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REPRODUCING FORMULAS FOR THE FOURIER-LIKE MULTIPLIERS OPERATORS IN q-RUBIN SETTING

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Abstract. The aim of this work is to study of the q^2-Fourier multiplier operators on $\mathbb{R}_q$ and we give for them Calderón's reproducing formulas and best approximation on the q^2-analogue Sobolev type space $\mathcal{H}_q$ using the theory of q^2-Fourier transform and reproducing kernels.

1. Introduction

The q^2-analogue differential-difference operator ∂_q, also called q-Rubin's operator defined on $\mathbb{R}_q$ in [11,12] by

$$\partial_q f(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & \text{if} \quad z \neq 0 \\ \\ \lim\limits_{z \to 0} \partial_q f(z) \quad \text{in} \quad \mathbb{R}_q & \text{if} \quad z = 0. \end{cases}$$

This operator has correct eigenvalue relationships for analogue exponential Fourier analysis using the functions and orthogonalities of [9].

The q^2-analogue Fourier transform we employ to make our constructions and results in this paper is based on analogue trigonometric functions and orthogonality results from [9] which have important applications to

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q-deformed quantum mechanics. This transform generalizing the usual Fourier transform, is given by

$$\mathcal{F}_q(f)(x) := K \int_{-\infty}^{+\infty} f(t)e(-itx;q^2)d_qt, \quad x \in \widetilde{\mathbb{R}}_q.$$

In this paper we study the Fourier multiplier operators $\mathcal{T}_m$ defined for $f \in L_q^2$ by

$$\mathcal{T}_m f(x) := \mathcal{F}_q^{-1}\left(m_a \mathcal{F}_q(f)\right)(x), \quad x \in \mathbb{R}_q,$$

where the function m_a is given by

$$m_a(x) = m(ax).$$

These operators are a generalization of the multiplier operators $\mathcal{T}_m$ associated with a bounded function m and given by $\mathcal{T}_m(\varphi) = \mathcal{F}^{-1}(m\mathcal{F}(\varphi))$, where $\mathcal{F}(\varphi)$ denotes the ordinary Fourier transform on $\mathbb{R}^n$. These operators made the interest of several Mathematicians and they were generalized in many settings, (see for instance [1, 2, 14, 18]).

This paper is organized as follows. In Section 2, we recall some basic harmonic analysis results related with the q-Rubin's operator ∂_q and we introduce preliminary facts that will be used later.

In section 3, we study the q^2-Fourier L^2-multiplier operators $\mathcal{T}_q$ and we give for them a Plancherel formula and pointwise reproducing formulas. Afterward, we give Calderón's reproducing formulas by using the theory of q^2-analogue Fourier transform.

The last section of this paper is devoted to giving best approximation for the operators $\mathcal{T}_q$ and good estimates of the associated extremal function on the q^2-analogue Sobolev type space $\mathcal{H}_q$ studied in [15–17].

2. Notations and preliminaries

Throughout this paper, we assume $0 < q < 1$ and we refer the reader to [5, 7] for the definitions and properties of hypergeometric functions. In this section we will fix some notations and recall some preliminary results. We put $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$ and $\widetilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}$. For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n = 1, 2, ...; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We denote also

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

A q-analogue of the classical exponential function is given by (see [11, 12])

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2), \tag{2.1}$$

where

$$\cos(z;q^2) = \sum_{n=0}^{+\infty} q^{n(n+1)} \frac{(-1)^n z^{2n}}{[2n]q!}, \quad \sin(z;q^2) = \sum_{n=0}^{+\infty} q^{n(n+1)} \frac{(-1)^n z^{2n+1}}{[2n+1]q!}, \tag{2.2}$$

satisfying the following inequality for all $x \in \mathbb{R}_q$

$$|\cos(x;q^2)| \le \frac{1}{(q;q)_\infty}, \quad \sin(x;q^2)| \le \frac{1}{(q;q)_\infty} \quad \text{and} \quad |e(ix;q^2)| \le \frac{2}{(q;q)_\infty}. \tag{2.3}$$

The q-differential-difference operators is defined as (see [11,12])

$$\partial_q f(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & \text{if} \quad z \neq 0 \\ \\ \lim\limits_{z\to 0} \partial_q f(z) \quad \text{in} \quad \mathbb{R}_q & \text{if} \quad z = 0 \end{cases}$$

and we denote a repeated application by

$$\partial_q^0 f = f, \quad \partial_q^{n+1} f = \partial_q(\partial_q^n f).$$

The q-Jackson integrals are defined by (see [6])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{+\infty} q^n f(aq^n),$$

$$\int_a^b f(x)d_qx = (1-q)\sum_{n=0}^{+\infty} q^n (bf(bq^n) - af(aq^n))$$

and

$$\int_{-\infty}^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{+\infty} q^n \left\{f(q^n) + f(-q^n)\right\},$$

provided the sums converge absolutely.

