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Solutions of Linear and Nonlinear Fractional Fredholm Integro-Differential Equations

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Abstract. The present paper analyzes a class of first-order fractional Fredholm integro differential equations in terms of Caputo fractional derivative. In the literature, such kind of fractional integro-differential equations have been solved using several numerical methods, while the exact solutions were not obtained. However, the exact solutions are obtained in this paper for various linear and nonlinear examples. It is shown that the exact solution of the linear problems is unique, while multiple exact solutions exist for the nonlinear ones. Moreover, the obtained results reduce to the classical ones in the relevant literature as the fractional order becomes unity. The obtained exact solutions can be further invested by other researchers to validate their numerical/approximation methods.

1. Introduction

The fractional calculus (FC) has gained observable interest in recent years due to its applications several fields [1-14]. The FC has been also extended to integro-differential equations (FIDEs) as observed in the literature [15-28], where various numerical and analytical methods were applied to solve for approximate solutions. We are concerned here with fractional Fredholm integro-differential equations (FFIDEs) of first-order. Although important results were reported [15-28] for FIDEs, obtaining the exact solution of FFIDEs is not an easy task, even for simple equations as will be shown later. So,

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we consider in this paper the following class of FFIDEs:

$$ {}_0^C D_x^{\alpha} u(x) = f(x) + \lambda \int_{b_1}^{b_2} K(x, \tau, u(\tau))\, d\tau, \quad 0 < \alpha \le 1, \tag{1.1}$$

$$ u(0) = h, \tag{1.2}$$

where h, λ, b_1 and b_2 are given constants, $f(x)$ is a given continuous function on $[b_1, b_2]$. The objective of this paper is to introduce a direct analytic approach for obtaining exact solutions for the class (1-2). It will be shown that the solution is unique when $K(x, \tau, u(\tau))$ is a linear function in the unknown function $u(\tau)$. In addition, it will be declared that multiple exact solutions exists when $K(x, \tau, u(\tau))$ is a nonlinear function in $u(\tau)$.

The Caputo definition is chosen as a fractional derivative in Eq. (1) and the structure of the paper is as follows. In section 2, we give the main aspects of the FC. In addition, a basic Lemma will be provided for the formal exact solution of the class (1-2). Sections 3 investigates the application of the present approach on several linear and nonlinear problems. Besides, the way of obtaining exact dual solution for the nonlinear case will be demonstrated in section 3. Moreover, it will be shown that the present exact solutions reduce to the classical ones as $\alpha \to 1$. Finally, section 5 outlines the conclusions.

2. Main aspects of FC

The Riemann-Liouville fractional integral of order α is defined as [1]:

$$ J_0^{\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} u(\tau) d\tau \qquad \alpha > 0. \tag{2.1}$$

The Caputo's FD of order α of a function $u(x)$ is defined by

$$ {}_0^C D_x^{\alpha} u(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} u^{(n)}(\tau) d\tau, \quad n-1 < \alpha \le n. \tag{2.2}$$

The J_0^{α} and ${}_0^C D_x^{\alpha}$ are related by:

$$ J_0^{\alpha} \left({}_0^C D_x^{\alpha} u(x) \right) = u(x) - \sum_{m=0}^{n-1} \frac{u^{(m)}(0)}{m!}\, x^m(0), \tag{2.3}$$

which is useful when solving FDEs/FIEs. A basic property of the J_0^{α} is

$$ J_0^{\alpha} (x^r) = \frac{\Gamma(r+1)}{\Gamma(\alpha+r+1)}\, x^{\alpha+r}, \quad r > -1. \tag{2.4}$$

The Mittag-Leffler function (MLF) of one-parameter is defined as

$$E_{\alpha}(z)=\sum_{m=0}^{\infty}\frac{z^{m}}{\Gamma(\alpha m+1)},\qquad z\in C, \tag{2.5}$$

while the two-parameter MLF is given as

$$E_{\alpha,\beta}(z)=\sum_{m=0}^{\infty}\frac{z^{m}}{\Gamma(\alpha m+\beta)},\qquad \alpha>0,\ \beta>0. \tag{2.6}$$

