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SYMMETRY ANALYSIS AND SOLITARY WAVE SOLUTIONS OF NONLINEAR ION-ACOUSTIC WAVES EQUATION

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Abstract. The problem of nonlinear ion-acoustic waves equation in a magnetized plasma, known as Zakharov-Kuznetsov equation, is investigated by using symmetry analysis. The carryover of the symmetry analysis has led to certain similarity reductions of this equation. Furthermore, exact solutions of similarity reductions are obtained by modified Exp-Function method with computational symbolic. Some figures are obtained to show the properties of the solutions.

1. Introduction

There are many well-known methods to obtain exact solutions $[1-5]$. In order to unite and widen various specialized solution method for partial differential equations Lie introduced the notion of continuous groups now know as Lie groups. contiuing his investigations he shown that partial differential equation can be reduced to many ordinary differential equations which is led to varied solutions . In the last century, the application of the Lie groups has been developed by a number of reserchers. Ovsiannikov [6], Olver [7], Ibragimov [8], and Bluman et al. [9] are some of the mathematicians who have huge number of studies in that field.

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Consider the nonlinear ion-acoustic waves equation which is called (1+3)-dimensional Zakharov-Kuznetsov (Zk) equation [10, 11] in the following form:

$$u_t + p_1 u\ u_x + p_2 u_{x,x,x} + p_3 u_{x,y,y} + p_4 u_{x,z,z} = 0 \tag{1}$$

where p_1, p_2, p_3 and p_4 are nonzero constants. ZK [10] is described the diffusion of nonlinear ion-acoustic waves in magnetized plasma [10]. This equation was devoted to study many properties including presence and stability of solitary wave solutions for the ZK model [10, 13 − 15].

2. Determination of the symmetries

Firstly, we shall conclude the similarity reductions using Lie group method [16 − 22]. In order to apply Lie group method we can write the one parameter Lie group of infinitesimal transformations as follow:

$$\begin{aligned} &t^* = t + \varepsilon A(t,x,y,z,u) + o(\varepsilon^2), x^* = x + \varepsilon B(t,x,y,z,u) + o(\varepsilon^2),\\ &y^* = y + \varepsilon C(t,x,y,z,u) + o(\varepsilon^2), z^* = z + \varepsilon D(t,x,y,z,u) + o(\varepsilon^2),\\ &u^* = u + \varepsilon E(t,x,y,z,u) + o(\varepsilon^2). \end{aligned} \tag{2}$$

If we set

$$\Delta = u_t + p_1 u\ u_x + p_2 u_{x,x,x} + p_3 u_{x,y,y} + p_4 u_{x,z,z} = 0 \tag{3}$$

where subscripts t, x, y and z to the function u denote differentiation with respect to these variables. The infinitesimal generator V associated with the above mentioned group of transformations can be presented as following expression

$$V = A\frac{\partial}{\partial t} + B\frac{\partial}{\partial x} + C\frac{\partial}{\partial y} + D\frac{\partial}{\partial z} + E\frac{\partial}{\partial u}, \tag{4}$$

when the following invariance condition is satisfied:

$$\Gamma^{(3)}(\Delta) = 0, \tag{5}$$

where $\Gamma^{(3)}$ is the third order prolongation of the operator V

$$\Gamma^{(3)} = V + E_{[x]}\frac{\partial}{\partial u_x} + E_{[t]}\frac{\partial}{\partial u_t} + E_{[xxx]}\frac{\partial}{\partial u_{xxx}} + E_{[xyy]}\frac{\partial}{\partial u_{xyy}} + E_{[xzz]}\frac{\partial}{\partial u_{xzz}}, \tag{6}$$

where the components $E_{[x]}, E_{[xx]}$, $E_{[xyy]}$, $E_{[xz]}$, $E_{[t]}$....can be determined from the following expressions:

$$\begin{aligned} E_{[x]} &= D_x E - u_t D_x A - u_x D_x B,\\ E_{[xt]} &= D_t E_{[s]} - u_{tx} D_t A - u_{xx} D_t B. \end{aligned} \tag{7}$$

