Int. J. Anal. Appl. (2022), 20:12

International Journal of Analysis and Applications

Some Results of Conditionally Sequential Absorbing and Pseudo Reciprocally Continuous Mappings in Probabilistic 2-Metric Space

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Abstract. The objective of this paper is to generate two results in probabilistic 2-metric space by using the concepts of conditionally sequential absorbing mappings and pseudo reciprocally continuous mappings. These results stand as generalizations of the theorem proved by V. K. Gupta, Arihant Jain and Rajesh Kumar. Further these two outcomes are justified by supporting examples.

1. Introduction

The metric space notion was introduced by Fréchet [4]. Afterwords many generalizations came into existence one such prominent one was Banach contraction principle. Gähler [5] used the notion of 2-metric space as generalization of metric space. Golet [6] presented the concept of probabilistic 2-metric space as generalization of 2- metric space and gave some fundamental concepts like convergence, continuity. Dwelling of fixed point results has got importance for researchers due to the newly emerging platforms like 2-metric space, fuzzy space, menger space and 2-menger space etc. In this aspect many fixed point theorems came into the light by using the concepts like compatibility, continuity and contraction. The notion of compatibility was coined in metric space by Jungck and B. E. Rhodes [9]. The weaker form of compatibility as weakly compatible mappings in 2-Menger space used V. K. Gupta et al. [7] and obtained some results. Further Abbas and B. E. Rhodes [1] by using the concept of

Received: Nov. 8, 2021.

2010 *Mathematics Subject Classification.* 47H10.

Key words and phrases. self-mappings; pseudo reciprocally continuous mappings; conditionally sequentially absorbing maps; probabilistic 2-metric space.

https://doi.org/10.28924/2291-8639-20-2022-12

ISSN: 2291-8639

occasionally weakly compatible mappings proved some fixed point theorems. In the recent past some of fixed theorems have been evolved without using the condition of continuity [14], [3], [13]. Out of these results weaker form continuity known as reciprocally continuity came into existence due to Pant er al. [11] and became instrumental for deriving some fruitful results in metric and fuzzy spaces under some weaker conditions. U. Mishra et al. [10] tried to find out coincidence points without using the concepts of compatibility and continuity conditions to generate some fixed point theorems in menger space. some more results witnessed by [2], [8] in menger space. In this process pseudo reciprocally continuous and conditionally sequential absorbing mappings were emerged in metric space [12] and resulted in the formation of some fixed point results. In this article we extend these notions to 2-menger space and generate two results in 2- menger space. For this we present some definitions and preliminaries.

2. Preliminaries

Definition 2.1. $\mathtt{F}:\mathtt{R}\to\mathtt{R}^+$ *is distribution function [13] if it is (i) non-decreasing, (ii) continuous from left, (iii) inf* $\{\mathtt{F}(t):t\in R\}=0$, *(iv) Sup* $\{\mathtt{F}(t):t\in R\}=1$. *The letter* $\mathtt{L}$ *is used to refer to a collection of all distribution functions.*

Definition 2.2. *A probabilistic 2-metric space (2-PM space) [7] is a pair* $(\mathtt{X},\mathtt{F})$ *with* $\mathtt{F}:\mathtt{X}\times\mathtt{X}\times\mathtt{X}\to\mathtt{L}$ *here* $\mathtt{L}$ *stands as the set of all distribution functions and the* $\mathtt{F}$ *value at* $(e,f,g)\in\mathtt{X}\times\mathtt{X}\times\mathtt{X}$ *is written as* $\mathtt{F}_{e,f,g}$ *and obeys the following properties:*
(a) $\mathtt{F}_{e,f,g}(0)=0$
(b) $\exists\, g\in\mathtt{X}$ *such that* $\mathtt{F}_{e,f,g}(t_\epsilon)<1, \forall\, e,f\in\mathtt{X}, e\neq f,$ *for some* $t_\epsilon>0$
(c) $\mathtt{F}_{e,f,g}(t_\epsilon)=1 \quad \forall\, t_\epsilon>0$ *if* $e=f=g$ *or* $e=f$ *or* $f=g$ *or* $e=g$
(d) $\mathtt{F}_{e,f,g}(t_\epsilon)=\mathtt{F}_{f,g,e}(t_\epsilon)=\mathtt{F}_{g,f,e,}(t_\epsilon)$
(e) $\mathtt{F}_{e,f,g}(t_x)=\mathtt{F}_{f,g,e}(t_y)=\mathtt{F}_{g,f,e,}(t_z)=1 \implies \mathtt{F}_{e,f,g}(t_x+t_y+t_z)=1.$

Definition 2.3. *The mapping* $t_\epsilon:[0,1]^3\to[0,1]$ *is a* $t-$ *norm [7] it has the following properties:*
(i) $t_\epsilon(0,0,0)=0$
(ii) $t_\epsilon(\upsilon,1,1)=\upsilon$
(iii) $t_\epsilon(a_0,b_0,c_0)=t_\epsilon(b_0,c_0,a_0)=t_\epsilon(c_0,a_0,b_0)$
(iv) $t_\epsilon(d,e,f)\geq t_\epsilon(d_1,e_1,f_1)$ *for* $d\geq d_1, e\geq e_1, f\geq f_1$
(v) $t_\epsilon(t_\epsilon(a_0,b_0,c_0)),r,s)=t_\epsilon(a_0,t_\epsilon(b_0,c_0,r),s)=t_\epsilon(a_0,b_0,t_\epsilon(c_0,r,s)).$

