RADAU QUADRATURE FOR AN ALMOST QUASI-HERMITE-FEJÉR-TYPE INTERPOLATION IN RATIONAL SPACES

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Abstract. In this paper, we have studied an almost quasi Hermite-Fejér-type interpolation in rational spaces. A Radau type quadrature formula has also been obtained for the same.

1. Introduction

Hermite Fejér and Quasi-Hermite-Fejér-type interpolation processes has been a subject of interest for several mathematicians. In almost all the cases the interpolatory polynomials are considered on the nodes which are the zeros of certain classical orthogonal polynomials. The main idea of the present paper is to construct a rational interpolation process and its corresponding quadrature formula with prescribed nodes based on the Chebyshev Markov fractions.

Chebyshev and Markov introduced rational cosine and sine fractions [3] which generalizes Chebyshev polynomials, possesses many similar properties ([2, 10, 11]) and are called Chebychev–Markov rational fractions. Different aspects of the rational generalization of Chebyshev polynomials are discussed in many works ([1, 12]).

In 1962, Rusak [9] initiated the study of interpolation processes by means of rational functions on the interval $[-1, 1]$. The nodes were taken to be the zeros of Chebyshev–Markov rational fractions. In [6] rational interpolation functions of Hermite-Fejér-type were constructed [7]. Min [4] was the first to consider the rational quasi-Hermite-type interpolation. He constructed the interpolatory function and proved its uniform convergence for the continuous functions on the segment with the restriction that the poles of the approximating rational functions should not have limit points on the interval $[-1, 1]$.

Recently, based on the ideas of [6] and using method that was different from that of [4], Rouba et. al. ([5], [8]) revisited the rational interpolation functions of Hermite-Fejér-type. They also proved the uniform convergence of the interpolation process for the function $f \in C[-1, 1]$ and obtained explicitly its corresponding Lobatto type quadrature formula.

In this paper, we have considered an almost quasi-Hermite-Fejér-type interpolation process on the zeros of the rational Chebyshev-Markov sine fraction on the semi closed interval $(-1, 1]$, that is, when the interpolatory condition is prescribed only at one of the end points. A Radau type quadrature formula corresponding to the interpolation process has also been obtained.

2. Preliminaries

Consider a set of points a_k, $k = 0, 1, \cdots, 2n - 1$ which are either real and $a_k \in (-1, 1)$ or be paired by complex conjugation. Also let $U_n(x)$ be the rational Chebyshev-Markov sine fraction,

$$(2.1) \qquad U_n(x) = \frac{\sin \mu_{2n}(x)}{\sqrt{1 - x^2}}$$

where,

$$(2.2) \qquad \mu_{2n}(x) = \frac{1}{2} \sum_{k=0}^{2n-1} \arccos \frac{x + a_k}{1 + a_k x},$$

$$(2.3) \qquad \mu'_{2n}(x) = -\frac{\lambda_{2n}(x)}{\sqrt{1 - x^2}},$$

$$(2.4) \qquad \lambda_{2n}(x) = \frac{1}{2} \sum_{k=0}^{2n-1} \frac{\sqrt{1 - a_k^2}}{1 + a_k x}, \quad n \in N.$$

Also

$$(2.5) \qquad U'_n(x) = \frac{-\cos \mu_{2n}(x) \lambda_{2n}(x) \sqrt{1 - x^2} + x \sin \mu_{2n}(x)}{(1 - x^2)^{3/2}}$$

and

$$(2.6) \qquad U'_n(x_k) = -\frac{\lambda_{2n}(x_k)}{(1-x_k^2)}.$$

Let $\mathcal{R}_{2n-1}(a_0, a_1, \cdots, a_{2n-1})$ be a rational space defined as

$$(2.7) \qquad \mathcal{R}_{2n-1}(a_0, a_1, \cdots, a_{2n-1}) := \left\{ \frac{p_{2n-1}(x)}{\prod_{k=0}^{2n-1}(1+a_k x)} \right\}$$

where $p_{2n-1}(x)$ is a polynomial of degree $\leq 2n-1$ and $\{a_k\}_{k=0}^{2n-1}$ are real and belong to $[-1,1]$ or are paired by complex conjugation.