The following properties are also hold:

$$E_{1,2}(z)=\left(e^{z}-1\right)/\left(z\right), \tag{2.7}$$

$$E_{2,1}(-z^{2})=\cos(z),\qquad E_{2,2}(-z^{2})=\frac{\sin(z)}{z}. \tag{2.8}$$

Lemma 1. *The analytic solution of the first-order FFIDE (1-2) is given by*

$$u(x)=h+a\lambda\left(\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right)+J_{0}^{\alpha}\left(f(x)\right), \tag{2.9}$$

provided that the fractional integral of $f(x)$, i.e., $J_{0}^{\alpha}(f(x))$, exists and a is the constant given by $a=\int_{b_1}^{b_2}K(x,\tau,u(\tau))\ d\tau$.

Proof: The bounded integral involved in Eq. (1) can be assumed as a constant. Besides, we assume that such integral is given by the constant a as

$$a=\int_{b_1}^{b_2}K(x,\tau,u(\tau))\ d\tau. \tag{2.10}$$

Operating with J_{0}^{α} on Eq. (1) and implementing (2), (5), and (14), it then follows

$$u(x)-u(0)=J_{0}^{\alpha}\left(a\lambda\right)+J_{0}^{\alpha}\left(f(x)\right), \tag{2.11}$$

or

$$u(x)=h+a\lambda J_{0}^{\alpha}\left(1\right)+J_{0}^{\alpha}\left(f(x)\right). \tag{2.12}$$

Calculating $J_{0}^{\alpha}(1)$ from Eq. (6) at $r=0$, we have $J_{0}^{\alpha}(1)=\frac{x^{\alpha}}{\Gamma(\alpha+1)}$. Substituting this last result into Eq. (14) we obtain Eq. (11) which completes the proofs. □

3. Examples

Example 1: Consider the FFIDE [29]

$$ {}_0^C D_x^\alpha u(x) = 2\left(1-\int_0^1 u(\tau)\, d\tau\right), \quad u(0)=0. \tag{3.1}$$

Let

$$a_1 = \int_0^1 u(\tau)\, d\tau, \tag{3.2}$$

where a_1 is an unknown constant. Accordingly, Eq. (15) becomes

$$ {}_0^C D_x^\alpha u(x) = 2\left(1-a_1\right). \tag{3.3}$$

Applying the integral operator J_0^α on Eq. (17) and making use of Eq. (5), we have

$$u(x) = u(0) + 2\left(1-a_1\right) J_0^\alpha(1), \tag{3.4}$$

or

$$u(x) = 2\left(1-a_1\right)\frac{x^\alpha}{\Gamma(\alpha+1)}. \tag{3.5}$$

The constant a_1 is evaluated by inserting (19) into (16), this yields

$$a_1 = 2\left(1-a_1\right)\int_0^1 \frac{\tau^\alpha}{\Gamma(\alpha+1)}\, d\tau = \frac{2\left(1-a_1\right)}{\Gamma(\alpha+2)}. \tag{3.6}$$

Solving Eq. (20) for a_1, we obtain

$$a_1 = \frac{2}{2+\Gamma(\alpha+2)}, \tag{3.7}$$

and hence, Eq. (19) becomes

$$u(x) = \left(\frac{2\Gamma(\alpha+2)}{\Gamma(\alpha+1)\left(2+\Gamma(\alpha+2)\right)}\right) x^\alpha. \tag{3.8}$$

As $\alpha \to 1$, Eq. (22) reduces to the exact solution $u(x)=x$ for the classical form of Eq. (15), given by $u'(x) = 2\left(1-\int_0^1 u(\tau)\, d\tau\right)$.

Example 2: Consider the FFIDE [29]

$$ {}_0^C D_x^\alpha u(x) = 3+6x+x\int_0^1 \tau u(\tau)\, d\tau, \quad u(0)=0. \tag{3.9}$$

Suppose that

$$a_2 = \int_0^1 \tau u(\tau)\, d\tau, \tag{3.10}$$

where a_2 is a constant to be determined, then Eq. (23) becomes

$$ {}_0^C D_x^{\alpha} u(x) = 3 + (6 + a_2)\, x. \tag{3.11}$$

Operating with J_0^{α} on Eq. (25), it then follows

$$u(x) = \frac{3x^{\alpha}}{\Gamma(\alpha+1)} + \frac{(6+a_2)x^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{3.12}$$