Definition 2.4. *A 2-Menger space [7] is a triplet* $(\mathtt{X},\mathtt{F},t_\epsilon)$ *where* $(\mathtt{X},\mathtt{F})$ *is a 2-PM space and* t_ϵ *is a t-norm having triangle inequality:*
$\mathtt{F}_{u,v,w}(t_x+t_y+t_z)\geq t(\mathtt{F}_{u,v,p}(t_x),\mathtt{F}_{u,p,w}(t_y),\mathtt{F}_{p,v,w}(t_z))$
$\forall w,p,v,u\in\mathtt{X}$ *and* $t_x,t_y,t_z\geq 0.$

Definition 2.5. *[7] A sequence* (p_n) *in 2-Menger space* $(\mathtt{X}, \mathtt{F}, t_\epsilon)$
(i) **converges** *to* β *if for each* $\epsilon > 0, t_\epsilon > 0, \exists N(\epsilon) \in N \implies F_{p_n,\beta,a}(\epsilon) > 1 - t_\epsilon$,
$\forall a \in \mathtt{X}$ *and* $n \geq N(\epsilon)$;
(ii) **Cauchy** *if for each* $\epsilon > 0, t_\epsilon > 0, \exists N(\epsilon) \in N \implies F_{p_n,p_m,a}(\epsilon) > 1 - t_\epsilon$,
$\forall a \in \mathtt{X}$ *and* $n, m \geq N(\epsilon)$;
(iii) if each cauchy sequence converges in $\mathtt{X}$ *then it is mentioned as complete 2-Menger space.*

Definition 2.6. *Self-mappings P, S in 2-Menger space* $(\mathtt{X}, \mathtt{F}, t_\epsilon)$ *are known as*
(a) **compatible** *[7] if* $\mathtt{F}_{PSx_n,SPx_n,a}(\beta) \to 1, \forall a \in \mathtt{X}$ *and* $\beta > 0$ *whenever a sequence* $(x_n) \in \mathtt{X}$ *such that* $Px_n, Sx_n \to \theta$ *where* θ *is some element of* $\mathtt{X}$ *as* $n \to \infty$.
(b) **Non compatible** *[14] if* $\lim_{n\to\infty} \mathtt{F}_{PSx_n,SPx_n,a}(\beta)$ *not exists or*
$\lim_{n\to\infty} \mathtt{F}_{PSx_n,SPx_n,a}(\beta) \neq 1 \forall a \in \mathtt{X}$ *and* $\beta > 0$ *whenever a sequence* $x_n \in \mathtt{X}$ *such that* $Px_n, Sx_n \to \theta$ *where* θ *is an element of* $\mathtt{X}$ *as* $n \to \infty$.
(c) **Weakly compatible** *[7] if commute at their coincidence points.*
(d) **Occasionally Weakly compatible (OWC)** *[1] if there is a coincidence point at which the mappings are commuting.*

Example 2.1. *Weakly compatible mappings always OWC but the converse may not be true.*
Define $\forall\, t_\epsilon \in [0, 1]$

$$\mathtt{F}_{\upsilon,\beta,\gamma}(t_1) = \begin{cases} \frac{t_\epsilon}{t_\epsilon + d(\upsilon,\beta)} & \text{if } t_\epsilon > 0 \\ 0, & \text{if } t_\epsilon = 0 \end{cases} \tag{2.1}$$

$\forall \upsilon, \beta$ *and fixed* $\gamma = 0, t_\epsilon > 0$.
By considering $\mathtt{X} = [-2, 2]$ *and d is usual distance on* $\mathtt{X}$ *then by* (2.1) $(\mathtt{X}, \mathtt{F}, t_\epsilon)$ *forms 2-Menger space.*
The mappings $E, H : \mathtt{X} \to \mathtt{X}$ *are defined as*

$$E(x) = 4^{-x} \quad \forall x \in [-2, 2] \tag{2.2}$$

$$H(x) = 4^{-x^2} \quad \forall x \in [-2, 2]. \tag{2.3}$$

From (2.2) *and* (2.3) *the mappings E,H have coincidence points 0, 1.*
At $x = 0, E(0) = H(0) = 1, EH(0) = E(1) = 4^{-1}$ *and* $HE(0) = H(1) = 4^{-1}$.
This shows that $E(0) = H(0) \implies EH(0) = HE(0)$.
At $x = 1, E(1) = H(1) = 4^{-1}, EH(0) = E(4^{-1}) = 4^{-4^{-1}}$ *and*
$HE(1) = H(4^{-1}) = 4^{-(4^{-1})^2}$.
This gives $E(1) = H(1)$ *but* $EH(1) \neq HE(1)$.
As a result the mappings E, H are OWC but not weakly compatible.

Definition 2.7. *Self-mappings P,S in 2-Menger space* $(\mathtt{X}, \mathtt{F}, t_\epsilon)$ *are known as*
(a) **Conditionally sequential absorbing***(CSA) [12] if, whenever sequence*

$\{< x_n >: \lim_{n\to\infty} Px_n = \lim_{n\to\infty} Sx_n\} \neq \phi \implies$ *there exists another sequence* $< y_n >$ *satisfying*
$\lim_{n\to\infty} Py_n = \lim_{n\to\infty} Sy_n = t(say)$ *such that* $\lim_{n\to\infty} \mathtt{F}_{Py_n,PSy_n,a}(\beta) = 1$ *and*
$\lim_{n\to\infty} \mathtt{F}_{Sy_n,SPy_n,a}(\beta) = 1 \quad \forall a \in \mathtt{X}$ *and* $\beta > 0$.