The rational fraction $U_n(x)$ can be expressed as

$$U_n(x) = \frac{P_{n-1}(x)}{\sqrt{\prod_{k=0}^{2n-1}(1+a_k x)}}$$

where $P_{n-1}(x)$ is an algebraic polynomial of degree $n-1$ with real coefficient. The fraction $U_n(x)$ has $n-1$ zeros on the interval $(-1,1)$ given by,

$$-1 < x_{n-1} < x_{n-2} < \cdots < x_2 < x_1 < 1, \quad \mu_{2n}(x_k) = k\pi, \ k = 1, 2, \cdots, n-1,$$

where $\mu_{2n}(x)$ is given by (2.2). Also, the rational function $\lambda_{2n}(x)$, given by (2.4), can be expressed as

$$\lambda_{2n}(x) = \frac{q_{2n-1}(x)}{\prod_{k=0}^{2n-1}(1+a_k x)}$$

where $q_{2n-1}(x)$ is a polynomial of degree atmost $2n-1$. It has no zeros on $[-1,1]$.

3. Almost Quasi-Hermite-Fejér-type interpolation

Let $x_0 = 1$. Then for any function $f \in C(-1,1]$ the almost quasi type Hermite interpolation function $H_n(x, f)$ satisfying the conditions

$$(3.1) \qquad H_n(x_k, f) = f(x_k), \quad k = 0, 1, \cdots, n-1,$$

$$(3.2) \qquad H'_n(x_k, f) = y_k, \quad k = 1, 2, \cdots, n-1,$$

is given by

$$(3.3) \qquad H_n(x, f) = \sum_{k=0}^{n-1} f(x_k) A_k(x) + \sum_{k=1}^{n-1} y_k B_k(x),$$

where $y_k, k = 1, 2, \cdots, n-1$ are arbitrarily given real numbers, $\{A_k(x)\}_{k=0}^{n-1}$ and $\{B_k(x)\}_{k=1}^{n-1}$ are fundamental functions satisfying the conditions

$$(3.4) \qquad \begin{cases} A_k(x_j) = \delta_{kj}, & j, k = 0, 1, \cdots, n-1, \\ A'_k(x_j) = 0, & j = 1, 2, \cdots, n-1, k = 0, 1, \cdots, n-1 \end{cases}$$

and

$$(3.5) \quad \begin{cases} B_k(x_j) = 0, \quad j = 0, 1, \cdots, n-1, k = 1, 2, \cdots, n-1, \\ B'_k(x_j) = \delta_{kj}, \quad j, k = 1, 2, \cdots, n-1. \end{cases}$$

4. Explicit Representation of the Fundamental Functions

Lemma 4.1. *The fundamental functions $\{A_k(x)\}_{k=0}^{n-1}$ satisfying the conditions (3.4) can be explicitly represented as for $k = 1, 2, \cdots, n-1$*

$$(4.1) \quad A_k(x) = \frac{(1+x_k)(1-x_k^2)(1-x)\{1-b_k(x-x_k)\}U_n^2(x)}{\lambda_{2n}(x_k)(x-x_k)^2\lambda_{2n}(x)},$$

where

$$(4.2) \quad b_k = \frac{2x_k - 1}{1 - x_k^2}$$

and

$$(4.3) \quad A_0(x) = \frac{U_n^2(x)}{\lambda_{2n}(x)\lambda_{2n}(1)}.$$

Proof. We will show that $A_k(x)$, $k = 0, 1, \cdots, n-1$ defined by (4.1) and (4.3) satisfy the conditions (3.4) Obviously for $k = 1, 2, \cdots, n-1$, $A_k(x_0) = 0$ and $A_k(x_j) = 0$, $j = 1, 2, \cdots, n-1$ when $j \neq k$. For $j = k$ using the l'Hospital's rule, we have

$$\begin{aligned} \lim_{x \to x_k} A_k(x) &= \frac{(1-x_k^2)}{\lambda_{2n}^2(x_k)} \left(\lim_{x \to x_k} \frac{\sin\mu_{2n}(x)}{(x-x_k)} \right)^2 \\ &= \frac{(1-x_k^2)}{\lambda_{2n}^2(x_k)} \left(\lim_{x \to x_k} \frac{-\lambda_{2n}(x)\cos\mu_{2n}(x)}{\sqrt{1-x^2}} \right)^2 \\ &= 1. \end{aligned}$$