From Eq. (24), we have

$$\begin{aligned} a_2 &= \int_0^1 \left(\frac{3\tau^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{(6+a_2)\tau^{\alpha+2}}{\Gamma(\alpha+2)} \right)\, d\tau, \\ &= \frac{3}{(\alpha+2)\Gamma(\alpha+1)} + \frac{(6+a_2)}{(\alpha+3)\Gamma(\alpha+2)}, \end{aligned} \tag{3.13}$$

which gives

$$a_2 = \frac{3\alpha^2 + 18\alpha + 21}{(\alpha+2)\left[(\alpha+3)\Gamma(\alpha+2) - 1\right]}. \tag{3.14}$$

Therefore,

$$u(x) = \frac{3x^{\alpha}}{\Gamma(\alpha+1)} + \left(6 + \frac{3\alpha^2 + 18\alpha + 21}{(\alpha+2)\left[(\alpha+3)\Gamma(\alpha+2) - 1\right]}\right) \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}, \tag{3.15}$$

and reduces, as $\alpha \to 1$, to $u(x) = 3x + 4x^2$ which is the same solution in Ref. [29] for the classical form: $u'(x) = 3 + 6x + x\int_0^1 \tau u(\tau)\, d\tau$.

Example 3: This example considers the FFIDE [29]

$$ {}_0^C D_x^{\alpha} u(x) = -1 + \cos x + \int_0^{\pi/2} \tau u(\tau)\, d\tau, \quad u(0) = 0, \tag{3.16}$$

which takes the form:

$$ {}_0^C D_x^{\alpha} u(x) = (a_3 - 1) + \cos x, \tag{3.17}$$

where a_3 is a constant defined by

$$a_3 = \int_0^{\pi/2} \tau u(\tau)\, d\tau. \tag{3.18}$$

Expressing $\cos x$ as Maclaurin series and then applying J_0^{α} on both sides of Eq. (31), gives

$$\begin{aligned} u(x) &= (a_3 - 1)\, J_0^{\alpha}(1) + J_0^{\alpha}\left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \right), \\ &= \frac{(a_3-1)\, x^{\alpha}}{\Gamma(\alpha+1)} + \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \times \frac{\Gamma(2m+1) x^{\alpha+2m}}{\Gamma(\alpha+2m+1)}, \end{aligned} \tag{3.19}$$

which is simplified as

$$u(x) = \frac{(a_3 - 1)\,x^{\alpha}}{\Gamma(\alpha+1)} + \sum_{m=0}^{\infty} \frac{(-1)^m x^{\alpha+2m}}{\Gamma(\alpha+2m+1)}, \tag{3.20}$$

or in terms of the MLF $E_{2,\alpha+1}(-x^2)$ as

$$u(x) = \frac{(a_3 - 1)\,x^{\alpha}}{\Gamma(\alpha+1)} + x^{\alpha} E_{2,\alpha+1}(-x^2). \tag{3.21}$$

From Eq. (32) and Eq. (35), we have

$$\begin{aligned} a_3 &= \int_0^{\pi/2} \left(\frac{(a_3 - 1)\,\tau^{\alpha+1}}{\Gamma(\alpha+1)} + \tau^{\alpha+1} E_{2,\alpha+1}(-\tau^2) \right) \, d\tau, \\ &= \frac{(a_3 - 1)\,(\pi/2)^{\alpha+2}}{(\alpha+2)\Gamma(\alpha+1)} + I, \end{aligned} \tag{3.22}$$

where the integral I is defined by

$$I = \int_0^{\pi/2} \tau^{\alpha+1} E_{2,\alpha+1}(-\tau^2) \, d\tau. \tag{3.23}$$

Solving Eq. (36) for a_3, we obtain

$$a_3 = -\frac{(\pi/2)^{\alpha+2}}{(\alpha+2)\Gamma(\alpha+1) - (\pi/2)^{\alpha+2}} + \frac{(\alpha+2)\Gamma(\alpha+1)}{(\alpha+2)\Gamma(\alpha+1) - (\pi/2)^{\alpha+2}} I. \tag{3.24}$$