(b) **Pseudo reciprocally continuous** *(PRC) [12] (w.r.t to conditionally sequential absorbing) if, whenever sequence* $\{< x_n >: \lim_{n\to\infty} Px_n = \lim_{n\to\infty} Sx_n\} \neq \phi \implies$ *there exist another sequence* $< y_n >$ *satisfying*
$\lim_{n\to\infty} Py_n = \lim_{n\to\infty} Sy_n = t(say)$ *then* $\lim_{n\to\infty} \mathtt{F}_{Py_n,PSy_n,a}(\beta) = 1$ *and*
$\lim_{n\to\infty} \mathtt{F}_{Sy_n,SPy_n,a}(\beta) = 1$ *such that* $\lim_{n\to\infty} \mathtt{F}_{PSy_n,Pt,a}(\beta) = 1$ *and*
$\lim_{n\to\infty} \mathtt{F}_{SPy_n,St,a}(\beta) = 1 \quad \forall a \in \mathtt{X}$ *and for some* $\beta > 0$.

Example 2.2. *By considering* $\mathtt{X} = [-3, 5]$ *and* d *is usual distance on* $\mathtt{X}$ *then by* (2.1) $(\mathtt{X}, \mathtt{F}, t_\epsilon)$ *forms 2-Menger space.*
The mappings $E, H : \mathtt{X} \to \mathtt{X}$ *are defined as*

$$E(x) = \begin{cases} -3 & \text{if } x \in [-3, 0) \\ 2^{-x} & \text{if } x \in [0, 5] \end{cases} \tag{2.4}$$

and

$$H(x) = \begin{cases} x & \text{if } x \in [-3, 0) \\ 2^{-2x} & \text{if } x \in [0, 5]. \end{cases} \tag{2.5}$$

From (2.4), (2.5) *-3 and 0 are intersecting points for the mappings E, H.*
At $x = 0, E(0) = H(0) = 1$,
$EH(0) = E(1) = \frac{1}{2}$ *and* $HE(0) = H(1) = \frac{1}{4} \implies EH(0) \neq HE(0)$.
Consequently the mapping H, E are not commuting at that coincidence point $x = 0$. *Hence these are not weakly compatible.*
Consider a sequence $< x_n >= \frac{\sqrt{3}}{n} \quad \forall \quad n \geq 1$ *then*

$$Ex_n = E(\frac{\sqrt{3}}{n}) = 2^{-(\frac{\sqrt{3}}{n})} \to 1 \tag{2.6}$$

and

$$Hx_n = H(\frac{\sqrt{3}}{n}) = 2^{-2(\frac{\sqrt{3}}{n})} \to 1 \tag{2.7}$$

as $n \to \infty$.
From (2.6), (2.7)

$$\implies \lim_{n\to\infty} Hx_n = \lim_{n\to\infty} Ex_n.$$

For a sequence $< y_n >= (-3 + \frac{3}{n} \quad \forall \quad n \geq 1$. *Then*

$$Ey_n = E(-3 + \frac{3}{n} = -3 \to -3, \tag{2.8}$$

$$Hy_n = H(-3 + \frac{3}{n}) = (-3 + \frac{3}{n}) \to -3. \tag{2.9}$$

as $n \to \infty$ *and*

$$EHy_n = E(-3 + \frac{3}{n}) = -3 \to -3, \tag{2.10}$$

$$HEy_n = H(-3) = -3 \to -3 \tag{2.11}$$

as $n \to \infty$.

From (2.8), (2.9), (2.10) *and* (2.11)

$$\lim_{n\to\infty} \mathtt{F}_{Ey_n,EHy_n,a}(\beta) = 1 \text{ and } \lim_{n\to\infty} \mathtt{F}_{Hy_n,HEy_n,a}(\beta) = 1. \tag{2.12}$$

Further

$$\lim_{n\to\infty} \mathtt{F}_{EHy_n,E(-3),a}(\beta) = 1 \text{ and } \lim_{n\to\infty} \mathtt{F}_{HEy_n,H(-3),a}(\beta) = 1. \tag{2.13}$$

From (2.12), (2.13) *the mappings* E, H *are CSA and PSC (w.r.t CSA) but not weakly compatible. Consequently conditionally sequential absorbing and Pseudo reciprocally continuous maps(w.r.t CSA) are weaker than weakly compatible mappings.*

We discuss the following examples to find the relation between conditionally sequential absorbing and non- compatible mappings.

Example 2.3. *By considering* $\mathtt{X} = (0, 3]$ *and* d *is usual distance on* $\mathtt{X}$ *then by* (2.1) $(\mathtt{X}, \mathtt{F}, t_\epsilon)$ *forms 2-Menger space.*

$E, H : \mathtt{X} \to \mathtt{X}$ *are defined as*

$$E(x) = \begin{cases} 1 - 5x & \text{if } 0 < x \le \frac{1}{10} \\ x^2, & \text{if } \frac{1}{10} < x \le 3 \end{cases} \tag{2.14}$$

and

$$H(x) = \begin{cases} 5x & \text{if } 0 < x \le \frac{1}{10} \\ 3, & \text{if } \frac{1}{10} < x \le 3 \end{cases} \tag{2.15}$$

By considering sequence $< x_n >= \frac{1}{10} \quad \forall n \ge 1$. *Then from* (2.14), (2.15)

$$Ex_n = E(\frac{1}{10}) = 1 - 5(\frac{1}{10}) = \frac{1}{2} \to \frac{1}{2}, \tag{2.16}$$

$$Hx_n = H(\frac{1}{10}) = 5(\frac{1}{10}) \to \frac{1}{2} \tag{2.17}$$

as $n \to \infty$.