In the following we denote by

- $\mathcal{C}_{q,0}$ the space of bounded functions on $\mathbb{R}_q$, continued at 0 and vanishing a ∞.
- $\mathcal{C}_q^p$ the space of functions p-times q-differentiable on $\mathbb{R}_q$ such that for all $0 \le n \le p$. $\partial_q^p f$ is continuous on $\mathbb{R}_q$,
- $\mathcal{D}_q$ the space of functions infinitely q-differentiable on $\mathbb{R}_q$ with compact supports.
- $\mathcal{S}_q$ stands for the q-analogue Schwartz space of smooth functions over $\mathbb{R}_q$ whose q-derivatives of all order decay at infinity. $\mathcal{S}_q$ is endowed with the topology generated by the following family of semi-norms:

$$\|u\|_{M,\mathcal{S}_q}(f) := \sup_{x\in\mathbb{R};k\le M} (1+|x|)^M |\partial_q^k u(x)| \quad \text{for all} \quad u \in \mathcal{S}_q \quad \text{and} \quad M \in \mathbb{N}.$$

- $\mathcal{S}'_q$ the space of tempered distributions on $\mathbb{R}_q$, it is the topological dual of $\mathcal{S}_q$.
- $L_q^p = \left\{ f : \|f\|_{q,p} = \left(\int_{-\infty}^{+\infty} |f(x)|^p d_qx\right)^{\frac{1}{p}} < \infty \right\}$.
- $L_q^\infty = \left\{ f : \|f\|_{q,\infty} = \sup_{x\in\mathbb{R}_q} |f(x)| < \infty \right\}$.

The q^2-Fourier transform was defined by R. L. Rubin defined in [11], as follow

$$\mathcal{F}_q(f)(x) = K \int_{-\infty}^{+\infty} f(t)e(-itx;q^2)d_qt, \quad x \in \widetilde{\mathbb{R}}_q$$

where

$$K = \frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty(1-q)^2}.$$

To get convergence of our analogue functions to their classical counterparts as $q \uparrow 1$ as in [9,12], we impose the condition that $1-q = q^{2m}$ for some integer m. Therefore, in the remainder of this paper, letting $q \uparrow 1$ subject to the condition

$$\frac{\log(1-q)}{\log(q)} \in 2\mathbb{Z}.$$

It was shown in ([4,11]) that the q^2-Fourier transform $\mathcal{F}_q$ verifies the following properties:

(a) If $f,\ uf(u) \in L_q^1$, then

$$\partial_q(\mathcal{F}_q)(f)(x) = \mathcal{F}_q(-iuf(u))(x).$$

(b) If $f,\ \partial_q f \in L_q^1$, then

$$\mathcal{F}_q(\partial_q(f))(x) = ix\mathcal{F}_q(f)(x). \tag{2.4}$$

(c) If $f \in L_q^1$, then $\mathcal{F}_q(f) \in \mathcal{C}_{q,0}$ and we have

$$\|\mathcal{F}_q(f)\|_{q,\infty} \leq \frac{2K}{(q;q)_\infty}\|f\|_{q,1}. \tag{2.5}$$

(d) If $f \in L_q^1$, then, we have the reciprocity formula

$$\forall t \in \mathbb{R}_q, \quad f(t) = K\int_{-\infty}^{+\infty} \mathcal{F}_q(f)(x)e(itx;q^2)d_qx. \tag{2.6}$$

(e) The q^2-Fourier transform $\mathcal{F}_q$ is an isomorphism from $\mathcal{S}_q$ onto itself and we have, for all $f \in \mathcal{S}_q$

$$\mathcal{F}_q^{-1}(f)(x) = \mathcal{F}_q(f)(-x) = \overline{\mathcal{F}_q(\overline{f})}(x). \tag{2.7}$$

(f) $\mathcal{F}_q$ is an isomorphism from L_q^2 onto itself, and we have

$$\|\mathcal{F}_q(f)\|_{2,q} = \|f\|_{q,2}, \quad \forall f \in L_q^2 \tag{2.8}$$

and

$$\forall t \in \mathbb{R}_q, \quad f(t) = K\int_{-\infty}^{+\infty} \mathcal{F}_q(f)(x)e(itx;q^2)d_qx.$$

The q-translation operator $\tau_{q;x}, x \in \mathbb{R}_q$ is defined on L_q^1 by (see [11])

$$\tau_{q,y}(f)(x) = K\int_{-\infty}^{+\infty} \mathcal{F}_q(f)(t)e(itx;q^2)e(ity;q^2)d_qt, \quad y \in \mathbb{R}_q,$$

$$\tau_{q,0}(f)(x) = (f)(x).$$

It was shown in [11] that the q-translation operator can be also defined on L_q^2. Furthermore, it verifies the following properties

(a) For $f, g \in L_q^1$, we have

$$\tau_{q,y} f(x) = \tau_{q,x} f(y), \quad \forall x, y \in \mathbb{R}_q$$

and

$$\int_{-\infty}^{+\infty} \tau_{q,y}(f)(-x) g(x) d_q x = \int_{-\infty}^{+\infty} f(x) \tau_{q,y}(g)(-x) d_q x, \quad \forall y \in \widetilde{\mathbb{R}}_q.$$

(b) For all $f \in L_q^1$ and all $y \in \mathbb{R}_q$, we have(see [3])

$$\int_{-\infty}^{+\infty} \tau_{q,y}(f)(x) d_q x = \int_{-\infty}^{+\infty} f(x) d_q x. \tag{2.9}$$

(c) For all $y \in \mathbb{R}_q$ and for all $f \in L_q^p$,$1 \le p \le \infty$, we have $\tau_{q,y}(f) \in L_q^p$ (see [3]) and

$$\|\tau_{q,y} f\|_{q,p} \le M \|f\|_{q,p}, \tag{2.10}$$

where

$$M = \frac{4(-q,q)_\infty}{(1-q)^2 q (q,q)_\infty} + 2C, \quad \text{with} \quad C = K^2 \|e(\cdot, q^2)\|_{\infty,q} \|e(\cdot, q^2)\|_{1,q}. \tag{2.11}$$