Therefore, $u(x)$ is finally given by

$$\begin{aligned} u(x) = &\left(-\frac{(\pi/2)^{\alpha+2}}{(\alpha+2)\Gamma(\alpha+1) - (\pi/2)^{\alpha+2}} + \frac{(\alpha+2)\Gamma(\alpha+1)}{(\alpha+2)\Gamma(\alpha+1) - (\pi/2)^{\alpha+2}} I - 1 \right) \times \\ &\frac{x^{\alpha}}{\Gamma(\alpha+1)} + x^{\alpha} E_{2,\alpha+1}(-x^2), \end{aligned} \tag{3.25}$$

and I is already defined by Eq. (37). The solution given by Eq. (39) reduces, as $\alpha \to 1$, to

$$\begin{aligned} u(x) &= \left(-\frac{(\pi/2)^3}{3 - (\pi/2)^3} + \frac{3}{3 - (\pi/2)^3} \int_0^{\pi/2} \tau^2 E_{2,2}(-\tau^2) \, d\tau - 1 \right) x + x E_{2,2}(-x^2), \\ &= \left(-\frac{(\pi/2)^3}{3 - (\pi/2)^3} + \frac{3}{3 - (\pi/2)^3} \int_0^{\pi/2} \tau \sin\tau \, d\tau - 1 \right) x + x \left(\frac{\sin x}{x} \right), \\ &= \left(-\frac{(\pi/2)^3}{3 - (\pi/2)^3} + \frac{(\pi/2)^3}{3 - (\pi/2)^3} \right) x + \sin x, \ \text{where} \ \int_0^{\pi/2} \tau \sin\tau \, d\tau = 1, \\ &= \sin x, \end{aligned} \tag{3.26}$$

which is the corresponding solution for the classical form: $u'(x) = -1 + \cos x + \int_0^{\pi/2} \tau u(\tau) \, d\tau$.

Example 4: In this example, we considers the FFIDE [29]:

$$ {}_0^C D_x^\alpha u(x) = -10x + \int_{-1}^{1} (x-\tau)\, u(\tau)\; d\tau, \quad u(0) = 1. \tag{3.27} $$

Assuming that

$$ a_4 = \int_{-1}^{1} u(\tau)\; d\tau, \quad a_5 = \int_{-1}^{1} \tau u(\tau)\; d\tau, \tag{3.28} $$

then Eq. (41) becomes

$$ {}_0^C D_x^\alpha u(x) = (a_4 - 10)\, x - a_5, \tag{3.29} $$

where a_4 and a_5 are constants. Following the same analysis of the previous examples, we obtain

$$ u(x) = 1 + \frac{(a_4 - 10)\, x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{a_5 x^\alpha}{\Gamma(\alpha+1)}. \tag{3.30} $$

Substituting Eq. (44) into Eqs. (42) and performing the associated integrals, we get the following system:

$$ \left(1 - \left(\frac{1-(-1)^\alpha}{\Gamma(\alpha+3)}\right)\right) a_4 + \left(\frac{1+(-1)^\alpha}{\Gamma(\alpha+2)}\right) a_5 = 2 - 10\left(\frac{1-(-1)^\alpha}{\Gamma(\alpha+3)}\right), \tag{3.31} $$

$$ \left(\frac{1+(-1)^\alpha}{(\alpha+3)\Gamma(\alpha+2)}\right) a_4 - \left(1 + \frac{1-(-1)^\alpha}{(\alpha+2)\Gamma(\alpha+1)}\right) a_5 = 10\left(\frac{1+(-1)^\alpha}{(\alpha+3)\Gamma(\alpha+2)}\right). \tag{3.32} $$