From (2.15), (2.16)

$$\implies \lim_{n\to\infty} Hx_n = \lim_{n\to\infty} Ex_n. \tag{2.18}$$

Then there is a sequence $< y_n >= \sqrt{3} \quad \forall \quad n \geq 1$. *Then*

$$Ey_n = E(\sqrt{3}) = 3 \to 3, \tag{2.19}$$

$$Hy_n = H(\sqrt{3}) = 3 \tag{2.20}$$

as $n \to \infty$.
From (2.19), (2.20)

$$\implies \lim_{n\to\infty} Hy_n = \lim_{n\to\infty} Ey_n. \tag{2.21}$$

$$EHy_n = E(3) = 9 \to 9, \tag{2.22}$$

$$HEy_n = H(3) = 3 \to 3 \tag{2.23}$$

as $n \to \infty$.
From (2.22), (2.23)

$$\lim_{n\to\infty} \mathtt{F}_{EHy_n, HEy_n, a}(t_\epsilon) \neq 1 \tag{2.24}$$

$\forall a \in \mathtt{X}$ *and* $t_\epsilon > 0$.
Hence from (2.21), (2.24) *the mappings* E, H *are non compatible.*
Moreover from (2.19)(2.22)

$$\lim_{n\to\infty} F_{Ey_n, EHy_n, a}(t_\epsilon) \neq 1 \tag{2.25}$$

and from (2.20),(2.23)

$$\lim_{n\to\infty} F_{Hy_n, HEy_n, a}(t_\epsilon) \neq 1. \tag{2.26}$$

From (2.24), (2.25) *and* (2.26) *demonstrate that the mappings* E, H *are non compatible but not conditionally sequential absorbing.*

Example 2.4. *By considering* $\mathtt{X} = (0, 13]$ *and* d *is usual distance on* $\mathtt{X}$ *then by* (2.1) $(\mathtt{X}, \mathtt{F}, t_\epsilon)$ *forms 2-Menger space.*
$E, H : \mathtt{X} \to \mathtt{X}$ *are defined as*

$$E(x) = \begin{cases} 5x & \text{if } 0 \leq x < 2 \\ 7, & \text{if } 2 \leq x \leq 8 \end{cases} \tag{2.27}$$

and

$$H(x) = \begin{cases} 6x & \text{if } 0 \leq x < 2 \\ x, & \text{if } 2 \leq x \leq 8 \end{cases} \tag{2.28}$$

By considering sequence $< x_n >= \frac{e}{3n} \quad \forall \quad n \geq 1$. *Then from* (2.27),(2.28)

$$Ex_n = E(\frac{e}{3n}) = 5(\frac{e}{3n}) \to 0, \tag{2.29}$$

$$Hx_n = H(\frac{e}{3n}) = 6(\frac{e}{3n}) \to 0 \tag{2.30}$$

as $n \to \infty$.

From (2.29), (2.30)

$$\implies \lim_{n\to\infty} Hx_n = \lim_{n\to\infty} Ex_n.$$

There exists another sequence $< y_n >= 7 \quad \forall n \quad \geq 1$.

Then from (2.27), (2.28)

$$Ey_n = E(7) = 7 \to 7, \tag{2.31}$$

$$Hy_n = H(7) = 7 \to 7 \tag{2.32}$$

as $n \to \infty$.

But

$$EHy_n = E(7) = 7 \to 7, \tag{2.33}$$

$$HEy_n = H(7) = 7 \to 7 \tag{2.34}$$

as $n \to \infty$.

From (2.33), (2.34)

$$\lim_{n\to\infty} \mathtt{F}_{EHy_n,HEy_n,a}(t_\epsilon) = 1 \tag{2.35}$$

$\forall a \in \mathtt{X}$ *and* $t_\epsilon > 0$.

Hence from (2.33) *the mappings* E, H *are compatible.*

Moreover from (2.31), (2.33)

$$\lim_{n\to\infty} \mathtt{F}_{Ey_n,EHy_n,a}(t_\epsilon) = 1 \tag{2.36}$$

and from (2.32), (2.34)

$$\lim_{n\to\infty} \mathtt{F}_{Hy_n,HEy_n,a}(t_\epsilon) = 1 \tag{2.37}$$

$\forall a \in \mathtt{X}$ *and* $t_\epsilon > 0$.

Resulting that from (2.35), (2.36) *and* (2.37) *the mappings* H, E *are compatible as well as conditionally sequential absorbing.*

The following theorem was proved in [7].

Theorem 2.1. *Let A,B,S and T be self -mappings on a complete probabilistic 2-metric space* $(\mathtt{X}, \mathtt{F}, t_\epsilon)$ *satisfying :*

(i) $A(\mathtt{X}) \subseteq T(\mathtt{X})$, $B(\mathtt{X}) \subseteq S(\mathtt{X})$

(ii) one of $A(\mathtt{X})$, $B(\mathtt{X})$, $T(\mathtt{X})$ *or* $S(\mathtt{X})$ *is complete*

(iii) pairs (A, S) and (B, T) are weakly compatible

(iv) $\mathtt{F}_{Ax,By,\gamma}(t_\epsilon) \geq r\mathtt{F}_{Sx,Ty,\gamma}(t_\epsilon)$ *for all x, y in* $\mathtt{X}$ *and* $t_\epsilon > 0$

where $r : [0, 1] \to [0, 1]$ *is some continuous function such that* $r(t_\epsilon) > t_\epsilon$ *for each* $o < t_\epsilon < 1$.
Then the mappings A, B, S and T have unique common fixed point in $\mathtt{X}$.