On differentiating (4.1) with respect x we get

$$\begin{aligned} A'_k(x) = \frac{(1+x_k)(1-x_k^2)}{\lambda_{2n}(x_k)} \Bigg[\frac{2\{1-b_k(x-x_k)\}}{(1+x)\lambda_{2n}(x)} \left(\frac{\sin\mu_{2n}(x)}{x-x_k}\right) \left(\frac{\sin\mu_{2n}}{x-x_k}\right)' \\ - \frac{b_k(1+x)\lambda_{2n}(x) + \{1-b_k(x-x_k)\}\{(1+x)\lambda'_{2n}(x) + \lambda_{2n}(x)\}}{(1+x)^2\lambda_{2n}^2(x)} \\ \times \left(\frac{\sin\mu_{2n}(x)}{x-x_k}\right)^2 \Bigg] \end{aligned}$$

then for $j \neq k$ we have $A'_k(x_j) = 0$, $j = 1, 2, \cdots, n-1$. For $j = k$,

$$\begin{aligned} \lim_{x \to x_k} A'_k(x) &= \frac{(1-x_k^2)}{\lambda_{2n}^2(x_k)} \Bigg[2 \lim_{x \to x_k} \left(\left(\frac{\sin\mu_{2n}(x)}{x-x_k}\right) \left(\frac{\sin\mu_{2n}(x)}{x-x_k}\right)' \right) \\ &\quad - \frac{b_k(1+x_k)\lambda_{2n}(x_k) + (1+x_k)\lambda'_{2n}(x_k) + \lambda_{2n}(x_k)}{(1+x_k)\lambda_{2n}(x_k)} \\ &\quad \times \left(\lim_{x \to x_k} \frac{\sin\mu_{2n}(x)}{x-x_k} \right)^2 \Bigg] \end{aligned}$$

We know that

(4.4)
$$\lim_{x \to x_k} \frac{\sin \mu_{2n}(x)}{(x - x_k)} = \mu'_{2n}(x_k) \cos \mu_{2n}(x_k) = -\frac{\lambda_{2n}(x_k)}{\sqrt{1 - x_k^2}}$$

and

(4.5)
$$\lim_{x \to x_k} \left(\frac{\sin \mu_{2n}(x)}{x - x_k} \right)' = \frac{1}{2} \cos \mu_{2n}(x_k) \mu''_{2n}(x_k)$$

where

(4.6)
$$\mu''_{2n}(x) = -\frac{(1 - x^2)\lambda'_{2n}(x) + x\lambda_{2n}(x)}{(1 - x^2)^{3/2}}$$

then

$$\lim_{x \to x_k} A'_k(x) = \left[\frac{2x_k - 1}{(1 - x_k^2)} - b_k \right] = 0$$

due to (4.2) which shows that $A_k(x)$, $k = 1, 2, \cdots, n - 1$, given by (4.1), satisfy all the conditions given by (3.4).

Similarly, for $A_0(x)$, given by (4.3), we have that $A_0(x_j) = 0$, $j = 1, \cdots, n - 1$. For $j = 0$ and using the fact that $U_n(1) = \lambda_{2n}(1)$, we have $A_0(x_0) = 1$. Again by differentiating (4.3) with respect x, we get $A'_0(x_j) = 0$, $j = 1, 2, \cdots, n - 1$. Thus $A_0(x)$ given by (4.3) satisfy the conditions (3.4) for $j = 0$, which completes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 4.2. *The fundamental functions* $\{B_k(x)\}_{k=1}^{n-1}$ *satisfying the conditions* (3.5) *can be explicitly represented as*

(4.7)
$$B_k(x) = \frac{(1 - x)(1 + x_k)(1 - x_k^2)U_n^2(x)}{\lambda_{2n}(x)\lambda_{2n}(x_k)(x - x_k)}.$$

