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A NOTE ON OLIVIER'S THEOREM AND CONVERGENCE IN ERDŐS-ULAM DENSITY

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Abstract. Olivier's Theorem says that if $\sum a_n$ is a convergent positive series and (a_n) is monotone decreasing, then $na_n \to 0$. Šalát and Toma [4] proved that the monotonicity condition can be omitted if the convergence of $(na_n)_n$ is replaced by the statistical convergence. The aim of this note is to give an alternative proof and generalization of this result.

1. Introduction

A classical Olivier's Theorem says that if $\sum a_n$ is a convergent positive series and (a_n) is monotone decreasing, then $na_n \to 0$.

T. Šalát and V. Toma proved in 2003 [4] that the monotonicity condition in the above result can be omitted if the convergence of $(na_n)_n$ is replaced by the statistical convergence. This result was generalized and extended by several authors, see e.g., [3] and [2].

The aim of this note is to give an alternative proof and a generalization of the result of Šalát and Toma, and extend a result of Niculescu and Prăjitură (see [3], Theorem 6) which we recall later.

From now on, we call a positive function $f : \mathbb{N} \to (0, \infty)$ weight function (or Erdős-Ulam function) if it satisfies

$$\sum_{n=1}^{\infty} f(n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{f(n)}{\sum_{j=1}^{n} f(j)} = 0\,.$$

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With respect to a weight function f the f-weighted densities are defined as follows. For $A \subset \mathbb{N}$ let

$$F(A,n) = \frac{\sum\limits_{j=1}^{n} f(j) \cdot \chi_A(j)}{\sum_{j=1}^{n} f(j)}\,,$$

where χ_A denotes the characteristic function of A. Now we define the lower and upper f-densities of A by

$$\underline{d}_f(A) = \liminf_{n\to\infty} F(A,n) \quad \text{and} \quad \overline{d}_f(A) = \limsup_{n\to\infty} F(A,n),$$

respectively. In the case when $\underline{d}_f(A) = \overline{d}_f(A)$ we say that A has the f-density property denoted by $d_f(A)$.

Note that the asymptotic density corresponds to $f(n) = 1$, while the logarithmic density does to $f(n) = 1/n$. The logarithmic density is related to the asymptotic density via the inequalities

$$0 \le \underline{d}_1(A) \le \underline{d}_{\frac{1}{n}}(A) \le \overline{d}_{\frac{1}{n}}(A) \le \overline{d}_1(A) \le 1\,.$$

Define the function f^* by

$$f^*(n) = \frac{f(n)}{\sum_{j=1}^{n} f(j)}\,. \tag{1.1}$$

The logarithmic density can be considered as a density derived from the asymptotic density by (1.1). This method can be extended for an arbitrary weighted density given by the weight function f to provide a new weight function f^* (and, consequently, a new weighted density). Moreover, for arbitrary $A \subset \mathbb{N}$ we have

$$\underline{d}_f(A) \le \underline{d}_{f^*}(A) \le \overline{d}_{f^*}(A) \le \overline{d}_f(A)\,, \tag{1.2}$$

see [1].

The concept of convergence in density is an extension of the concept of statistical convergence. A sequence (a_n) converges to a number α in density d_f, which we denote as $(d_f)\text{–}\lim_{n\to\infty} a_n = \alpha$, provided the set

$$A_\varepsilon = \{n \in \mathbb{N} : |a_n - \alpha| \ge \varepsilon\}$$

has zero f-density, i.e., $d_f(A_\varepsilon) = 0$.

Now, we can rewrite the result of Šalát and Toma as

$$\text{if} \quad \sum a_n \quad \text{is a convergent positive series, then} \quad (d_1)\text{–}\lim_{n\to\infty} n a_n = 0. \tag{1.3}$$

Niculescu and Prăjitură [3] studied an analogous question for the harmonic density. They stated that

$$\text{if} \quad \sum a_n \quad \text{is a convergent positive series, then} \quad (d_{\frac{1}{n}})\text{–}\lim_{n\to\infty} (n \ln n) a_n = 0. \tag{1.4}$$

We generalize these results above.

2. Results

In the proof of our theorem we will use the following observation.

Lemma 2.1. *Let f be an Erdős-Ulam function and f^* is defined by (1.1). Let A be an infinite set of positive integers such that $\sum_{k\in A} f^*(k)$ is convergent. Then $d_f(A) = 0$.*

Proof. From the assertion of the lemma $d_{f^*}(A) = 0$ follows immediately. But inequality (1.2) does not give any information on the behavior of $\overline{d}_f(A)$. Taking into account that the upper density of a set does not change by removing finitely many elements. This observation, together with the fact that the tail of a convergent series tends to zero shows

$$\overline{d}_f(A) = \lim_{n\to\infty}\left(\limsup_{m\to\infty}\frac{\sum_{k\in A\cap[n,m]} f(k)}{\sum_{k=1}^{m} f(k)}\right) \le \lim_{n\to\infty}\left(\lim_{m\to\infty}\sum_{k\in A\cap[n,m]}\frac{f(k)}{\sum_{j=1}^{k} f(j)}\right)$$
$$= \lim_{n\to\infty}\left(\lim_{m\to\infty}\sum_{k\in A\cap[n,m]} f^*(k)\right) \le \lim_{n\to\infty}\sum_{k\in A\cap[n,\infty)} f^*(k) = 0.$$

□

Hence $d_f(A) = 0$.