The solution of the system (45-46) can be obtained as

$$ a_4 = \frac{\Delta_1}{\Delta}, \qquad a_5 = \frac{\Delta_2}{\Delta}, \tag{3.33} $$

where Δ, Δ_1, and Δ_2 are given by the determinants:

$$ \Delta = \begin{vmatrix} 1 - \dfrac{1-(-1)^\alpha}{\Gamma(\alpha+3)} & \dfrac{1+(-1)^\alpha}{\Gamma(\alpha+2)} \\ \\ \dfrac{1+(-1)^\alpha}{(\alpha+3)\Gamma(\alpha+2)} & -1 - \dfrac{1-(-1)^\alpha}{(\alpha+2)\Gamma(\alpha+1)} \end{vmatrix}, \tag{3.34} $$

and

$$ \Delta_1 = \begin{vmatrix} 2 - 10\left(\frac{1-(-1)^\alpha}{\Gamma(\alpha+3)}\right) & \dfrac{1+(-1)^\alpha}{\Gamma(\alpha+2)} \\ \\ 10\left(\frac{1+(-1)^\alpha}{(\alpha+3)\Gamma(\alpha+2)}\right) & -1 - \dfrac{1-(-1)^\alpha}{(\alpha+2)\Gamma(\alpha+1)} \end{vmatrix}, \tag{3.35} $$

$$ \Delta_2 = \begin{vmatrix} 1 - \dfrac{1-(-1)^\alpha}{\Gamma(\alpha+3)} & 2 - 10\left(\frac{1-(-1)^\alpha}{\Gamma(\alpha+3)}\right) \\ \\ \dfrac{1+(-1)^\alpha}{(\alpha+3)\Gamma(\alpha+2)} & 10\left(\frac{1+(-1)^\alpha}{(\alpha+3)\Gamma(\alpha+2)}\right) \end{vmatrix}. \tag{3.36} $$

Hence, the solution of Eq. (41) is finally given by

$$u(x)=1+\left(\frac{\Delta_1}{\Delta}-10\right)\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}-\left(\frac{\Delta_2}{\Delta}\right)\frac{x^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.37}$$

As $\alpha\to 1$, $u(x)$ given by Eq. (44) implies that

$$u(x)=1+\frac{1}{2}\left(\left[\frac{\Delta_1}{\Delta}\right]_{\alpha\to 1}-10\right)x^2-\left[\frac{\Delta_2}{\Delta}\right]_{\alpha\to 1}x. \tag{3.38}$$

Calculating Δ, Δ_1, and Δ_2 when $\alpha\to 1$, we obtain

$$\Delta=-\frac{10}{9},\quad \Delta_1=\frac{20}{9},\quad \Delta_2=0. \tag{3.39}$$

Thus, Eq. (52) becomes

$$u(x)=1-6x^2, \tag{3.40}$$

which is the corresponding solution of the classical form: $u'(x)=-10x+\int_{-1}^{1}(x-\tau)\,u(\tau)\,d\tau$.

Example 5: In order to show how to apply the present direct approach to solving nonlinear FFIDEs, we consider here a simple example, given by the nonlinear FFIDE:

$${}_0^C D_x^{\alpha}u(x)=\int_0^1 u^2(\tau)\,d\tau,\quad u(0)=0, \tag{3.41}$$

which can be written as

$${}_0^C D_x^{\alpha}u(x)=a_6, \tag{3.42}$$

where a_6 is a constant defined by

$$a_6=\int_0^1 u^2(\tau)\,d\tau. \tag{3.43}$$

On solving Eq. (56), we have

$$u(x)=\frac{a_6x^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.44}$$

Evaluating a_6 from Eq. (57), we obtain

$$a_6=a_6^2\int_0^1\frac{\tau^{2\alpha}}{(\Gamma(\alpha+1))^2}\,d\tau, \tag{3.45}$$

which leads to

$$a_6\left(\frac{a_6}{(2\alpha+1)\,(\Gamma(\alpha+1))^2}-1\right)=0. \tag{3.46}$$

Solving this equation for a_6, we obtain

$$a_6=0,\qquad a_6=(2\alpha+1)\,(\Gamma(\alpha+1))^2\,. \tag{3.47}$$

The first value $a_6 = 0$ leads $u(x) = 0$, which is a trivial solution. While the value $a_6 = (2\alpha + 1)\left(\Gamma(\alpha+1)\right)^2$ gives

$$u(x) = (2\alpha+1)\Gamma(\alpha+1)x^{\alpha}, \tag{3.48}$$

as a second solution. As $\alpha \to 1$, we obtain the corresponding solution $u(x) = 3x$ for the classical form $u'(x) = \int_0^1 u^2(\tau)\ d\tau$.