Proof. Obviously, $B_k(x_j) = 0$, $k = 1, 2, \cdots, n - 1$, $j = 0, 1, \cdots, n - 1$ and for $j \neq k$, $B'_k(x_j) = 0$, $j, k = 1, 2, \cdots, n - 1$. For $j = k$,

$$
\begin{aligned}
\lim_{x \to x_k} B'_k(x) &= \frac{(1 - x_k^2)^2}{\lambda_{2n}^2(x_k)} \lim_{x \to x_k} \left(\frac{U_n^2(x)}{(x - x_k)} \right) \\
&= \frac{(1 - x_k^2)^2}{\lambda_{2n}^2(x_k)} \lim_{x \to x_k} \left(\frac{2U_n(x)U'_n(x)}{(x - x_k)} - \left(\frac{U_n(x)}{x - x_k} \right)^2 \right) \\
&= \frac{(1 - x_k^2)^2}{\lambda_{2n}^2(x_k)} (U'_n(x_k))^2 \\
&= 1,
\end{aligned}
$$

due to (2.6), which proves that $B_k(x)$, $k = 1, 2, \cdots, n - 1$ given by (4.7) satisfy all the conditions given by (3.5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

From Lemma 4.1 and Lemma 4.2 it follows that $H_n(f, x)$ satisfying the conditions (3.1) is an almost quasi Hermite interpolation.

Theorem 4.1. *The function $H_n(f, x)$ is a rational function of degree atmost $2n - 1$ that is*

(4.8)
$$H_n(f, x) \in \mathcal{R}_{2n-1}(a_0, a_1, \cdots, a_{2n-1}).$$

Proof. Since $U_n \in \mathcal{P}_{n-1}(a_0, a_1, \cdots, a_{2n-1})$, we can express it as

$$U_n(x) := \frac{S_{n-1}(x)}{(S_n^*(x))^{1/2}}$$

where $S_n^*(x) := (x - a_0)(x - a_1) \cdots (x - a_{2n-1})$, $S_{n-1}(x) := c_{n-1}(x - x_1)(x - x_2) \cdots (x - x_{n-1})$ and c_{n-1} depends on n and $\{a_k\}_{k=0}^{2n-1}$. So, we have

(4.9)
$$\ell_k(x) = \left(\frac{S_n^*(x_k)}{S_n^*(x)}\right)^{1/2} q_k(x), \quad k = 1, 2, \cdots, n-1,$$

where

(4.10)
$$q_k(x) := \frac{S_{n-1}(x)}{S_{n-1}'(x_k)(x - x_k)}, \quad k = 1, 2, \cdots, n-1.$$

Thus, $\ell_k(x) \in \mathcal{P}_{n-1}(a_0, a_1, \cdots, a_{2n-1})$, thus by (3.3), (4.1) and (4.7) we easily find that

(4.11)
$$H_n(f, x) = \frac{t_{2n-1}(x)}{q_{2n-1}(x)}$$

where $t_{2n-1}(x)$ is a polynomial of degree $\leq 2n - 1$, which proves the lemma. $\qquad\qquad\square$

Let $y_k = 0$, $k = 1, 2, \cdots, n-1$ then (3.3) reduces to

(4.12)
$$H_n(f, x) = \sum_{k=1}^{n-1} f(x_k)A_k(x) + f(1)A_0(x)$$

which is an almost quasi Hermite Fejér interpolation function for $f \in C[-1, 1]$.