Substituting (3) into invariance condition (5), yields an identity components $A_x, A_{xx}, B_t, B_x, ...$ hence we collect the coefficients of u_x, $u_{x,x,x}$,... and equate it to zero, which led to obtain a system of linear differential equations of the infinitesimals A, B, C and E

$$A_x = A_y = A_z = A_u = A_{t,t} = 0,$$
$$B_y = B_z = B_u = B_{t,t} = 0, B_x = \frac{1}{3}A_t,$$
$$C_t = C_x = B_u = 0, C_y = \frac{1}{3}A_t,\ C_z = \frac{-p_3}{p_4}D_y,$$
$$D_t = D_x = D_u = D_{y,y} = 0, D_z = \frac{1}{3}A_t,$$
$$-p_1^2u^2B_x + p_1uA_x - p_1uB_x - p_1E + A_t = 0,$$

$$3p_2C_{x,x} + p_3C_{y,y} + p_4C_{z,z} = 0, \tag{8}$$

Solving resulting of partial differential equations system, we got:

$$\begin{aligned} A &= c_1t + c_2, & B &= \frac{1}{3}c_1x + p_1c_7t + c_4 \\ C &= \frac{1}{3}c_1y - c_3\frac{p_3}{p_4}z + c_5, & D &= \frac{1}{3}c_1z + c_3y + c_6, \\ E &= \frac{-2}{3}c_1u + c_7. \end{aligned} \tag{9}$$

We can be easily write the vector field operator V from (9) as

$$V = V_1(c_1) + V_2(c_2) + V_3(c_3) + V_4(c_4) + V_5(c_5) + V_6(c_6) + V_7(c_7), \tag{10}$$

where

$$\begin{aligned} V_1 &= t\frac{\partial}{\partial t} + \frac{1}{3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u}\right), \\ V_2 &= \frac{\partial}{\partial t}, \quad V_3 = \frac{-p_3}{p_4}z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \\ V_4 &= \frac{\partial}{\partial x}, \quad V_5 = \frac{\partial}{\partial y}, \quad V_6 = \frac{\partial}{\partial z}, \\ V_7 &= p_1t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \end{aligned} \tag{11}$$

The commutator relations are given in Table 1.

Table 1: The commutator table

	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	0	$-V_2$	0	$\frac{-1}{3}V_4$	$\frac{-1}{3}V_5$	$\frac{-1}{3}V_6$	$\frac{2}{3}V_7$
V_2	V_2	0	0	0	0	0	p_1V_4
V_3	0	0	0	0	$-V_6$	$\frac{p_3}{p_4}V_5$	0
V_4	$\frac{1}{3}V_4$	0	0	0	0	0	0
V_5	$\frac{1}{3}V_5$	0	0	V_6	0	0	0
V_6	$\frac{1}{3}V_6$	0	$\frac{-p_3}{p_4}V_5$	0	0	0	0
V_7	$\frac{-2}{3}V_7$	$-p_1V_4$	0	0	0	0	0

From the commutator relations in table 1, we utilized the following six non-equivalent possibilities of Lie algebra

(I)$V_1 + m_1V_2 + m_2V_4 + m_3V_5 + m_4V_6 + m_5V_7$,

(II)$V_2 + m_1V_4 + m_2V_5 + m_3V_6$

(III)$V_4 + m_1V_5 + m_2V_7$

(IV)$V_2 + m_1V_5 + m_2V_7$

(V)$V_2 + m_1V_4 + m_2V_7$

(VI)$V_2 + m_1V_4 + m_2V_5$

3. Reductions and exact solutions

In order to obtain the invariant transformation, we can write the characteristic equation as follow

$$\frac{dt}{A(t,x,y,z,u)} = \frac{dx}{B(t,x,y,z,u)} = \frac{dy}{C(t,x,y,z,u)} = \frac{dz}{D(t,x,y,z,u)} = \frac{du}{E(t,x,y,z,u)}. \tag{12}$$

This equation is solved for the above six cases the invariant variables, then the corresponding reductions to partial differential equations are obtained and by using the similarity transformations the govern partial differential equations reduced to ordinary differential equations

Table 2: The invariant variables and their corresponding partial differential equations

Case	The invariant variables				Corresponding partial differential equations
	ζ_1	ζ_2	ζ_3	u	
I(i)	$\frac{x-\frac{m_5}{2}p_1t+n}{(t+m_1)^{\frac{1}{3}}}$	$\frac{y+n_3}{(t+m_1)^{\frac{1}{3}}}$	$\frac{z+n_4}{(t+m_1)^{\frac{1}{3}}}$	$\frac{F}{(t+m_1)^{\frac{2}{3}}}+\frac{m_5}{2}$	$2F+\zeta_1F_{\zeta_1}+\zeta_2F_{\zeta_2}+\zeta_3F_{\zeta_3}-3p_1FF_{\zeta_1}$ $-3p_2F_{\zeta_1\zeta_1\zeta_1}-3p_3F_{\zeta_1\zeta_2\zeta_2}-3p_4F_{\zeta_1\zeta_3\zeta_3}=0.$