Now we generalize above theorem in next section.

3. Main results

Theorem 3.1. *Let A, B, S and T be mappings on a complete probabilistic 2-metric space* $(\mathtt{X}, \mathtt{F}, t_\epsilon)$ *to itself satisfying*

$$A(\mathtt{X}) \subseteq T(\mathtt{X}), B(\mathtt{X}) \subseteq S(\mathtt{X}) \tag{3.1}$$

the pair of mappings (A, S) pseudo reciprocally continuous (w,r.t. conditionally sequential absorbing) and conditionally sequential absorbing and (B, T) is occasionally weakly compatible

$$\mathtt{F}_{Ax,By,\gamma}(t_\epsilon) \geq r(\mathtt{F}_{Sx,Ty,\gamma}(t_\epsilon)) \tag{3.2}$$

whenever x, y in X and $t_\epsilon > 0$ *for some continuous self-map on [0, 1] such that* $r(t_\epsilon) > t_\epsilon$ *for each* $o < t_\epsilon < 1$.
Then A, B, S and T have unique common fixed point in $\mathtt{X}$.

Proof. By using (3.1) the sequence $< y_n >$ derived as

$$< y_{2n} >= Ax_{2n} = Tx_{2n+1} \tag{3.3}$$

$$< y_{2n+1} > = Bx_{2n+1} = Sx_{2n+2}. \tag{3.4}$$

Now our claim is to show $< y_n >$ is a Cauchy sequence.
By taking the values $x = x_{2n}, y = x_{2n+1}$ in (3.2) we get

$$\mathtt{F}_{Ax_{2n},Bx_{2n+1},\gamma}(t_\epsilon) \geq r(\mathtt{F}_{Sx_{2n},Tx_{2n+1},\gamma}(t_\epsilon))$$

$$\mathtt{F}_{y_{2n},y_{2n+1},\gamma}(t_\epsilon) \geq r(\mathtt{F}_{y_{2n-1},y_{2n},\gamma}(t_\epsilon)) > \mathtt{F}_{y_{2n-1},y_{2n},\gamma}(t_\epsilon).$$

In general we have

$$\mathtt{F}_{y_{n+1},y_n,\gamma}(t_\epsilon) > \mathtt{F}_{y_n,y_{n-1},\gamma}(t_\epsilon)$$

for all $n \geq 1$.
Then we have $\{\mathtt{F}_{y_{n+1},y_n,\gamma}(t_\epsilon),\ \forall\ n \geq 1\}$ is an increasing sequence of positive real numbers bounded above by 1 therefore it must be converge to a limit say $L \leq 1$.

If $L < 1$ then $\mathtt{F}_{y_{n+1},y_n,\gamma}(t_\epsilon) = L > r(1) > 1$ which is a conflict. Hence $L = 1$.

Therefore for all n and p $\mathtt{F}_{y_{n+p},y_n,\gamma}(t_\epsilon) = 1$.

Thus the cauchyness of the sequence (y_n) in complete space $\mathtt{X}$ so it has limit $z \in \mathtt{X}$, results every sub sequence has the same limit z.

That is from (3.2) and (3.4)

$$Ax_{2n}, Sx_{2n} \to z \tag{3.5}$$

$$Tx_{2n+1}, Bx_{2n+1} \to z$$

as $n \to \infty$. Use the notion

$L\{A, S\} = \{< x_n >: \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n\}$.

Since the pair (A, S) is conditionally sequential absorbing from (3.5)

$L\{A, S\} \neq \phi \implies \exists < y_n >$ such that

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \theta(say) \tag{3.6}$$

$$\implies \lim_{n\to\infty} \mathtt{F}_{Ay_n,ASy_n,\gamma}(t_\epsilon) = 1 \text{ and } \mathtt{F}_{Sy_n,SAy_n,\gamma}(t_\epsilon) = 1 \tag{3.7}$$

$\forall \gamma \in \mathtt{X}$, $t_\epsilon > 0$.

Also the pair (A, S) satisfies pseudo reciprocally continuous means whenever

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \theta(say)$$

$$\implies \lim_{n\to\infty} \mathtt{F}_{Ay_n,ASy_n,\gamma}(t_\epsilon) = 1 \text{ and } \lim_{n\to\infty} \mathtt{F}_{Sy_n,SAy_n,\gamma}(t_\epsilon) = 1$$

such that

$$\lim_{n\to\infty} ASy_n = A(\theta) \text{ and } \lim_{n\to\infty} SAy_n = S(\theta). \tag{3.8}$$

Using (3.6) and (3.8) in (3.7) we get

$$A\theta = S\theta = \theta.$$

But $A\theta$ is element in $A(\mathtt{X})$ by (3.1) there exists η such that

$$\theta = S\theta = A\theta = T\eta. \tag{3.9}$$

Claim $B\eta = T\eta$.

By putting $x = \theta, y = \eta$ in (3.2)

$$\mathtt{F}_{A\theta,B\eta,\gamma}(t_\epsilon) \geq r(\mathtt{F}_{S\theta,T\eta,\gamma}(t))$$

From (3.9)

$$\mathrm{F}_{A\theta,B\eta,\gamma}(t_\epsilon) \geq r(\mathrm{F}_{S\theta,S\theta,\gamma}(t_\epsilon)) = r(1) = 1. \tag{3.10}$$

$\implies A\theta = B\eta = T\eta$.

The pair (B, T) is occasionally weakly compatible gives $BT\eta = TB\eta \implies B\theta = T\theta$ from (3.9).

Claim $\theta = B\theta$.

