turn generalizes Theorem 2 by Wittmann [21] from Hilbert spaces to uniformly convex Banach spaces.

In 2005, Chidume and Shahzad [1]) introduced the following conception. Recall that a family $\{T_i\}_{i=1}^N \colon C \to C$ with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy *Condition* (B) on C if there is a nondecreasing function $f \colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$

$$\max_{1 \leq i \leq N} \{\|x - T_i x\|\} \geq f(d(x, F)). \tag{3.51}$$

Based on Condition (B), Qin et al. [14] introduced the following conception for two finite families of mappings. Recall that two families $\{S_i\}_{i=1}^N \colon C \to C$ and $\{T_i\}_{i=1}^N \colon C \to C$ with $F = \cap_{i=1}^N F(S_i) \bigcap \cap_{i=1}^N F(T_i) \neq \emptyset$ are said to satisfy *Condition* (B') if there is a nondecreasing function $f \colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$

$$\max_{1 \leq i \leq N} \{\|x - S_i x\| + \|x - T_i x\|\} \geq f(d(x, F)). \tag{3.52}$$

Note that *Condition* (B') defined above reduces to the *Condition* (B) [1] if we choose $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$.

Finally, an application of the convergence criteria established in Theorem 3.1 is given below to obtain yet another strong convergence result in our setting.

4. **Application**

Theorem 4.1. Let C be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_i \colon C \to C$ be a uniformly $L_{t,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping and let $S_i \colon C \to C$ be a uniformly $L_{s,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping for each $1 \leq i \leq N$. Assume that $F = \cap_{i=1}^N F(T_i) \bigcap \cap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a iterative sequence generated in (2.12). Put

$$A_n = \max\left\{0, \sup_{p \in F,\ n \geq 1} \left(\left\|T_{i(n)}^{h(n)} x_n - p\right\| - \|x_n - p\| \right) : i \in \mathcal{N}\right\}$$

and

$$B_n \quad = \quad \max\Big\{0, \sup_{p \in F,\ n \geq 1} \Big(\Big\| S_{i(n)}^{h(n)} x_{n-1} - p \Big\| - \| x_{n-1} - p \| \Big) : i \in \mathcal{N} \Big\},$$

such that $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$. Assume that the following restrictions are satisfied:

(a) there exist constants $a, b, c, d \in (0,1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$, and $c \leq \gamma_n \leq d < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If $\{S_1, S_2, \ldots, S_N\}$ and $\{T_1, T_2, \ldots, T_N\}$ satisfy *Condition* (B'), then the iterative sequence $\{x_n\}$ converges strongly to some point in F.

Proof. As in the proof of Theorem 3.2, (3.48) holds. Taking $\liminf$ on both sides of *Condition* (B') and using (3.48), we have that $\liminf_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0.\infty)$, it follows that $\liminf_{n\to\infty} d(x_n, F) = 0$. Now by Theorem 3.1, $x_n \to p \in F$, that is, $\{x_n\}$ converges strongly to a point in F. This completes the proof.

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then Theorem 4.1 is reduced to the following result:

Corollary 4.1. Let C be a nonempty closed convex subset of a Banach space E. Let $T_i : C \to C$ be a uniformly $L_{t,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping for each $1 \leq i \leq N$. Assume that $F = \cap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a iterative sequence generated in (2.13). Put

$$A_n \quad = \quad \max\Big\{0, \sup_{p \in F,\ n \geq 1} \Big(\Big\| T_{i(n)}^{h(n)} x_n - p \Big\| - \| x_n - p \| \Big) : i \in \mathcal{N} \Big\},$$

such that $\sum_{n=1}^{\infty} A_n < \infty$. Assume that the following restrictions are satisfied:

(a) there exist constants $a, b, c \in (0,1)$ such that $a \leq \alpha_n + \beta_n$ and $b \leq \gamma_n \leq c < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If $\{T_1, T_2, \ldots, T_N\}$ satisfies *Condition* (B), then the iterative sequence $\{x_n\}$ converges strongly to some point in F.

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \ldots, N\}$, then Theorem 4.1 is reduced to the following result:

Corollary 4.2. Let C be a nonempty closed convex subset of a Banach space E. Let $S_i\colon C \to C$ be a uniformly $L_{s,i}$-Lipschitz and asymptotically quasi-nonexpansive type mapping for each $1 \leq i \leq N$. Assume that $F = \cap_{i=1}^{N} F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a iterative sequence generated in (2.14). Put

$$B_n \quad = \quad \max\Big\{0, \sup_{p\in F,\ n\geq 1} \Big(\Big\| S_{i(n)}^{h(n)} x_{n-1} - p \Big\| - \|x_{n-1} - p\| \Big) : i \in \mathcal{N} \Big\},$$

such that $\sum_{n=1}^{\infty} B_n < \infty$. Assume that the following restrictions are satisfied:

(a) there exist constants $a, b, c, d \in (0,1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$ and $c \leq \gamma_n$, for all $n \geq 1$;

(b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If $\{S_1, S_2, \ldots, S_N\}$ satisfies *Condition* (B), then the iterative sequence $\{x_n\}$ converges strongly to some point in F.

Remark 4.1. (i) Our results extend the corresponding results of Ud-din and Khan [4] to the case of more general class of asymptotically quasi-nonexpansive mappings considered in this paper.

(ii) Our results also generalize and improve the corresponding results of Sun [19], Wittmann [21] and Xu and Ori [24] to the case of more general class of nonexpansive, asymptotically quasi-nonexpansive mappings and general implicit iterative process for two finite families of mappings considered in this paper.

(iii) Our results also extend the corresponding results of [1, 3, 7, 15] and many others.

Example 4.1. Let E be the real line with the usual norm $|.|$ and $K = [0,1]$. Define $T\colon K \to K$ by

$$T(x) = sin\, x, \quad x \in [0,1],$$

for $x \in K$. Obviously $T(0) = 0$, that is, 0 is a fixed point of T, that is, $F(T) = \{0\}$. Now we check that T asymptotically quasi-nonexpansive type mapping. In fact, if $x \in [0,1]$ and $p = 0 \in [0,1]$, then

$$|T(x) - p| = |T(x) - 0| = |sin\, x - 0| = |sin\, x| \leq |x| = |x - 0| = |x - p|,$$

that is,

$$|T(x) - p| \leq |x - p|.$$

Thus, T is quasi-nonexpansive. It follows that T is asymptotically quasi-nonexpansive with the constant sequence $\{k_n\} = \{1\}$ for each $n \geq 1$ and hence it is asymptotically quasi-nonexpansive type mapping (by remark 2.1). But the converse does not hold in general.

5. **Conclusion**

The class of asymptotically quasi-nonexpansive type mapping is more general than the class of nonexpansive, quasi-nonexpansive, asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings. Therefore the results presented in this paper are improvement and generalization of several known results in the existing literature.

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