5. Radau-type quadrature formula

For a given function f defined on $[-1, 1]$, we define the function

(5.1)
$$G_n(x, f) = \sum_{k=0}^{n-1} f(x_k)h_k(x)$$

where,

$$h_k(x) = \frac{1 - x}{1 - x_k}\left[1 - \left(\frac{U_n''(x_k)}{U_n'(x_k)} - \frac{1}{(1 - x_k)}\right)(x - x_k)\right]\ell_k^2(x),$$
$$k = 1, 2, \cdots, n-1$$

and

$$h_0(x) = \frac{U_n^2(x)}{U_n^2(1)}.$$

We have that $G_n(f, x) \in \mathcal{R}_{2n-1}(a_0, a_1, \cdots, a_{2n-1})$. Also the rational function $G_n(f, x)$ is an almost quasi Hermite Fejér interpolation function. Let

$$(5.2) \qquad A_k = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} h_k(x) dx, \quad k = 1, 2, \cdots, n-1$$

and

$$(5.3) \qquad A_0 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{U_n^2(x)}{U_n^2(1)} dx$$

then the Radau-type quadrature formula is given by

$$(5.4) \qquad \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = A_0 f(1) + \sum_{k=1}^{n-1} A_k f(x_k).$$

With respect to this quadrature formula, we have the following

Theorem 5.1. *The quadrature formula* (5.4) *can be expressed as*

$$(5.5) \qquad \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{2\pi}{\lambda_{2n}(1)} f(1) + \sum_{k=1}^{n-1} \frac{\pi}{\lambda_{2n}(x_k)} f(x_k).$$

Lemma 5.1. *For* $k = 1, 2, \cdots, n-1$,

$$(5.6) \qquad \int_{-1}^{1} (1-x)(x-x_k)\ell_k^2(x) dx = 0.$$

Proof. We have that for $k = 1, 2, \cdots, n-1$,

$$\ell_k^2(x) = \frac{U_n^2(x)}{(U_n')^2(x_k)(x-x_k)^2}$$

$$(5.7) \qquad\qquad = \frac{(1-x_k^2)^2 \sin^2 \mu_{2n}(x)}{\lambda_{2n}^2(x_k)(1-x^2)(x-x_k)^2}.$$

Also,

$$(5.8) \qquad U_n(1) = \lim_{x \to 1} \frac{\sin \mu_{2n}(x)}{\sqrt{1-x^2}} = \lambda_{2n}(1)$$

and

$$(5.9) \qquad U_n(-1) = (-1)^{n+1} \lambda_{2n}(-1).$$

By these equalities, the left hand side of (5.6) can be represented as

$$(5.10) \qquad I_k = \int_{-1}^{1} \frac{\sin^2 \mu_{2n}(x)}{(1+x)\sqrt{1-x^2}(x-x_k)} dx$$

Consider the transformation

$$(5.11) \qquad x = \frac{1-y^2}{1+y^2}$$

which gives

$$(5.12) \qquad dx = -\frac{4y}{(1+y^2)^2} dy,$$

$$(5.13) \qquad x - x_k = -\frac{2(y^2 - y_k^2)}{(1 + y^2)(1 + y_k^2)},$$

$$(5.14) \qquad 1 + x = \frac{2}{1 + y^2},$$

$$(5.15) \qquad \sqrt{1 - x^2} = \frac{2y}{1 + y^2}.$$

We know that,

$$(5.16) \qquad \sin \mu_{2n} \left(\frac{1 - y^2}{1 + y^2}\right) = \sin \phi_{2n}(y)$$

where $\sin \phi_{2n}(y)$ is a Bernstein sine fraction

$$(5.17) \qquad \sin \phi_{2n}(y) = \frac{1}{2i}\left(\chi_n(y) - \chi_n^{-1}(y)\right)$$

where $\chi_n(y) = \prod_{j=0}^{2n-1} \frac{y - z_j}{y - \bar{z}_j}$ and z_k are the roots of the equations $y^2 + (1 + a_k)(1 - a_k)^{-1} = 0$, $\mathcal{I}z_k > 0$, $k = 0, 1, \cdots, 2n - 1$. Taking into account the assumptions on the parameters a_k, $k = 0, 1, \cdots, 2n - 1$, we have the following: 1) $z_0 = i$, 2) if a_k and a_l are paired by complex conjugation, then the corresponding numbers z_k and z_l are symmetric with respect to the imaginary axis. Besides, the function $\sin \phi_{2n}(y)$ has zeros at $\pm y_k$, $y_k = \sqrt{(1 - x_k)/(1 + x_k)}$, $k = 1, 2, \cdots, n - 1$. Thus,

$$
\begin{aligned}
I_k &= -\frac{1 + y_k^2}{4} \int_{-\infty}^{\infty} \frac{(1 + y^2) \sin^2 \phi_{2n}(y)}{y^2 - y_k^2} dy \\
(5.18) \qquad &= -\frac{1 + y_k^2}{4} \lim_{z \to y_k, \mathcal{I}z_k > 0} J_k(z)
\end{aligned}
$$

where

$$(5.19) \qquad J_k(z) = \int_{-\infty}^{\infty} \frac{(1 + y^2) \sin^2 \phi_{2n}(y)}{y^2 - z^2} dy.$$