By taking $x = y = \theta$ in (3.2)

$$\mathrm{F}_{A\theta,B\theta,\gamma}(t_\epsilon) \geq r(\mathrm{F}_{S\theta,T\theta,\gamma}(t_\epsilon))$$

using (3.9) and $B\theta = T\theta$

$$\mathrm{F}_{\theta,B\theta,\gamma}(t_\epsilon) \geq r(\mathrm{F}_{\theta,B\theta,\gamma}(t_\epsilon)) > \mathrm{F}_{\theta,B\theta,\gamma}(t_\epsilon)$$

$$\mathrm{F}_{\theta,B\theta,\gamma}(t_\epsilon) > \mathrm{F}_{\theta,B\theta,\gamma}(t_\epsilon)$$

which is absurd. Hence $\theta = B\theta$.

Resulting

$$\theta = B\theta = T\theta = A\theta = S\theta. \tag{3.11}$$

Therefore θ is the required common fixed point.

Uniqueness: Suppose θ_1 is another fixed common fixed point for the mappings A, S, B and T.

Claim $\theta = \theta_1$.

Suppose if $\theta \neq \theta_1$,

then by taking $x = \theta, y = \theta_1$ in (3.2)

$$\mathrm{F}_{A\theta,B\theta_1,\gamma}(t_\epsilon) \geq r(\mathrm{F}_{S\theta,T\theta_1,\gamma}(t_\epsilon)).$$

This gives

$$\mathrm{F}_{\theta,\theta_1,\gamma}(t_\epsilon) \geq r(\mathrm{F}_{\theta,\theta_1,\gamma}(t_\epsilon)) > \mathrm{F}_{\theta,\theta_1,\gamma}(t_\epsilon)$$

implies

$$\mathrm{F}_{\theta,\theta_1,\gamma}(t_\epsilon) > \mathrm{F}_{\theta,\theta_1,\gamma}(t_\epsilon)$$

which is absurd. Hence $\theta = \theta_1$.

As a result four self mappings have unique common fixd point in $\mathtt{X}$.

$\square$

This result can be justified by the following example.

Example 3.1. *By considering* $\mathtt{X} = [-2, 3]$ *and* d *is usual distance on* $\mathtt{X}$ *then by* (2.1) $(\mathtt{X}, \mathtt{F}, t_\epsilon)$ *forms 2-Menger space.*
The mappings $A, B, S, T : \mathtt{X} \to \mathtt{X}$ *are defined as*

$$A(x) = B(x) = \begin{cases} -2 & \text{if } x \in [-2, 0) \\ e^{-x^2}, & \text{if } x \in [0, 3] \end{cases} \text{ and} \tag{3.12}$$

$$S(x) = T(x) = \begin{cases} \frac{x^3}{4} & \text{if } x \in [-2, 0) \\ e^{-3x}, & \text{if } x \in [0, 3]. \end{cases} \tag{3.13}$$

From (3.12) *and* (3.13) $A(\mathtt{X}) = \{-2\} \cup [e^{-9}, 1], S(\mathtt{X}) = [-2, 0) \cup [e^{-9}, 1]$ *implies*
$A(\mathtt{X}) \subseteq T(\mathtt{X}), B(\mathtt{X}) \subseteq S(\mathtt{X})$.
Clearly -2 and 0 are coincidence points for the mappings A, S.
At $x = -2, S(-2) = A(-2)$

$$AS(-2) = A(-2) = -2, \tag{3.14}$$

$$SA(-2) = SS(-2) = -2. \tag{3.15}$$

From (3.14) *and* (3.15)

$$AS(-2) = SA(-2).$$

At $x = 0, A(0) = S(0) = 1$ *and*

$$AS(0) = A(1) = e^{-1}, \tag{3.16}$$

$$SA(0) = S(1) = e^{-3}. \tag{3.17}$$

From (3.16) *and* (3.17)

$$AS(0) \neq SA(0). \tag{3.18}$$

From (3.18) *the pair (A, S) is not weakly compatible but OWC.*
Considering a sequence $< x_n >= \frac{\sqrt{2}}{n} \quad \forall \quad n \geq 1$ *then*

$$Ax_n = A(\frac{\sqrt{2}}{n}) = e^{-(\frac{\sqrt{2}}{n})^2} \to 1, \tag{3.19}$$

$$Sx_n = S(\frac{\sqrt{2}}{n}) = e^{-3(\frac{\sqrt{2}}{n})} \to 1 \tag{3.20}$$

as $n \to \infty$.
From (3.19) *and* (3.20)

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n. \tag{3.21}$$

From (3.21) $L\{A,S\} \neq \phi \implies \exists < y_n >$ *such that* $(y_n) = -2 + \frac{\sqrt{3}}{n} \quad \forall n \geq 1$ *such that*

$$Ay_n = A(-2 + \frac{\sqrt{3}}{n}) = -2 \to -2 \tag{3.22}$$

and

$$Sy_n = S(-2 + \frac{\sqrt{3}}{n}) = \frac{(-2 + \frac{\sqrt{3}}{n})^3}{4} \to -2 \tag{3.23}$$

as $n \to \infty$ *and*

$$ASy_n = A(\frac{(-2 + \frac{\sqrt{3}}{n})^3}{4}) = -2 \to -2, \tag{3.24}$$

$$SAy_n = S(-2) = \frac{(-2)^3}{4} = -2 \to -2 \tag{3.25}$$

as $n \to \infty$.