(d) $\tau_{q;y} f$ is an isomorphism for $f \in L_q^2$ onto itself and we have

$$\|\tau_{q,y} f\|_{q,2} \le \frac{2}{(q,q)_\infty} \|f\|_{q,2}, \quad \forall y \in \widetilde{\mathbb{R}}_q. \tag{2.12}$$

(e) Let $f \in L_q^2$, then

$$\mathcal{F}_q(\tau_{q,y} f)(\lambda) = e(i\lambda y; q^2) \mathcal{F}_q(f)(\lambda), \quad \forall y \in \widetilde{\mathbb{R}}_q. \tag{2.13}$$

The q-convolution product is defined by using the q-translation operator, as follow For $f \in L_q^2$ and $g \in L_q^1$, the q-convolution product is given by

$$f * g(y) = K \int_{-\infty}^{+\infty} \tau_{q,y} f(x) g(x) d_q x.$$

The q-convolution product satisfying the following properties:

(a) $f * g = g * f$.

(b) $\forall f, g \in L_q^1 \cap L_q^2, \quad \mathcal{F}_q(f *_q g) = \mathcal{F}_q(f) \mathcal{F}_q(g)$.

(c) $\forall f, g \in \mathcal{S}_q, \quad f *_q g \in \mathcal{S}_q$.

(d) $f * g \in L_q^2$ if and only if $\mathcal{F}_q(f) \mathcal{F}_q(g) \in L_q^2$ and we have

$$\mathcal{F}_q(f * g) = \mathcal{F}_q(f) \mathcal{F}_q(g).$$

(e) Let $f, g \in L_q^2$. Then we have

$$\|f * g\|_{q,2}^2 = K \|\mathcal{F}_q(f) \mathcal{F}_q(g)\|_{q,2}^2, \tag{2.14}$$

and

$$f * g = \mathcal{F}_q^{-1} \left(\mathcal{F}_q(f) \mathcal{F}_q(g) \right). \tag{2.15}$$

(f) If $f, g \in L_q^1$ then $f * g \in L_q^1$ and

$$\|f * g\|_{q,1} = KM\|f\|_{q,1}\|g\|_{q,1}. \tag{2.16}$$

3. L^2-Multiplier operators for the q-Rubin-Fourier transform

In this section we study the q^2-Fourier-multiplier operators and we establish theirs Calderón's reproducing formulas in L^2-case.

Definition 3.1. *Let $a \in \mathbb{R}_q^+$, $m \in L_q^2$ and f a smooth function on $\mathbb{R}_q$. We define the q^2-Fourier L^2-multiplier operators $\mathcal{T}_m$ for a regular function f on $\mathbb{R}_q$ as follow*

$$\mathcal{T}_m f(x) = \mathcal{F}_q^{-1}\left(m_a \mathcal{F}_q(f)\right)(x), \quad x \in \mathbb{R}_q, \tag{3.1}$$

where the function m_a is given by

$$m_a(x) = m(ax).$$

Remark 3.1. *Let $a \in \mathbb{R}_q^+$, $m \in L_q^2$ and f, we can write the operator $\mathcal{T}_m$ as*

$$\mathcal{T}_m f(x) = \mathcal{F}_q^{-1}(m_a) * f(x), \quad x \in \mathbb{R}_q, \tag{3.2}$$

where

$$\mathcal{F}_q^{-1}(m_a)(x) = \frac{1}{a}\mathcal{F}_q^{-1}(m)(\frac{x}{a}).$$

Proposition 3.1. *(i) If $m \in L_q^2$ and $f \in L_q^1$, then $\mathcal{T}_m f \in L_q^2$, and we have*

$$\|\mathcal{T}_m f\|_{q,2} \leq \frac{2K}{\sqrt{a}(q,q)_\infty}\|m\|_{q,2}\|f\|_{q,1}.$$

(ii) If $m \in L_q^\infty$ and $f \in L_q^2$, then $\mathcal{T}_m f \in L_q^2$, and we have

$$\|\mathcal{T}_m f\|_{q,2} \leq \|m\|_{\infty,q}\|f\|_{q,2}.$$

(iii) If $m \in L_q^2$ and $f \in L_q^2$, then $\mathcal{T}_m f \in L_q^\infty$, and we have

$$\mathcal{T}_m f(x) = K\int_{-\infty}^{\infty} m(a\xi)\mathcal{F}_q(f)(\xi)e(i\xi x; q^2)d_q\xi, \quad x \in \mathbb{R}_q$$

and

$$\|\mathcal{T}_m f\|_{q,\infty} \leq \frac{2K}{\sqrt{a}(q,q)_\infty}\|m\|_{q,2}\|f\|_{q,2}.$$

Proof. i) Let $m \in L^2_q$, and $f \in L^1$. From the definition of the q^2-Fourier L^2-multiplier operators (3.1) and relations (2.5) and (2.8) we get that the function $\mathcal{T}_m f$ belongs to L^2_q, and we have

$$\begin{aligned}\|\mathcal{T}_m f\|_{q,2} &= \|m_a \mathcal{F}_q(f)\|_{q,2}\\ &\leq \frac{1}{\sqrt{a}}\|m\|_{q,2}\|\mathcal{F}_q(f)\|_{q,\infty}\\ &\leq \frac{2K}{\sqrt{a}(q,q)_\infty}\|m\|_{q,2}\|f\|_{q,1}.\end{aligned}$$

ii) The result follows from the Plancherel Theorem for the Rubin operator.