From (5.17) we get

$$(5.20) \qquad \sin^2 \phi_{2n}(y) = -\frac{1}{4}\left(\chi_n^2(y) - 2 + \chi_n^{-2}(y)\right)$$

due to which, we have

$$(5.21) \qquad J_k(z) = -\frac{1}{4}\left(J_{k1}(z) - 2J_{k2}(z) + J_{k3}(z)\right)$$

where

$$J_{k1}(z) = \int_{-\infty}^{\infty} \frac{(1 + y^2)\chi_n^2(y)}{y^2 - z^2} dy,$$

$$J_{k2}(z) = \int_{-\infty}^{\infty} \frac{(1 + y^2)}{y^2 - z^2} dy$$

and

$$J_{k3}(z) = \int_{-\infty}^{\infty} \frac{(1+y^2)\chi_n^{-2}(y)}{y^2 - z^2} dy.$$

Since $J_{k1}(z)$ has only singular point $y = z$ in the upper half plane. Thus by the residue theorem we have

$$
\begin{aligned}
J_{k1}(z) &= 2\pi i \lim_{y \to z} \frac{(1+y^2)\chi_n^2(y)}{(y+z)} \\
(5.22) &\qquad = \frac{1+z^2}{z}\chi_n^2(z)\pi i.
\end{aligned}
$$

Similarly,

$$(5.23) \qquad J_{k3}(z) = \frac{1+z^2}{z}\chi_n^{-2}(z)\pi i.$$

Also, $J_{k2}(z)$ has only singular point $y = z$ in the upper half plane, therefore by the residue theorem, we have

$$(5.24) \qquad J_{k2}(z) = 2\pi i \lim_{y \to z} \frac{(1+y^2)}{(y+z)} = \frac{1+z^2}{z}\pi i.$$

Putting the value of $J_{k1}(z)$, $J_{k3}(z)$ and $J_{k2}(z)$ from (5.22), (5.23) and (5.24) respectively in (5.21) we get

$$(5.25) \qquad J_k(z) = -\frac{1}{4}\left(\frac{1+z^2}{z}\chi_n^2(z)\pi i - 2\frac{1+z^2}{z}\pi i + \frac{1+z^2}{z}\chi_n^{-2}(z)\pi i\right)$$

which by (5.18) gives

$$(5.26) \qquad I_k = \frac{1+y_k^2}{16} \lim_{z \to y_k, \mathcal{I} z_k > 0}\left(\frac{1+z^2}{z}\chi_n^2(z)\pi i + \frac{1+z^2}{z}\chi_n^{-2}(z)\pi i - 2\frac{1+z^2}{z}\pi i\right).$$

Since $\chi_n(y_k) = 1$, thus it follows that $I_k = 0$, which completes the proof of the lemma. $\qquad\square$

Lemma 5.2. *For $k = 1, 2, \cdots, n-1$,*

$$(5.27) \qquad A_k = \frac{\pi}{\lambda_{2n}(x_k)}.$$

Proof. Due to Lemma 5.1 and by putting the value of $\ell_k^2(x)$, $k = 1, 2, \cdots, n-1$, from (5.7) in (5.2), we have

$$(5.28) \qquad A_k = \frac{(1+x_k)(1-x_k^2)}{\lambda_{2n}^2(x_k)}\int_{-1}^{1}\frac{\sin^2\mu_{2n}(x)}{(x-x_k)^2(1+x)\sqrt{1-x^2}}dx.$$