Case	The invariant variables				Corresponding partial differential equations
	ζ_1	ζ_2	ζ_3	u	
I(ii)	$\frac{x-\frac{m_5}{2}p_1t+n}{(t+m_1)^{\frac{1}{3}}}$	$\frac{y+n_3}{(t+m_1)^{\frac{1}{3}}}$	$\frac{z+n_4}{(t+m_1)^{\frac{1}{3}}}$	$\frac{F}{(t+m_1)^{\frac{2}{3}}}+\frac{m_5}{2}$	If we put $p_1FF_{\zeta_1}+(p_2F_{\zeta_1\zeta_1\zeta_1}+p_3F_{\zeta_2\zeta_2}$ $+p_4F_{\zeta_3\zeta_3})_{\zeta_1}=0$, we conclude that $[2F+\zeta_1F_{\zeta_1}+\zeta_2F_{\zeta_2}+\zeta_3F_{\zeta_3}]=0$
II	$x-m_1t$	$y-m_2t$	$z-m_3t$	F	$m_1F_{\zeta_1}+m_2F_{\zeta_2}+m_3F_{\zeta_3}-p_1FF_{\zeta_1}$ $-p_2F_{\zeta_1\zeta_1\zeta_1}-p_3F_{\zeta_1\zeta_2\zeta_2}-p_4F_{\zeta_1\zeta_3\zeta_3}=0,$
III	t	$y-m_1x$	$z-m_2x$	F	$F_{\zeta_1}-p_1F(m_1F_{\zeta_1}+m_2F_{\zeta_2})-m_1(p_2m_1^2$ $+p_3)F_{\zeta_2\zeta_2\zeta_2}-m_2(3p_2m_1^2+p_3m_1)$ $F_{\zeta_2\zeta_2\zeta_3}-m_1(3p_2m_2^2+p_4m_1)F_{\zeta_2\zeta_3\zeta_3}$ $-m_2(3p_2m_2^2+p_4)F_{\zeta_3\zeta_3\zeta_3}=0$
IV	x	$y-m_1t$	$z-m_2t$	F	$m_1F_{\zeta_2}+m_2F_{\zeta_3}-p_1FF_{\zeta_1}-p_2F_{\zeta_1\zeta_1\zeta_1}$ $-p_3F_{\zeta_1\zeta_2\zeta_2}-p_4F_{\zeta_1\zeta_3\zeta_3}=0,$
V	$x-m_1t$	y	$z-m_2t$	F	$m_1F_{\zeta_1}+m_2F_{\zeta_3}-p_1FF_{\zeta_1}-p_2F_{\zeta_1\zeta_1\zeta_1}$ $-p_3F_{\zeta_1\zeta_2\zeta_2}-p_4F_{\zeta_1\zeta_3\zeta_3}=0,$
VI	$x-m_1t$	$y-m_2t$	z	F	$m_1F_{\zeta_1}+m_2F_{\zeta_2}-p_1FF_{\zeta_1}-p_2F_{\zeta_1\zeta_1\zeta_1}$ $-p_3F_{\zeta_1\zeta_2\zeta_2}-p_4F_{\zeta_1\zeta_3\zeta_3}=0,$

where $n_3=3m_3$, $n_4=3m_4$, $n=3m_2-\frac{9}{2}p_1m_2m_5$

Case I(i):

In this case, we put $\theta = k_1\zeta_1 + k_2\zeta_2 + k_3\zeta_3$,then, the equation can be written in the form:

$$2F + \theta F' - 3p_1k_1FF' - 3k_1(p_2k_1^2 + p_3k_2^2 + p_4k_3^2)F''' = 0. \tag{13}$$

To obtain the solution for the ODE corresponding to this case, we assume that this solution takes the following form

$$F = a_0 + a_1\theta + a_2\theta^2 + \frac{b_1}{\theta} + \frac{b_2}{\theta^2}. \tag{14}$$

Substituting Eq. (14) into Eq. (13), equating to zero the coefficients of all powers of θ yields a set of algebraic equations for a_0, a_1, a_1, b_1, b_2,solving the system of algebraic equations with the aid of Maple, we obtain the following results:- $a_0{=}a_1{=}b_1{=}b_2{=}0$, $a_1{=}\frac{1}{\mathrm{k_1p_1}}$, then, the final solution of Eq. (1) can be written in the form:

$$u(t, x, y, z) = \frac{1}{\mathrm{k_1p_1(t{+}m_1)}}[k_1(x - \frac{m_5}{2}p_1t + n) + k_2(y + n_3) + k_3(z + n_4)] + \frac{m_5}{2}. \tag{15}$$