iii) Let $m \in L^2_q$, and $f \in L^2_q$, then from inversion formula we get $\mathcal{T}_m f \in L^\infty_q$, and by relation (2.5) we obtain

$$\|\mathcal{T}_m f\|_{q,\infty} \leq \frac{2K}{(q,q)_\infty}\|m_a \mathcal{F}_q(f)\|_{q,1}$$

then, using Hölder's inequality, we get

$$\|\mathcal{T}_m f\|_{q,\infty} \leq \frac{2K}{\sqrt{a}(q,q)_\infty}\|m\|_{q,2}\|f\|_{q,2}.$$

□

In the following, we give Plancherel and pointwise reproducing inversion formulas for the q^2-Fourier-multiplier operators $\mathcal{T}_m$.

Theorem 3.1. *Let m be a function in L^2_q satisfying the admissibility condition:*

$$\int_0^\infty |m_a(x)|^2 \frac{d_q a}{a} = 1, \quad x \in \mathbb{R}_q. \tag{3.3}$$

i)Plancherel formula: For all f in L^2_q, we have

$$\int_0^\infty \|\mathcal{T}_m f\|^2_{q,2}\frac{d_q a}{a} = K\int_{-\infty}^\infty |f(x)|^2 d_q(x).$$

ii) First Calderón's formula: Let f be a function in L^1_q such that $\mathcal{F}_q f$ in L^1_q then we have

$$f(x) = \int_0^\infty \left(\mathcal{T}_m f * \mathcal{F}_q^{-1}(\overline{m_a})\right)(x)\frac{d_q a}{a}, \quad x \in \mathbb{R}_q.$$

Proof. i) According to identity (2.14) and relation (3.2) we have

$$\begin{aligned}\int_0^\infty \|\mathcal{T}_m f\|^2_{q,2}\frac{d_q a}{a} &= \int_0^\infty \|\mathcal{F}_q^{-1}(m_a) * f\|^2_{q,2}\frac{d_q a}{a}\\ &= K\int_0^\infty \|m_a \mathcal{F}_q(f)\|^2_{q,2}\frac{d_q a}{a}\\ &= K\int_{-\infty}^\infty |\mathcal{F}_q(x)|^2\left(\int_0^\infty |m_a|^2\frac{d_q a}{a}\right)d_q x.\end{aligned}$$

The result follows from Plancherel Theorem (2.8) and the assumption (3.3).

ii) Let f be a function in L^1_q, then

$$\int_0^{\infty}\left(\mathcal{T}_m f * \mathcal{F}_q^{-1}(\overline{m_a})\right)(x)\frac{d_qa}{a} = \int_0^{\infty}\left(K\int_{-\infty}^{\infty}\mathcal{T}_m f(y)\tau_{q,x}\left(\mathcal{F}_q^{-1}(\overline{m_a})\right)(y)d_qy\right)\frac{d_qa}{a}.$$

From Proposition 3.1 i), relation (2.12) and Plancherel Theorem, it is obvious that $\mathcal{T}_m f, \tau_{q,x}\left(\mathcal{F}_q^{-1}(\overline{m_a})\right) \in L^2_q$. After that, according to relation (2.13), identity (3.1) and Plancherel Theorem of the q^2-Fourier transform, we obtain

$$\int_0^{\infty}\left(\mathcal{T}_m f * \mathcal{F}_q^{-1}(\overline{m_a})\right)(x)\frac{d_qa}{a} = K\int_0^{\infty}\left(\int_{-\infty}^{\infty}e(ixy;q^2)\mathcal{F}_q(f)(y)|m_a(y)|^2d_qy\right)\frac{d_qa}{a}.$$

Since

$$\int_0^{\infty}\left(\int_{-\infty}^{\infty}|e(ixy;q^2)\mathcal{F}_q(f)(y)||m_a(y)|^2d_qy\right)\frac{d_qa}{a} \leq \|\mathcal{F}_q(f)\|_{q,1} \leq \infty,$$

then, by Fubini's theorem, we have

$$\begin{aligned}\int_0^{\infty}\left(\mathcal{T}_m f * \mathcal{F}_q^{-1}(\overline{m_a})\right)(x)\frac{d_qa}{a} &= K\int_{-\infty}^{\infty}e(ixy;q^2)\mathcal{F}_q(y)\left(\int_0^{\infty}|m_a(y)|^2\frac{d_qa}{a}\right)d_qy \\ &= K\int_{-\infty}^{\infty}e(ixy;q^2)\mathcal{F}_q(y)d_qy = f(x).\end{aligned}$$

□

We need the following technical lemma to establish the Calderón's reproducing formulas for the q^2-Fourier L^2-multiplier operators.