We find the integrals

$$(5.29) \qquad I_k^* = \int_{-1}^{1}\frac{\sin^2\mu_{2n}(x)}{(x-x_k)^2(1+x)\sqrt{1-x^2}}dx, \quad k = 1, 2, \cdots, n-1,$$

by using the transformation (5.11), (5.12), (5.13), (5.14), (5.15) and (5.16), we have

$$I_k^* = \frac{(1+y_k^2)^2}{8}\int_{-\infty}^{\infty}\frac{(1+y^2)^2\sin^2\phi_{2n}(y)}{(y^2-y_k^2)^2}dy.$$

Consider the auxiliary integral

$$J_k^*(z) = \int_{-\infty}^{\infty}\frac{(1+y^2)^2\sin^2\phi_{2n}(y)}{(y^2-z^2)^2}dy$$

then I_k^* can be written as

(5.30)
$$I_k^* = \frac{(1 + y_k^2)^2}{8} \lim_{z \to y_k, \Im z_k > 0} J_k^*(z).$$

which due to (5.20) can be expressed as

(5.31)
$$I_k^* = \frac{(1 + y_k^2)^2}{32} \lim_{z \to y_k, \Im z_k > 0} (I_{1k}^*(z) - 2I_{2k}^*(z) + I_{3k}^*(z))$$

where,

$$I_{1k}^*(z) = \int_{-\infty}^{\infty} \frac{(1 + y^2)^2}{(y^2 - z^2)^2} \chi_n^2(y) dy,$$

$$I_{2k}^*(z) = \int_{-\infty}^{\infty} \frac{(1 + y^2)^2}{(y^2 - z^2)^2} dy$$

and

$$I_{3k}^*(z) = \int_{-\infty}^{\infty} \frac{(1 + y^2)^2}{(y^2 - z^2)^2} \chi_n^{-2}(y) dy.$$

Since $z_0 = i$ the integrand of $I_{1k}^*(z)$ has only singular point $y = z$ in the upper half plane. Thus by the residue theorem, we have

$$
\begin{aligned}
I_{1k}^*(z) &= 2\pi i \lim_{y \to z} \frac{d}{dy} \left(\frac{(1 + y^2)^2}{(y + z)^2} \chi_n^2(y) \right) \\
(5.32) &= 2\pi i \lim_{y \to z} \left[\chi_n^2(y) \frac{d}{dy} \frac{(y^2 + 1)^2}{(y + z)^2} + \frac{(y^2 + 1)^2}{(y + z)^2} \frac{d}{dy} \chi_n^2(y) \right].
\end{aligned}
$$

Since,

$$\chi_n(y) = \prod_{j=0}^{2n-1} \frac{y - z_j}{y - \bar{z}_j}$$

which by logarithmic differentiation gives

$$\frac{d}{dy} \chi_n(y) = \chi_n(y) \sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y - z_j)(y - \bar{z}_j)}.$$

Also,

$$\frac{d}{dy} \left(\frac{(y^2 + 1)^2}{(y + z)^2} \right) = \frac{2y^4 + 4y^3 z + 4yz - 2}{(y + z)^3}.$$

Therefore,

(5.33)
$$
\begin{aligned}
I_{1k}^*(z) &= 2\pi i \chi_n^2(z) \left[\frac{3z^4 + 2z^2 - 1}{4z^3} \right. \\
&\quad \left. + \frac{(z^2 + 1)^2}{2z^2} \sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(z - z_j)(z - \bar{z}_j)} \right].
\end{aligned}
$$

Similarly,

(5.34)
$$
\begin{aligned}
I_{3k}^*(z) &= 2\pi i \chi_n^{-2}(z) \left[\frac{3z^4 + 2z^2 - 1}{4z^3} \right. \\
&\quad \left. + \frac{(z^2 + 1)^2}{2z^2} \sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(z - z_j)(z - \bar{z}_j)} \right].
\end{aligned}
$$

Again by residue theorem we have

$$
\begin{aligned}
I_{2k}^*(z) &= 2\pi i \lim_{y\to z} \frac{2y^4 + 4y^3 z + 4yz - 2}{(y+z)^3} \\
&= 2\pi i \left(\frac{3z^4 + 2z^2 - 1}{4z^3} \right).
\end{aligned}
$$