Case I(ii): In this case we have to solve the following two PDEs

$$p_1FF_{\zeta_1}{+}p_2F_{\zeta_1\zeta_1\zeta_1}{+}p_3F_{\zeta_1\zeta_2\zeta_2}{+}p_4F_{\zeta_1\zeta_3\zeta_3} = 0, \tag{16}$$

$$2F{+}\zeta_1F_{\zeta_1}{+}\zeta_2F_{\zeta_2}{+}\zeta_3F_{\zeta_3} = 0. \tag{17}$$

We now introduce the simplified form of Lie-group transformations namely, the scaling group of transformation

$$F = e^{\epsilon}\ \bar{F}, \quad \zeta_1 = e^{\epsilon_1}\ \bar{\zeta}_1, \quad \zeta_2 = e^{\epsilon_2}\ \bar{\zeta}_2, \quad \zeta_3 = e^{\epsilon_3}\ \bar{\zeta}_3. \tag{18}$$

Substituting from (18) into (17) we have $\epsilon_1 = \epsilon_2 = \epsilon_3 = -\epsilon$.

This mean that (16) is invariant under the transformation (18) and the characteristic equation can be written as

$$\frac{d\zeta_1}{\zeta_1} = \frac{d\zeta_2}{\zeta_2} = \frac{d\zeta_3}{\zeta_3} = \frac{-dF}{F} \tag{19}$$

We get the similarity variables

$$\eta_1 = \frac{\zeta_1}{\zeta_2}, \quad \eta_2 = \frac{\zeta_3}{\zeta_2}, \quad F = \frac{f}{\zeta_2^2} \tag{20}$$

Substituting from (20) into (17) we find that it is satisfied. Also substituting from (20) to Eq.(16) we obtain

$$\begin{aligned} &p_1ff_{\eta_1} + p_2f_{\eta_1\eta_1\eta_1} + 12p_3f_{\eta_1} + 8p_3\eta_1f_{\eta_1} + 8p_3\eta_2f_{\eta_2} \\ &p_3\eta_1^2f_{\eta_1\eta_1\eta_1} + p_3\eta_2^2f_{\eta_1\eta_1\eta_2} + 2p_3\eta_1\eta_2f_{\eta_1\eta_2\eta_2} + p_4f_{\eta_1\eta_2\eta_2} = 0. \end{aligned} \tag{21}$$

By using $\theta = \eta_1 + \eta_2$ (21) can be written in form

$$p_1ff' + p_2f''' + 4p_3(3 + 2\theta)f' + (p_3\theta^2 + p_4)f''' + 4\theta + \theta^2 + 2 - \theta = 0. \tag{22}$$

Using the same method in the previous case, hence, we have obtained the following exact solution to ODE corresponding this case in the form $f = \frac{-12(p_2+p_4)}{p_1\theta^2}$, then the solution of Zk equation is

$$u(t,x,z) = \frac{m_5}{2} - \frac{12(p_2+p_4)}{p_1(n+n_4-\frac{m_5}{2}t+x+z)^2}. \tag{23}$$

Case II: We take $\theta = k_1\zeta_1 + k_2\zeta_2 + k_3\zeta_3$, then equation of case (II) can be written in the form

$$\begin{aligned}&(k_1m_1+k_2m_2+k_3m_3)F' - p_1k_1FF'\\ &-k_1(p_2k_1^2+p_3k_2^2+p_4k_3^2)F''' = 0.\end{aligned} \tag{24}$$

To utilize the solution for the ODE corresponding to this case, we used modified Exp-Function method [13,23], which is expressed in the form:

$$\begin{aligned}F(\theta) &= \frac{\sum\limits_{n=-c}^{p} a_n[\phi(\theta)]^n}{\sum\limits_{m=-d}^{q} b_m[\phi(\theta)]^m}\\ &= \frac{a_{-c}[\phi(\theta)]^{-c}+...+a_p[\phi(\theta)]^p}{b_{-d}[\phi(\theta)]^{-d}+...+b_q[\phi(\theta)]^q}\end{aligned} \tag{25}$$

where $\phi(\theta)$ satisfies the following Riccati equation

$$\phi'(\theta) = A + B\ \phi(\theta) + C\ \phi^2(\theta). \tag{26}$$

see [24-25].