Lemma 3.1. *Let m be a function in $L^2_q \cap L^\infty_q$ satisfy the admissibility condition (3.3). Then the function*

$$\Phi_{\gamma,\delta}(x) = \int_\gamma^\delta |m(ax)|^2\frac{d_qa}{a}$$

belongs to L^2_q for all $0 < \gamma < \delta < \infty$ and we have

$$\Phi_{\gamma,\delta}(x) \in L^2_q \cap L^\infty_q.$$

Proof. Using Hölder's inequality for the measure $\frac{d_qa}{a}$, we get

$$|\Phi_{\gamma,\delta}(x)|^2 \leq \ln(\delta/\gamma)\int_\gamma^\delta |m(ax)|^4\frac{d_qa}{a}, \quad x \in \mathbb{R}_q.$$

Therefore,

$$\begin{aligned}\|\Phi_{\gamma,\delta}\|^2_{q,2} &\leq \ln(\delta/\gamma)\int_\gamma^\delta\left(\int_{-\infty}^{\infty}|m(ax)|^4d_qx\right)\frac{d_qa}{a} \\ &\leq \ln(\delta/\gamma)\int_\gamma^\delta\left(\int_{-\infty}^{\infty}|m(x)|^4d_qx\right)\frac{da}{a^2} \\ &\leq \left(\frac{1}{\gamma}-\frac{1}{\delta}\right)\ln(\delta/\gamma)\,\|m\|^2_{q,2}\|m\|^2_{q,\infty} < \infty.\end{aligned}$$

On the other hand, from the admissibility condition (3.3), we get

$$\|\Phi_{\gamma,\delta}\|_{q,\infty} \leq 1,$$

which completes the proof. □

Theorem 3.2. *(Second Calderón's formula) Let* $f \in L^2_q$, $m \in L^2_q \cap L^\infty_q$ *satisfy the admissibility condition (3.3) and* $0 < \gamma < \delta < \infty$. *Then the function*

$$f_{\gamma,\delta}(x) = \int_\gamma^\delta \left(\mathcal{T}_m f * \mathcal{F}_q^{-1}(\overline{m_a})\right)(x)\frac{d_q a}{a}, \quad x \in \mathbb{R}_q$$

belongs to L^2_q *and satisfies*

$$\lim_{(\gamma,\delta)\to(0,\infty)} \|f_{\gamma,\delta} - f\|_{q,2} = 0. \tag{3.4}$$

Proof. Let f be a function in L^2_q, and $m \in L^2_q \cap L^\infty_q$, then

$$\int_0^\infty \left(\mathcal{T}_m f * \mathcal{F}_q^{-1}(\overline{m_a})\right)(x)\frac{d_q a}{a} = \int_0^\infty \left(K \int_{-\infty}^\infty \mathcal{T}_m f(y)\tau_{q,x}\left(\mathcal{F}_q^{-1}(\overline{m_a})\right)(y) d_q y\right)\frac{d_q a}{a}.$$

According to Proposition 3.1, relation (2.12) and Plancherel Theorem, it is obvious that $\mathcal{T}_m f, \tau_{q,x}\left(\mathcal{F}_q^{-1}(\overline{m_a})\right) \in L^2_q$. Then, from relation (2.13) and the identity (3.1), we obtain

$$f_{\gamma,\delta}(x) = K\int_\gamma^\delta \left(\int_{-\infty}^\infty e(ixy, q^2)\mathcal{F}_q(f)(y)|m_a(y)|^2 d_q y\right)\frac{d_q a}{a}.$$

By Fubini-Tonnelli's theorem, Hölder's inequality and Lemma 3.1, we get

$$\begin{aligned}
\int_\gamma^\delta \left(\int_{-\infty}^\infty |e(ixy, q^2)\mathcal{F}_q(f)(y)||m_a(y)|^2 d_q y\right)\frac{d_q a}{a} &\leq \frac{2}{(q,q)_\infty}\int_{-\infty}^\infty |\mathcal{F}_q(f)(y)|\Phi_{\gamma,\delta}(y) d_q y \\
&\leq \frac{2}{(q,q)_\infty}\|f\|_{q,2}\|\Phi_{\gamma,\delta}\|_{q,2} < \infty.
\end{aligned}$$

Then, according to Fubini's theorem and the inversion formula, we have

$$\begin{aligned}
f_{\gamma,\delta}(x) &= K\int_{-\infty}^\infty e(ixy, q^2)\mathcal{F}_q(f)(y)\left(\int_\gamma^\delta |m_a(y)|^2\frac{d_q a}{a}\right) d_q y \\
&= K\int_{-\infty}^\infty e(ixy, q^2)\mathcal{F}_q(f)(y)\Phi_{\gamma,\delta}(y) d_q y \\
&= \mathcal{F}_q^{-1}\left[\mathcal{F}_q(f)\Phi_{\gamma,\delta}\right](x).
\end{aligned}$$

On the other hand, the function $\Phi_{\gamma,\delta}$ belongs to L^∞_q which allows to see that $f_{\gamma,\delta}$ belongs to L^2_q and using the identity (2.15), we obtain

$$\mathcal{F}_q(f_{\gamma,\delta}) = \mathcal{F}_q(f)\Phi_{\gamma,\delta}.$$

By the Plancherel formula we get

$$\|f_{\gamma,\delta} - f\|^2_{q,2} = \int_{-\infty}^\infty |\mathcal{F}_q(f)(y)|^2(1 - \Phi_{\gamma,\delta}(y))^2 d_q y.$$

The the admissibility condition (3.3) leads to

$$\lim_{(\gamma,\delta)\to(0,\infty)} \Phi_{\gamma,\delta}(y) = 1, \quad y \in \mathbb{R}_q$$

and

$$|\mathcal{F}_q(f)(y)|^2(1 - \Phi_{\gamma,\delta}(y))^2 \leq |\mathcal{F}_q(f)(y)|^2.$$

Finally, the relation (3.4) follows from the dominated convergence theorem. □

4. The extremal function associated with q^2-Fourier L^2-multiplier operators

In this section, we study the extremal function associated to the q^2-Fourier L^2-multiplier operators.