(5.35)

By (5.33), (5.34), (5.35), (5.31) and (5.30) and taking into account that $\chi_n^2(y_k) = 1$ it follows that

$$
I_k^* = -\frac{\pi i(1 + y_k^2)^4}{32 y_k^2} \sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)}..
$$

Since, $y_k = \sqrt{(1 - x_k)/(1 + x_k)}$ and $z_k = i\sqrt{(1 + a_k)/(1 - a_k)}$, thus by simple calculation, we have,

$$
\begin{aligned}
\sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)} &= \sum_{j=0}^{2n-1} \left(\frac{1}{y_k - z_j} - \frac{1}{y_k - \bar{z}_j} \right) \\
&= \sum_{j=0}^{2n-1} \frac{i\sqrt{(1 + a_j)}\sqrt{(1 - a_j)}}{1 + a_j x_k} \left(\frac{2}{1 + y_k^2} \right) \\
&= \frac{4i\lambda_{2n}(x_k)}{(1 + y_k^2)}
\end{aligned}
$$

(5.36)

thus

$$
I_k^* = \frac{\pi \lambda_{2n}(x_k)(1 + y_k^2)^3}{8 y_k^2}.
$$

Therefore by (5.28), the lemma follows. $\qquad\square$

Lemma 5.3. *For A_0, defined by (5.3), we have*

(5.37)
$$
A_0 = \left(\frac{\pi}{\lambda_{2n}(1)} \right).
$$

Proof. By using the transformation (5.11), (5.12), (5.15) and (5.16), we have

$$
A_0 = \frac{1}{4U_n^2(1)} \int_{-\infty}^{\infty} \frac{1 + y^2}{y^2} \sin^2 \phi_{2n}(y) dy
$$

which, due to (5.20), can expressed as

(5.38)
$$
A_0 = -\frac{1}{16 U_n^2(1)} (I_1 - 2I_2 + I_3)
$$

where

$$
I_1 = \int_{-\infty}^{\infty} \frac{1 + y^2}{y^2} \chi_n^2(y) dy,
$$

$$
I_2 = \int_{-\infty}^{\infty} \frac{1 + y^2}{y^2} dy
$$

and

$$
I_3 = \int_{-\infty}^{\infty} \frac{1 + y^2}{y^2} \chi_n^{-2}(y) dy.
$$

Since for I_1, $y = 0$ is the only singular point in the upper half plane. Thus by the residue theorem, we have

$$(5.39) \qquad \begin{aligned} I_1 &= 2\pi i \lim_{y \to 0} \frac{d}{dy} \left\{ (1 + y^2) \chi_n^2(y) \right\} \\ &= 4\pi i \sum_{j=0}^{2n-1} \left(\frac{1}{\bar{z}_j} - \frac{1}{z_j} \right). \end{aligned}$$

Similarly,

$$(5.40) \qquad I_3 = 4\pi i \sum_{j=0}^{2n-1} \left(\frac{1}{\bar{z}_j} - \frac{1}{z_j} \right).$$

The integrand of $I_2(z)$ has only singular point $y = 0$ in the upper half plane. Thus by the residue theorem, we have

$$(5.41) \qquad I_2 = 2\pi i \lim_{y \to 0} \frac{d}{dy} (1 + y^2) = 0.$$

Hence using (5.40), (5.40) and (5.41) in (5.38) we get

$$A_0 = -\frac{\pi i}{2\lambda_{2n}^2(1)} \sum_{j=0}^{2n-1} \left(\frac{1}{\bar{z}_j} - \frac{1}{z_j} \right)$$

and since

$$\sum_{j=0}^{2n-1} \left(\frac{1}{\bar{z}_j} - \frac{1}{z_j} \right) = -\frac{4}{i} \lambda_{2n}(1)$$

hence

$$(5.42) \qquad A_0 = \frac{2\pi}{\lambda_{2n}(1)}$$

which in turn proves the lemma. $\qquad \square$

By Lemma 5.2, Lemma 5.3 and (5.4), Theorem 5.1 follows.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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