We can freely choose the values of n and m in (25), that the solution does not depend on the balancing of the highest order linear and nonlinear terms [24].

$$F''' = \frac{a_1[\phi(\theta)]^{-c-8d-3}+...+a_2[\phi(\theta)]^{p+8d+3}}{b_1[\phi(\theta)]^{-9d}+...+b_2[\phi(\theta)]^{9q}}, \tag{27}$$

$$FF' = \frac{a_3[\phi(\theta)]^{-2c-7d-3}+...+a_4[\phi(\theta)]^{2p+7d+3}}{b_3[\phi(\theta)]^{-9d}+...+b_4[\phi(\theta)]^{9q}}, \tag{28}$$

where a_i and b_i are determined coefficients only for simplicity. From balancing the lowest order and highest order of ϕ (27-28) we obtain $-c-8d-3 = -2c-7d-3$, which leads to the limit $c = d$ and $p+8d+3 = 2p+7d+3$, which leads to the limit $p = q$. For simplicity, we set p=q=1, we have

$$\begin{aligned}F &= \frac{a_{-1}[\phi(\theta)]^{-1}+a_0+a_1[\phi(\theta)]}{b_{-1}[\phi(\theta)]^{-1}+b_0+b_1[\phi(\theta)]}.\\ &= \frac{a_{-1}+a_0[\phi(\theta)]+a_1[\phi(\theta)]^2}{b_{-1}+b_0[\phi(\theta)]+b_1[\phi(\theta)]^2}\end{aligned} \tag{29}$$

Substituting (29) into (24), equating to zero the coefficients of all powers of $\phi(\theta)$ yields a set of algebraic equations for a_i and b_i. By aid Maple we solve this algebraic equations, we get:

$$\begin{aligned} a_{-1} &= [\frac{1}{k_1p_1}(k_1m_1+k_2m_2+k_3m_3 \\ &\quad - k_1(B^2+8AC)(p_2k_1^2+p_3k_2^2+p_4k_3^2))]b_{-1}, \\ a_0 &= [\frac{-12BC}{p_1}(p_2k_1^2+p_3k_2^2+p_4k_3^2)]b_{-1}, \\ a_1 &= [\frac{-12C^2}{p_1}(p_2k_1^2+p_3k_2^2+p_4k_3^2)]b_{-1},\ b_0{=}b_1{=}0. \end{aligned} \tag{30}$$

The corresponding traveling wave solutions to (1) are:

Case 1: A$\neq$ 0, B$\neq$ 0, C$\neq$ 0.

$$\begin{aligned} u(t,x,y,z) &= \frac{1}{k_1p_1}(k_1m_1+k_2m_2+k_3m_3-k_1(B^2+8AC)(p_2k_1^2+p_3k_2^2+p_4k_3^2)) \\ &\quad - \frac{12BC}{p_1}(p_2k_1^2+p_3k_2^2+p_4k_3^2)[\frac{-B}{2C}+\frac{\sqrt{4AC-B^2}}{2C} \\ &\quad \tan(\frac{1}{2}(\sqrt{4AC-B^2}(k_1(\text{x-m}_1\text{t})+k_2(\text{y-m}_2\text{t})+k_3(\text{z-m}_2\text{t})+\text{d}_0)))] \\ &\quad - \frac{12C^2}{p_1}(p_2k_1^2+p_3k_2^2+p_4k_3^2)[\frac{-B}{2C}+\frac{\sqrt{4AC-B^2}}{2C} \\ &\quad \tan(\frac{1}{2}(\sqrt{4AC-B^2}(k_1(\text{x-m}_1\text{t})+k_2(\text{y-m}_2\text{t})+k_3(\text{z-m}_2\text{t})+\text{d}_0)))]^2. \end{aligned} \tag{31}$$

Case 2: A= 0, B$\neq$ 0, C$\neq$ 0.

$$\begin{aligned} u(t,x,y,z) &= \frac{1}{k_1p_1}(k_1m_1+k_2m_2+k_3m_3-k_1(B^2+8AC)(p_2k_1^2+p_3k_2^2+p_4k_3^2)) \\ &\quad - \frac{12BC}{p_1}(p_2k_1^2+p_3k_2^2+p_4k_3^2)\left[\frac{-B\exp(B(k_1(\text{x-m}_1\text{t})+k_2(\text{y-m}_2\text{t})+k_3(\text{z-m}_2\text{t}))+Bd_0)}{C\exp(B(k_1(\text{x-m}_1\text{t})+k_2(\text{y-m}_2\text{t})+k_3(\text{z-m}_2\text{t}))+Bd_0)-1}\right] \\ &\quad - \frac{12C^2}{p_1}(p_2k_1^2+p_3k_2^2+p_4k_3^2)\left[\frac{-B\exp(B(k_1(\text{x-m}_1\text{t})+k_2(\text{y-m}_2\text{t})+k_3(\text{z-m}_2\text{t}))+Bd_0)}{C\exp(B(k_1(\text{x-m}_1\text{t})+k_2(\text{y-m}_2\text{t})+k_3(\text{z-m}_2\text{t}))+Bd_0)-1}\right]^2. \end{aligned} \tag{32}$$