From (3.22) , (3.23) . (3.24), (3.25)

$$\lim_{n\to\infty} ASy_n = \lim_{n\to\infty} Ay_n \text{ and } \lim_{n\to\infty} SAy_n = \lim_{n\to\infty} Sy_n \tag{3.26}$$

$$\lim_{n\to\infty} ASy_n = A(-2) \text{ and } \lim_{n\to\infty} SAy_n = S(-2). \tag{3.27}$$

From (3.26) *and* (3.27) *the joint pairs (A, S), (B, T) are non-compatible pseudo reciprocally continuous (w,r.t. conditionally sequentially absorbing) and conditionally sequential absorbing having unique common fixed point at* $x = -2$*. Further these joint pairs (A, S), (B, T) are not weakly compatible and satisfy all the conditions of Theorem(3.1).*

Now we present another generalization of Theorem(2.1) on an incomplete 2-menger space.

Theorem 3.2. *Let A, B, S and T be mappings on a 2-menger space* $(\mathtt{X}, \mathtt{F}, t_\epsilon)$ *to itself satisfying*
(a) the pairs (A, S) and (B, T) non-compatible pseudo reciprocally continuous (w,r.t. CSA) and conditionally sequential absorbing
(b)

$$\mathtt{F}_{Ax,By,\gamma}(t_\epsilon) \geq r(\mathtt{F}_{Sx,Ty,\gamma}(t_\epsilon))$$

whenever x, y in X and $t_\epsilon > 0$
for some continuous self-map on [0, 1] such that $r(t_\epsilon) > t_\epsilon$ *for each* $o < t_\epsilon < 1$.
Then A, B, S and T have unique common fixed point in $\mathtt{X}$*. Moreover all these mappings are discontinuous at their fixed point.*

Proof. Since the pairs (A, S) are non-compatible implies some sequence $< x_n >$ with

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \theta(say) \tag{3.28}$$

for some $\theta \in \mathtt{X}$

$$\implies \lim_{n\to\infty} \mathtt{F}_{ASy_n,ASy_n,\gamma}(\beta) \text{ not exist or } \lim_{n\to\infty} \mathtt{F}_{Sy_n,SAy_n,\gamma}(\beta) \neq 1.$$

Since the pair (A, S) is conditionally sequential absorbing from (3.28)
$L\{A, S\} \neq \phi \implies$ there exists sequence $< y_n >$ such that

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \theta(say)$$

$$\implies \lim_{n\to\infty} \mathtt{F}_{Ay_n,ASy_n,\gamma}(t_\epsilon) = 1 \text{ and } \lim_{n\to\infty} \mathtt{F}_{Sy_n,SAy_n,\gamma}(t_\epsilon) = 1$$

$\forall \gamma \in \mathtt{X}$, $t_\epsilon > 0$.
Also the pair (A, S) satisfy pseudo reciprocally continuous means whenever

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \theta(say) \tag{3.29}$$

$$\implies \lim_{n\to\infty} F_{Ay_n,ASy_n,\gamma}(t_\epsilon) = 1 \text{ and } \lim_{n\to\infty} F_{Sy_n,SAy_n,\gamma}(t_\epsilon) = 1 \tag{3.30}$$

such that

$$\lim_{n\to\infty} ASy_n = A(\theta) \text{ and } \lim_{n\to\infty} SAy_n = S(\theta). \tag{3.31}$$

Using from (3.29) , (3.31) in (3.30) we get

$$A\theta = S\theta = \theta. \tag{3.32}$$

Since the pair (B, T) is non-compatible implies some sequence (x_n) with

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \eta(say) \tag{3.33}$$

for some $\eta \in X$

$$\implies \lim_{n\to\infty} \mathtt{F}_{BTx_n,TBx_n,\gamma}(t_\epsilon) \text{ does not exist or } \lim_{n\to\infty} \mathtt{F}_{BTx_n,TBx_n,\gamma}(t_\epsilon) \neq 1.$$

Since the pair (B, T) is conditionally sequential absorbing from (3.33)
$L\{B, T\} \neq \phi \implies$ there exists sequence $< y_n >$ such way that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w(say)$$

$$\implies \lim_{n\to\infty} \mathtt{F}_{By_n,BTy_n,\gamma}(t_\epsilon) = 1 \text{ and } \lim_{n\to\infty} \mathtt{F}_{Ty_n,TBy_n,\gamma}(t_\epsilon) = 1$$

$\forall \gamma \in \mathrm{X}, \beta > 0.$

Also the pair (B, T) is pseudo reciprocally continuous implies whenever

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} STy_n = w(say) \tag{3.34}$$

$$\implies \lim_{n\to\infty} \mathrm{F}_{By_n,BTy_n,\gamma}(t_\epsilon) = 1 \text{ and } \lim_{n\to\infty} \mathrm{F}_{Ty_n,TBy_n,\gamma}(t_\epsilon) = 1 \tag{3.35}$$

such that

$$\lim_{n\to\infty} BTy_n = Bw \text{ and } \lim_{n\to\infty} TBy_n = Tw. \tag{3.36}$$

Using (3.34) and (3.26) in (3.25) we get

$$Bw = Tw = w. \tag{3.37}$$

Claim $w = \theta$.

On contrary if $w \neq \theta$ put $x = \theta$ and $y = w$ in (3.2)

$$\mathrm{F}_{A\theta,Bw,\gamma}(t_\epsilon) \geq r(\mathrm{F}_{S\theta,Tw,\gamma}(t_\epsilon))$$

$$\implies \mathrm{F}_{\theta,w,\gamma}(t_\epsilon) \geq r(\mathrm{F}_{\theta,w,\gamma}(t)) > \mathrm{F}_{\theta,w,\gamma}(t_\epsilon).$$

From (3.32) and (3.37)

$$\implies \mathrm{F}_{\theta,w,\gamma}(t_\epsilon) > \mathrm{F}_{\theta,w,\gamma}(t_\epsilon)$$

which is contradiction hence $\theta = w$.