Let $s \in \mathbb{R}$ and $1 \leq p < \infty$, the q^2-analogue Sobolev type spaces is defined in [15] by

$$\mathcal{W}_q^{s,p} = \left\{u \in \mathcal{S}_q' : (1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}_q(u) \in L_q^p\right\}.$$

In the particular case $p = 2$, we denote $\mathcal{W}_q^{s,p}$ by $\mathcal{H}_q^s$ which provided with the inner product

$$\langle u, v\rangle_{\mathcal{H}_q^s} = \int_{-\infty}^{+\infty} (1 + \xi^2)^s \mathcal{F}_q(u)(\xi)\overline{\mathcal{F}_q(v)(\xi)} d_q\xi$$

and the norm

$$\|u\|_{\mathcal{H}_q^s} := \sqrt{\langle u, u\rangle_{\mathcal{H}_q^s}}.$$

$\mathcal{H}_q^s$ is a Hilbert space satisfying the following properties

(a) $\mathcal{H}_q^0 = L_q^2$.

(b) For all $s > 0$ the space $\mathcal{H}_q^s$ is continuously contained in L_q^2 and we have

$$\|f\|_{q,2} \leq \|f\|_{\mathcal{H}_q^s}. \tag{4.1}$$

Proposition 4.1. *Let m be a function in L_q^∞. Then the q^2-Fourier L^2-multiplier operators $\mathcal{T}_m$ are bounded and linear from $\mathcal{H}_q^s$ into L_q^2 and we have for all $f \in \mathcal{H}_q^s$*

$$\|\mathcal{T}_m f\|_{q,2} \leq \|m\|_{q,\infty}\|f\|_{\mathcal{H}_q^s}.$$

Proof. Let $f \in \mathcal{H}_q^s$. According to Proposition 3.1 (ii), the operator $\mathcal{T}_m$ belongs to L_q^2 and we have

$$\|\mathcal{T}_m f\|_{q,2} \leq \|m\|_{q,\infty}\|f\|_{q,2}.$$

On the other hand, by the inequality (4.1) we have $\|f\|_{q,2} \leq \|f\|_{\mathcal{H}_q^s}$, which gives the result. □

Definition 4.1. *Let $\eta > 0$ and let m be a function in L^∞_q. We denote by $\langle u, v\rangle_{\mathcal{H}^s_q,\eta}$ the inner product defined on the space $\mathcal{H}^s_q$ by*

$$\langle f, g\rangle_{\mathcal{H}^s_q,\eta} = \eta\langle f, g\rangle_{\mathcal{H}^s_q} + \langle \mathcal{T}_m f, \mathcal{T}_m g\rangle_{q,2} \tag{4.2}$$

and the norm

$$\|f\|_{\mathcal{H}^s_q,\eta} = \sqrt{\langle f, f\rangle_{\mathcal{H}^s_q,\eta}}.$$

It is easy to show the following results.

Proposition 4.2. *Let m be a function in L^∞_q and f in $\mathcal{H}^s_q$*

(i) The norm $\|\cdot\|_{\mathcal{H}^s_q,\eta}$ satisfies:

$$\|f\|^2_{\mathcal{H}^s_q,\eta} = \eta\|f\|^2_{\mathcal{H}^s_q} + \|\mathcal{T}_m f\|^2_{q,2}.$$

(ii) The norms $\|\cdot\|_{\mathcal{H}^s_q,\eta}$ and $\|\cdot\|_{\mathcal{H}^s_q}$ are equivalent and we have

$$\sqrt{\eta}\,\|f\|_{\mathcal{H}^s_q} \leq \|f\|_{\mathcal{H}^s_q,\eta} \leq \sqrt{\eta + \|m\|^2_{q,\infty}}\,\|f\|_{\mathcal{H}^s_q}.$$

Theorem 4.1. *Let $s > \frac{1}{2}$ and m be a function in L^∞_q. Then the Hilbert space $(\mathcal{H}^s_q, \langle\cdot,\cdot\rangle_{\mathcal{H}^s_q,\eta})$ has the following reproducing Kernel*

$$\Psi_{s,\eta}(x, y) = \int_{-\infty}^{\infty} \frac{e(ix\xi, q^2)e(-iy\xi, q^2)}{\eta(1 + |\xi|^2)^s + |m_a(\xi)|^2} d_q(\xi), \tag{4.3}$$

such that

(i) For all $y \in \mathbb{R}_q$, the function $x \mapsto \Psi_{s,\eta}(x, y)$ belongs to $\mathcal{H}^s_q$.

(ii) For all $f \in \mathcal{H}^s_q$ and $y \in \mathbb{R}_q$, we have the reproducing property

$$\langle f, \Psi_{s,\eta}(\cdot, y)\rangle_{\mathcal{H}^s_q,\eta} = f(y).$$

(iii) The Hilbert space $(\mathcal{H}^s_q, \langle\cdot,\cdot\rangle_{\mathcal{H}^s_q})$ has the following reproducing Kernel

$$\Psi_s(x, y) = \int_{-\infty}^{\infty} \frac{e(ix\xi, q^2)e(-iy\xi, q^2)}{(1 + |\xi|^2)^s} d_q(\xi). \tag{4.4}$$

Proof. (i) Let $y \in \mathbb{R}_q$ and $s > \frac{1}{2}$. From the relation (2.3), we show that the function

$$\varphi_y : \xi \longrightarrow \frac{e(-iy\xi, q^2)}{\eta(1 + |\xi|^2)^s + |m_a(\xi)|^2}$$

belongs to $L^1_q \cap L^2_q$. Hence the function $\Psi_{s,\eta}$ is well defined and by the inversion formula, we obtain

$$\Psi_{s,\eta}(x, y) = \mathcal{F}^{-1}_q(\varphi_y)(x), \quad x \in \mathbb{R}_q.$$