Case 3: A= $\frac{1}{2}$, B= 0, C= $\frac{1}{2}$.

$$\begin{aligned} u(t,x,y,z) &= \frac{1}{k_1p_1}(k_1m_1+k_2m_2+k_3m_3-2k_1(p_2k_1^2+p_3k_2^2+p_4k_3^2)) \\ &\quad - \frac{3}{p_1}(p_2k_1^2+p_3k_2^2+p_4k_3^2)\left[\frac{\tan(k_1(\text{x-m}_1\text{t})+k_2(\text{y-m}_2\text{t})+k_3(\text{z-m}_2\text{t}))}{1\pm\sec(k_1(\text{x-m}_1\text{t})+k_2(\text{y-m}_2\text{t})+k_3(\text{z-m}_2\text{t}))}\right]^2. \end{aligned} \tag{33}$$

Case III: We have two subcases

Subcase(a) We take $\theta = k_1\zeta_1 + k_2\zeta_2 + k_3\zeta_3$, then equation of case (III) can be written in the form

$$k_1F' - p_1(k_2m_1 + k_3m_2)FF' - (k_2m_1 + k_3m_2)$$
$$(p_2(k_2m_1 + k_3m_2)^2 + p_3k_2^2 + p_4k_3^2)F''' = 0. \quad (34)$$

Substituting (29) into (34), equating to zero the coefficients of all powers of $\phi(\theta)$ yields a set of algebraic equations for a_i and b_i i.e i=-1:1. By aid Maple we solve this algebraic equations, yields

$$a_1 = b_{-1} = b_0 = 0,\ B = 0,\ a_0 \text{ is arbitrary}$$
$$k_1 = (k_2m_1 + k_3m_2)(a_0p_1 + 8AC(k_2m_1k_3m_2p_2$$
$$+ k_1^2(m_1^2p_2 + p_3) + k_3^2(m_2^2p_3 + p_4))),$$
$$a_{-1} = [\frac{-12\ A^2}{p_1}(k_2m_1k_3m_2p_2 + k_1^2(m_1^2p_2 + p_3)$$
$$+ k_3^2(m_2^2p_3 + p_4))]b_1. \quad (35)$$

We apply the related $\phi(\theta)$ functions for this choice of A, B and C.

Using the cases in Appendix A wherein A= 1, C= 1, yields

$$u(t, x, y, z) = a_0 - \frac{12}{p_1}(k_2m_1k_3m_2p_2$$
$$+ k_1^2(m_1^2p_2 + p_3) + k_3^2(m_2^2p_3 + p_4))$$
$$\frac{1}{\tan^2(k_1t + k_2(\text{y-m}_1\text{x}) + k_3(\text{z-m}_2\text{x})).} \quad (36)$$

where $\mathrm{k}_1 = (k_2m_1 + k_3m_2)(a_0p_1 + 8(k_2m_1k_3m_2p_2 + k_1^2(m_1^2p_2 + p_3) + k_3^2(m_2^2p_3 + p_4)))$.

For A= $\frac{1}{2}$, C= $\frac{-1}{2}$, we get the following solutions

$$u(t, x, y, z) = a_0 + \frac{3}{p_1}(k_2m_1k_3m_2p_2$$
$$+ k_1^2(m_1^2p_2 + p_3) + k_3^2(m_2^2p_3 + p_4))$$
$$[\frac{1 \pm \sec\text{h}(k_1t + k_2(\text{y-m}_1\text{x}) + k_3(\text{z-m}_2\text{x}))}{\tanh(k_1t + k_2(\text{y-m}_1\text{x}) + k_3(\text{z-m}_2\text{x}))}]^2 \quad (37)$$

where $\mathrm{k}_1 = (k_2m_1 + k_3m_2)(a_0p_1 - 2(k_2m_1k_3m_2p_2 + k_1^2(m_1^2p_2 + p_3) + k_3^2(m_2^2p_3 + p_4)))$

Subcase(b) Using the scaling transformation to case III

$$F = e^{\epsilon}\ \bar{F}, \quad \zeta_1 = e^{\epsilon_1}\ \bar{\zeta}_1, \quad \zeta_2 = e^{\epsilon_2}\ \bar{\zeta}_2, \quad \zeta_3 = e^{\epsilon_3}\ \bar{\zeta}_3. \quad (38)$$

Substituting from Eq.(38) into case III we have $\frac{-2}{3}\epsilon_1 = -2\epsilon_2 = -2\epsilon_3 = \epsilon$.