Uniqueness follows as in Theorem(3.1).

Suppose A is continuous at w from (3.29) then

$$\lim_{n\to\infty} Sy_n = \theta \implies \lim_{n\to\infty} ASy_n = A\theta(say).$$

From (3.31)

$$\lim_{n\to\infty} SAy_n = S\theta$$

but $A\theta = S\theta = \theta$

$$\implies \lim_{n\to\infty} ASy_n = \lim_{n\to\infty} SAy_n \tag{3.38}$$

(3.38) demonstrates that (A, S) is compatible pair, despite the fact that it is non-compatible. Therefore A should be discontinuous at w. Similarly the other mappings are also discontinuous at w. □

To justify our theorem, we now present a supporting example.

Example 3.2. *By considering* $\mathtt{X} = (-2, 22)$ *and* d *is usual distance on* $\mathtt{X}$ *then by* (2.1) $(\mathtt{X}, \mathtt{F}, t_\epsilon)$ *forms 2-Menger space.*
The mappings $A, B, S, T : \mathtt{X} \to \mathtt{X}$ *are defined as*

$$A(x) = B(x) = \begin{cases} \frac{x^2}{2} & \textit{if } x \in (-2, 2] \\ 3, & \textit{if } x \in (2, 22) \end{cases} \textit{ and} \tag{3.39}$$

$$S(x) = T(x) = \begin{cases} 2 & \textit{if } x \in (-2, 2] \\ \log x, & \textit{if } x \in (2, 22) \end{cases} \tag{3.40}$$

From (3.39),(3.40) e^3 *and 2 are intersecting points for the mappings A, S.*
$Atx = 2, A(2) = S(2) = 2$ *and* $AS(2) = A(2) = 2 = S(2) = SA(2)$.
At $x = e^3$,

$$S(e^3) = A(e^3) = 3,$$

$$AS(e^3) = A(3) = 3. \tag{3.41}$$

$$SA(e^3) = S(3) = \log 3. \tag{3.42}$$

(3.41) *and* (3.42)

$$AS(e^2) \neq SA(e^2). \tag{3.43}$$

(3.43) *shows that this pair (A, S) is not weakly compatible.*
Considering a sequence $< x_n >= e^3 + \frac{3}{n} \quad \forall n \geq 1$. *Then*

$$Ax_n = A(e^2 + \frac{3}{n}) = 3 \to 3,$$

$$Sx_n = S(e^3 + \frac{3}{n}) = \log(e^3 + \frac{3}{n}) \to 3$$

as $n \to \infty$ *and*

$$ASx_n = A(\log(e^3 + \frac{3}{n})) = 3 \to 3, \tag{3.44}$$

$$SAx_n = S(3) = \log 3 \to \log 3 \tag{3.45}$$

as $n \to \infty$.
(3.44),(3.45) *demonstrate that the pair (A, S) is non-compatible implies there exists another sequence* $< y_n >= 2 - \frac{\sqrt{2}}{n} \quad \forall n \geq 1$ *such that*

$$Ay_n = A(2 - \frac{\sqrt{2}}{n}) = \frac{(2 - \frac{\sqrt{2}}{n})_2}{2} \to 2, \tag{3.46}$$

$$Sy_n = S(2 - \frac{\sqrt{2}}{n}) = 2 \to 2 \tag{3.47}$$

as $n \to \infty$ and

$$ASy_n = A((2) = 2 \to 2, \tag{3.48}$$

$$SAy_n = S((\frac{(2 - \frac{\sqrt{2}}{n})_2}{2}) = 2 \to 2 \tag{3.49}$$

as $n \to \infty$. From (3.46),(3.48)

$$\lim_{n\to\infty} ASy_n = \lim_{n\to\infty} Ay_n. \tag{3.50}$$

From (3.47),(3.49)

$$\lim_{n\to\infty} SAy_n = \lim_{n\to\infty} Sy_n. \tag{3.51}$$

Further

$$\lim_{n\to\infty} ASy_n = A(2), \tag{3.52}$$

$$\lim_{n\to\infty} SAy_n = S(2). \tag{3.53}$$

From (3.50),(3.51),(3.52) *and* (3.53) *the joint pairs (A, S), (B, T) are non-compatible pseudo reciprocally continuous (w,r.t. conditionally sequential absorbing) and conditionally sequential absorbing, having unique common fixed point at $x = 2$. Further the maps A, S, B and T have discontinuity at $x = 2$. Moreover $A(\mathtt{X}), S(\mathtt{X}), B(\mathtt{X}) and T(\mathtt{X})$ are not closed sub spaces and also the pairs of (A, S), (B, T) are not weakly compatible and satisfy all the conditions of Theorem (3.2).*

4. Conclusion

In this paper we improve Theorem (2.1) in two ways:

(i) In Theorem (3.1) the concepts of pseudo reciprocally continuous and conditionally sequential absorbing mapping are being used in place of weakly compatible mappings in the first pair and OWC mappings in place of weakly compatible mappings in the second pair.

(ii) In Theorem (3.2) the concepts of non-compatible pseudo reciprocally continuous and conditionally sequential absorbing mappings are being used in place of weakly compatible mappings and further the completeness of X is being removed. Moreover all the mappings are discontinuous at their fixed point.

Further these two results are justified with appropriate examples.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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