Theorem 2.1. *Let f be an Erdős-Ulam function. If $\sum a_n$ is a convergent positive series, then*

$$(d_f)\text{-}\lim_{n\to\infty}\frac{\sum_{k=1}^{n} f(k)}{f(n)}a_n = 0\,. \tag{2.1}$$

Proof. Fix $\varepsilon > 0$, and consider the set

$$A_\varepsilon = \{n\in\mathbb{N} : \frac{\sum_{k=1}^{n} f(k)}{f(n)}a_n \ge \varepsilon\}\,.$$

Since

$$\varepsilon\sum_{n\in A_\varepsilon} f^*(n) = \varepsilon\sum_{n\in A_\varepsilon}\frac{f(n)}{\sum_{k=1}^{n} f(k)} \le \sum_{n\in A_\varepsilon} a_n \le \sum_{n\in\mathbb{N}} a_n < \infty,$$

applying Lemma 2.1 we immediately get that the set A_ε has zero f-density. Then (2.1) holds and the proof is completed. □

Corollary 2.1. *If we consider the asymptotic density in (2.1), then we conclude (1.3). Similarly, the logarithmic density (if $f(n) = 1/n$) leads to (1.4). For $f(n) = 1/(n\ln n)$ (the case of loglog-density), we obtain*

$$\text{if } \sum a_n \text{ is a convergent positive series, then } (d_{\frac{1}{n\ln n}})\text{-}\lim_{n\to\infty} n(\ln n)(\ln\ln n)a_n = 0.$$

Roughly speaking, if $\sum a_n$ is a convergent positive series, then the fast growing of the weight function f guarantees a less speed convergence of (a_n) to zero in density d_f.

For example, let $f(n) = e^{\sqrt{n}}/(2\sqrt{n})$. In this case $\sum_{k=1}^{n} f(k) \sim e^{\sqrt{n}}$ and we have

$$\textit{if} \quad \sum a_n \quad \textit{is a convergent positive series, then} \quad (d_f)\text{–}\lim_{n\to\infty} \sqrt{n} a_n = 0.$$

Next, we show that (1.3) is best possible in the sense that we cannot replace (d_1)–$\lim_{n\to\infty} na_n = 0$ with (d_1)–$\lim_{n\to\infty} n\omega_n a_n = 0$, where ω_n is an arbitrary sequence tending to infinity.

Theorem 2.2. *Let (ω_n) be an increasing sequence, tending to infinity. Then there exists a sequence (a_n) of positive terms, such that $\sum a_n$ converges and (d_1)–$\lim_{n\to\infty} n\omega_n a_n \neq 0$.*

Proof. The construction of (a_n) is based on the fact that

$$\lim_{m\to\infty} \sum_{k=m}^{2m} \frac{1}{k\omega_k} \leq \lim_{m\to\infty} \frac{1}{\omega_m} \sum_{k=m}^{2m} \frac{1}{k} = \lim_{m\to\infty} \frac{\ln 2}{\omega_m} = 0\,. \tag{2.2}$$

Using (2.2) we are able to define an increasing sequence (m_i) for that

$$m_{i+1} > 2m_i \qquad \text{and} \qquad \sum_{k=m_i}^{2m_i} \frac{1}{k\omega_k} < \frac{1}{2^i}\,, \qquad i = 1, 2, \dots\,.$$

Define the sequence (a_n) as

$$a_n = \begin{cases} \frac{1}{n^2\omega_n} & \text{if} \quad n \in \mathbb{N} \setminus \bigcup_{i=1}^{\infty} [m_i, 2m_i] \\ \frac{1}{n\omega_n} & \text{if} \quad n \in \bigcup_{i=1}^{\infty} [m_i, 2m_i]. \end{cases}$$

Then $\sum a_n$ converges since

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n\in\mathbb{N}\setminus\cup_{i=1}^{\infty}[m_i,2m_i]} \frac{1}{n^2\omega_n} + \sum_{n\in\cup_{i=1}^{\infty}[m_i,2m_i]} \frac{1}{n\omega_n} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{i=1}^{\infty} \sum_{k=m_i}^{k=2m_i} \frac{1}{k\omega_k} < \frac{\pi^2}{6} + \sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{\pi^2}{6} + 1\,. \end{aligned}$$

We are going to show that (d_1)–$\lim_{n\to\infty} n\omega_n a_n = 0$ fails. Fix $\varepsilon \in (0,1)$ and consider the set

$$A_\varepsilon = \{n \in \mathbb{N} : n\omega_n a_n \geq \varepsilon\}\,.$$

Then for any $n \in [m_i, 2m_i]$ we have $n\omega_n a_n = 1$ and therefore the set A_ε does not have zero asymptotic density. □

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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