Then, the characteristic equation can be written as

$$\frac{-2}{3}\frac{d\zeta_1}{\zeta_1} = \frac{-2d\zeta_2}{\zeta_2} = \frac{-2d\zeta_3}{\zeta_3} = \frac{dF}{F} \quad (39)$$

We get the similarity variables

$$\eta_1 = \frac{\zeta_2}{\zeta_1^{\frac{1}{3}}}, \quad \eta_2 = \frac{\zeta_3}{\zeta_1^{\frac{1}{3}}}, \quad F = \frac{f}{\zeta_1^{\frac{2}{2}}} \tag{40}$$

By substituting into case III we get

$$\begin{aligned}
&\frac{2}{3}f + \frac{1}{3}\eta_1 f_{\eta_1} + \frac{1}{3}\eta_2 f_{\eta_2} + m_1 p_1 f f_{\eta_1} + m_2 p_1 f f_{\eta_2} \\
&+ m_1(m_1^2 p_2 + p_3) f_{\eta_1\eta_1\eta_1} + m_2(3m_1^2 p_2 + p_3) f_{\eta_1\eta_1\eta_2} \\
&+ m_1(3m_2^2 p_2 + p_4) f_{\eta_1\eta_2\eta_2} + m_2(m_2^2 p_2 + p_4) f_{\eta_2\eta_2\eta_2} = 0.
\end{aligned} \tag{41}$$

By using $\theta = k_1\eta_1 + k_2\eta_2$,(41) can be written in form

$$\begin{aligned}
&\frac{2}{3}f + \frac{1}{3}\theta f' + p_1(m_1 k_1 + m_2 k_2) f f' \\
&+ m_1 k_1^3 (m_1^2 p_2 + p_3) f''' + m_2 k_1^2 k_2 (3m_1^2 p_2 + p_3) f''' \\
&+ m_1 k_2^2 k_1 (3m_2^2 p_2 + p_4) f''' + m_2 k_2^3 (m_2^2 p_2 + p_4) f''' = 0.
\end{aligned} \tag{42}$$

To find the solution for the ODE corresponding to this case, we assume that this solution takes the following form

$$f = a_0 + a_1\theta + a_2\theta^2 + \frac{b_1}{\theta} + \frac{b_2}{\theta^2}, \tag{43}$$

where a_0, a_1, a_2, $b_1$and b_2 are arbitrary constants, Substituting from (43) into (42) and collecting the various powers of θ then equating them to zero, we get system of algebraic equations in the constants a_0, a_1, a_2, $b_1$and b_2 . Solving this system with the aid of Maple program, we get the following solutions:

$$\begin{aligned}
&a_0 = a_2 = b_1 = b_2 = 0, \\
&k_1 = \frac{-(1 + a_1 p_1 m_2 k_2)}{a_1 p_1 m_1}.
\end{aligned} \tag{44}$$

Then, we have obtained the following new exact solution for (1)

$$u(t,x,y,z) = \frac{a_2}{t}\left(k_1(\text{y-m}_1\text{x}) + k_2(\text{z-m}_2\text{x})\right). \tag{45}$$

Cases IV: We take the transformation $\theta = k_1\zeta_1 + k_2\zeta_2 + k_3\zeta_3$,we get

$$\begin{aligned}
&(k_2 m_1 + k_3 m_2)F' - p_1 k_1 F F' \\
&- 3k_1(p_2 k_1^2 + p_3 k_2^2 + p_4 k_3^2)F''' = 0.
\end{aligned} \tag{46}$$

To obtain the solution for the ODE corresponding to this case, substituting (29) into (46), equating to zero the coefficients of all powers of $\phi(\theta)$ yields a set of algebraic equations for a_0, a_i, b_i, i=1, 2. By aid

Maple we solve this algebraic equations, yields

$$a_0 = \frac{1}{12p_1k_1A^2}(\mathrm{a}_{-1}k_1p_1(B^2+8AC)+12\mathrm{b}_1A^2(m_1k_2+m_2k_3))$$
$$p_2 = -\frac{1}{12k_1A^2}(\frac{p_1\mathrm{a}_{-1}}{\mathrm{b}_1}+12A^2(p_3k_2^2+p_4k_3^2)),\ \mathrm{a}_0 = \frac{\mathrm{a}_{-1}B}{p_1}$$
$$\mathrm{b}_{-1} = \mathrm{b}_0 = 0. \tag{47}$$