On the other hand, using Plancherel theorem, we get that $\Psi_{s,\eta}(\cdot, y)$ belongs to L^2_q and we have

$$\mathcal{F}_q\left(\Psi_{s,\eta}(\cdot, y)\right)(\xi) = \frac{e(-iy\xi, q^2)}{\eta(1 + |\xi|^2)^s + |m_a(\xi)|^2}, \quad \xi \in \mathbb{R}_q. \tag{4.5}$$

Therefore, by the identity (2) we obtain

$$|\mathcal{F}_q(\Psi_{s,\eta}(\cdot,y))(\xi)| \leq \frac{(q,q)_\infty^{-1}}{2\eta(1+|\xi|^2)^s},$$

and

$$\|\Psi_{s,\eta}(\cdot,y)\|^2_{\mathcal{H}^s_q} \leq (2\eta(q,q)_\infty)^{-2}\|(1+|\cdot|^2)^{-s}\|_{q,1} < \infty.$$

This proves that for every $y \in \mathbb{R}_q$, the function $\Psi_{s,\eta}(\cdot,y)$ belongs to $\mathcal{H}^s_q$.

(ii) Let $f \in \mathcal{H}^s_q$ and $y \in \mathbb{R}_q$. According to the definition of inner product (4.2) and identity (4.5), we obtain

$$\langle f, \Psi_{s,\eta}(\cdot,y)\rangle_{\mathcal{H}^s_q,\eta} = \int_{-\infty}^{\infty} e(ix\xi,q^2)\mathcal{F}_q(\xi)d_q(\xi).$$

On the other hand, the function $\xi \longmapsto (1+|\xi|^2)^{-s/2}$ belongs to L^2_q for all $s > 1/2$. Therefore, the function $\mathcal{F}_q(f)$ belongs to L^1_q and we have

$$\langle f, \Psi_{s,\eta}(\cdot,y)\rangle_{\mathcal{H}^s_q,\eta} = f(y).$$

(iii) The result is obtained by taking m a null function and $\eta = 1$. □

The main result of this section can be stated as follows.

Theorem 4.2. *Let $s > \frac{1}{2}$ and m be a function in L^∞_q and $a > 0$. For any $h \in L^2_q$ and for any $\eta > 0$, there exists a unique function $f^*_{\eta,h,a}$, where the infimum*

$$\inf_{f\in\mathcal{H}^s_q}\left\{\eta\|f\|^2_{\mathcal{H}^s_q} + \|h - \mathcal{T}_m f\|^2_{q,2}\right\} \tag{4.6}$$

is attained. Moreover the extremal function $f^*_{\eta,h,a}$ is given by

$$f^*_{\eta,h,a}(y) = \int_{\infty}^{\infty} h(x)\overline{\Theta_{s,\eta}(x,y)}d_qx, \tag{4.7}$$

where

$$\Theta_{s,\eta}(x,y) = \int_{-\infty}^{\infty} \frac{m_a(\xi)e(ix\xi,q^2)}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2} e(-iy\xi,q^2)d_q\xi.$$

Proof. The existence and unicity of the extremal function $f^*_{\eta,h,a}$ satisfying (4.6) is given by [8, 10, 13]. On the other hand from Theorem 4.1 we have

$$f^*_{\eta,h,a}(y) = \langle h, \mathcal{T}_m(\Psi_{s,\eta}(\cdot,y))\rangle_{q,2}.$$

From Proposition 3.1 and identity (4.5) we obtain

$$\begin{aligned}\Theta_{s,\eta}(x,y) &= \mathcal{T}_m(\Psi_{s,\eta}(\cdot,y))(x)\\ &= \int_{-\infty}^{\infty} \frac{m_a(\xi)e(ix\xi,q^2)}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2} e(-iy\xi,q^2)d_q\xi.\end{aligned}$$

□

Theorem 4.3. *Let $s > \frac{1}{2}$ and m be a function in L^∞_q and $h \in L^2_q$. Then the extremal function $f^*_{\eta,h,a}$ satisfies the following properties:*

$$\mathcal{F}_q(f^*_{\eta,h,a})(\xi) = \frac{\overline{m_a(\xi)}}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2}\mathcal{F}_q(h)(\xi), \quad \xi \in \mathbb{R}_q$$

and

$$\|f^*_{\eta,h,a}\|^2_{\mathcal{H}^s_q} \leq \frac{1}{4\eta}\|h\|^2_{q,2}.$$

Proof. Let $y \in \mathbb{R}_q$, then the function

$$g_y : \xi \longmapsto \frac{m_a(\xi)e(-iy\xi, q^2)}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2}$$

belongs to $L^1_q \cap L^2_q$ and by the inversion formula we obtain

$$\Theta_{s,\eta}(x, y) = \mathcal{F}_q^{-1}(g_y)(x), \quad x \in \mathbb{R}_q.$$

Hence, by Plancherel formula, we have $\Theta_{s,\eta}(\cdot, y)$ belongs to L^2_q and

$$\begin{aligned} f^*_{\eta,h,a}(y) &= \int_{-\infty}^{\infty} \mathcal{F}_q(h)(\xi)\overline{g_y(\xi)}d_q\xi \\ &= \int_{-\infty}^{\infty} \frac{\overline{m_a(\xi)}\mathcal{F}_q(h)(\xi)}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2}e(iy\xi, q^2)d_q\xi. \end{aligned}$$