Example 6: We consider an additional nonlinear example:

$${}_0^C D_x^{\alpha} u(x) = 10x - 5 + \int_0^1 u^2(\tau)\ d\tau, \quad u(0) = 0, \tag{3.49}$$

Following the above analysis, we can obtain

$$u(x) = \frac{(a_7-5)x^{\alpha}}{\Gamma(\alpha+1)} + \frac{10x^{\alpha+1}}{\Gamma(\alpha+2)}, \tag{3.50}$$

and a_7 is given as

$$a_7 = \int_0^1 u^2(\tau)\ d\tau. \tag{3.51}$$

Substituting Eq. (64) into Eq. (65), yields

$$a_7 = \int_0^1 \left[\frac{(a_7-5)^2\tau^{2\alpha}}{(\Gamma(\alpha+1))^2} + \frac{20(a_7-5)\tau^{2\alpha+1}}{\Gamma(\alpha+1)\Gamma(\alpha+2)} + \frac{100\tau^{2\alpha+2}}{(\Gamma(\alpha+2))^2}\right]\ d\tau. \tag{3.52}$$

Performing this integral, we find that a_7 is governed by the equation:

$$a_7 = \frac{(a_7-5)^2}{(2\alpha+1)(\Gamma(\alpha+1))^2} + \frac{10(a_7-5)}{(\Gamma(\alpha+2))^2} + \frac{100}{(2\alpha+3)(\Gamma(\alpha+2))^2}. \tag{3.53}$$

Eq. (67) can be rewritten as

$$A(a_7-5)^2 + B(a_7-5) + C = 0, \tag{3.54}$$

where

$$A = \frac{(a_7-5)^2}{(2\alpha+1)(\Gamma(\alpha+1))^2}, \quad B = \frac{10}{(\Gamma(\alpha+2))^2} - 1, \quad C = \frac{100}{(2\alpha+3)(\Gamma(\alpha+2))^2} - 5. \tag{3.55}$$

Solving Eq. (68) for the constant a_7, we get

$$a_7 = 5 + \frac{1}{2A}\left(-B \pm \sqrt{B^2 - 4AC}\right). \tag{3.56}$$

From Eq. (64) and Eq. (70), we obtain

$$u(x) = \frac{1}{2A}\left(-B \pm \sqrt{B^2-4AC}\right)\left(\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right) + \frac{10x^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{3.57}$$

It can be seen from Eq. (71) that there are two different solutions for the present nonlinear example, the first one is given by

$$u_1(x) = \frac{1}{2A}\left(-B + \sqrt{B^2 - 4AC}\right)\left(\frac{x^\alpha}{\Gamma(\alpha+1)}\right) + \frac{10x^{\alpha+1}}{\Gamma(\alpha+2)}, \tag{3.58}$$

while the second solution is

$$u_2(x) = -\frac{1}{2A}\left(B + \sqrt{B^2 - 4AC}\right)\left(\frac{x^\alpha}{\Gamma(\alpha+1)}\right) + \frac{10x^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{3.59}$$

In order to check these two solutions, we evaluate them as $\alpha \to 1$. In this case, we have from Eqs. (69) that

$$A = \frac{1}{3}, \quad B = \frac{3}{2}, \quad C = 0. \tag{3.60}$$

Hence,

$$(u_1(x))_{\alpha\to 1} = 5x^2, \tag{3.61}$$

and

$$(u_2(x))_{\alpha\to 1} = -\frac{9}{2}x + 5x^2, \tag{3.62}$$

The solutions (75) and (76) are the same obtained one in Ref. [29] for the classical nonlinear version $u'(x) = 10x - 5 + \int_0^1 u^2(\tau)\ d\tau$.

4. Conclusion

A class of first-order FFIDEs was investigated in terms of Caputo definition in FC. The analytic solutions of several linear and nonlinear examples were obtained. For the linear FFIDEs, a unique solution was obtained, while multiple solutions were obtained for the nonlinear FFIDEs. It was shown that the linear problems posses unique solution of, while the nonlinear ones posses multiple solutions. Furthermore, as the fractional order is unity, the results agree with to the corresponding classical problems. This study may deserve extensions to further FFIDEs of higher-order.

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