On the other hand, the function

$$F : \xi \longmapsto \frac{\overline{m_a(\xi)}\mathcal{F}_q(h)(\xi)}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2}$$

belongs to $L^1_q \cap L^2_q$ and by the inversion formula we obtain

$$f^*_{\eta,h,a}(y) = \mathcal{F}_q^{-1}(F)(y).$$

Afterwards, by Plancherel formula, it follows that $f^*_{\eta,h,a}$ belongs to L^2_q and we have

$$\mathcal{F}_q(f^*_{\eta,h,a})(\xi) = \frac{\overline{m_a(\xi)}\mathcal{F}_q(h)(\xi)}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2}, \quad \xi \in \mathbb{R}_q.$$

Hence

$$\begin{aligned} (1+|\xi|^2)^s \left|\mathcal{F}_q(f^*_{\eta,h,a})(\xi)\right|^2 &= (1+|\xi|^2)^s \left|\frac{\overline{m_a(\xi)}\mathcal{F}_q(h)(\xi)}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2}\right|^2 \\ &\leq (1+|\xi|^2)^s \frac{\left|\overline{m_a(\xi)}\mathcal{F}_q(h)(\xi)\right|^2}{4\eta(1+|\xi|^2)^s|m_a(\xi)|^2} \\ &\leq \frac{1}{4\eta}\left|\mathcal{F}_q(h)(\xi)\right|^2. \end{aligned}$$

Finally, using Plancherel theorem, we obtain

$$\|f^*_{\eta,h,a}\|^2_{\mathcal{H}^s_q} \leq \frac{1}{4\eta}\|h\|^2_{q,2}.$$

□

Theorem 4.4. *(Third Calderón's formula). Let $s > \frac{1}{2}$, m be a function in L_q^∞ and $f \in \mathcal{H}_q^s$. The extremal function given by*

$$f_{\eta,a}^*(y) = \int_{-\infty}^{\infty} \mathcal{T}_m f(x) \overline{\Theta_{s,\eta}(x,y)} d_q x \tag{4.8}$$

satisfies

$$\lim_{\eta \to 0^+} \|f_{\eta,a}^* - f\|_{\mathcal{H}_q^s} = 0.$$

Moreover, $\{f_{\eta,a}^*\}_{\eta>0}$ converges uniformly to f when η converge to 0^+.

Proof. Let $f \in \mathcal{H}_q^s$, $h = \mathcal{T}_m f$ and $f_{\eta,a}^* = f_{\eta,h,a}^*$. According to Proposition 4.1 the function h belongs to L_q^2. From the definition of the q^2-Fourier-multiplier operators $\mathcal{T}_m$ and Theorem 4.3, we obtain

$$\mathcal{F}_q(f_{\eta,a}^*)(\xi) = \frac{|m_a(\xi)|^2}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2} \mathcal{F}_q(f)(\xi), \quad \xi \in \mathbb{R}_q.$$

Hence, it follows that

$$\mathcal{F}_q(f_{\eta,a}^* - f)(\xi) = \frac{-\eta(1+|\xi|^2)^s}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2} \mathcal{F}_q(f)(\xi), \quad \xi \in \mathbb{R}_q. \tag{4.9}$$

Therefore,

$$\|f_{\eta,a}^* - f\|_{\mathcal{H}_q^s}^2 = \int_{-\infty}^{\infty} \frac{\eta^2(1+|\xi|^2)^{3s}(\xi)|\mathcal{F}_q(f)(\xi)|^2}{\left(\eta(1+|\xi|^2)^s + |m_a(\xi)|^2\right)^2} d_q x.$$

Then, from the dominated convergence theorem and the following inequality

$$\frac{\eta^2(1+|\xi|^2)^{3s}|\mathcal{F}_q(f)(\xi)|^2}{\left(\eta(1+|\xi|^2)^s + |m_a(\xi)|^2\right)^2} \leq (1+|\xi|^2)^s |\mathcal{F}_q(f)(\xi)|^2,$$

we deduce that

$$\lim_{\eta \to 0^+} \|f_{\eta,a}^* - f\|_{\mathcal{H}_q^s} = 0.$$

On the other hand, the function $\xi \longmapsto (1+|\xi|^2)^{-s/2}$ belongs to L_q^2 for all $s > 1/2$. Therefore, the function $\mathcal{F}_q(f)$ belongs to $L_q^1 \cap L_q^2$ for all $f \in \mathcal{H}_q^s$. Then, according to (4.9) and the inversion formula for the q^2-Fourier transform, we get

$$f_{\eta,a}^*(y) - f(y) = K \int_{-\infty}^{\infty} \frac{-\eta(1+|\xi|^2)^s \mathcal{F}_q(f)(\xi)}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2} e(iy\xi, q^2) d_q x.$$

By using the dominated convergence theorem and the fact

$$\frac{\eta(1+|\xi|^2)^s |\mathcal{F}_q(f)(\xi)|^2}{\eta(1+|\xi|^2)^s + |m_a(\xi)|^2} \leq |\mathcal{F}_q(f)(\xi)|,$$

we deduce that

$$\lim_{\eta \to 0^+} \sup_{y \in \mathbb{R}_q} \|f_{\eta,a}^*(y) - f(y)\| = 0.$$

which completes the proof of the Theorem. □

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