Using choices for A, B and C, then we obtain the follwing exact solutions of (1)

$$u(t,x,y,z) = \frac{1}{3p_1k_1}(2\mathrm{a}_{-1}k_1p_1+3\mathrm{b}_1(m_1k_2+m_2k_3)) + \frac{\mathrm{a}_{-1}}{\mathrm{b}_1\left(\tan(k_1x+k_2(\text{y-m}_1\text{t})+k_3(\text{z-m}_2\text{t}))\pm\sec(k_1x+k_2(\text{y-m}_1\text{t})+k_3(\text{z-m}_2\text{t}))\right)^2}, \tag{48}$$

where A= $\frac{1}{2}$, B= 0, C= $\frac{1}{2}$ and $p_2 = -\frac{1}{3k_1}(\frac{p_1\mathrm{a}_{-1}}{\mathrm{b}_1}+3(p_3k_2^2+p_4k_3^2))$.

$$u(t,x,y,z) = \frac{1}{3p_1k_1}(-2\mathrm{a}_{-1}k_1p_1+3\mathrm{b}_1(m_1k_2+m_2k_3)) + \frac{\mathrm{a}_{-1}}{\mathrm{b}_1\left(\tanh(k_1x+k_2(\text{y-m}_1\text{t})+k_3(\text{z-m}_2\text{t}))\pm\mathrm{sec}\mathbf{h}(k_1x+k_2(\text{y-m}_1\text{t})+k_3(\text{z-m}_2\text{t}))\right)^2}, \tag{49}$$

where A= $\frac{1}{2}$, B= 0, C= $\frac{-1}{2}$ and $p_2 = -\frac{1}{3k_1}(\frac{p_1\mathrm{a}_{-1}}{\mathrm{b}_1}+3(p_3k_2^2+p_4k_3^2))$.

4. Conclusion

Symmetry analysis and modified Exp method were successfully used to obtain new solitary wave solutions for ZK equation. The solutions have physical structures and depend on the real parameters. finally, new type solutions of Riccati were obtained in family 1-4.

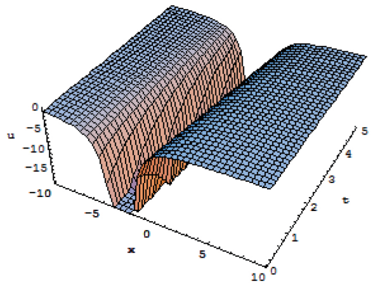

Fig. 1: Singular wave solution of (24), where $p_1=p_2=p_3=p_4=n=m_1=m_2=m_3=n_3=1, z=1$.

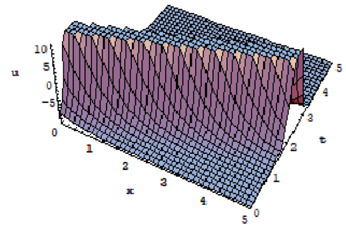

Fig. 2: Periodic solution of (31), where $p_1=p_2=p_3=p_4=m_1=m_2=m_3=n_3=1$, $k_1=k_2=k_3=1$ and $y=z=1$.

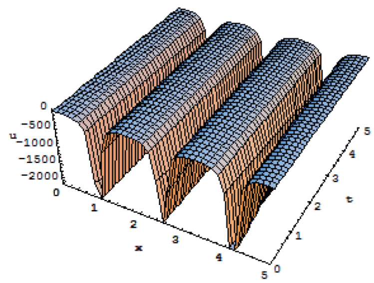

Fig. 3: Periodic solution of (36), where $p_1=p_2=p_3=p_4=m_1=m_2=m_3=n_3=1$, $a_0=k_1=k_2=k_3=1$ and $y=z=1$.

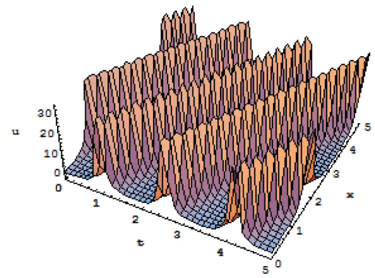

Fig. 4: Periodic solution of (48), where $p_1=p_2=p_3=p_4=m_1=m_2=m_3=n_3=1$, $k_1=k_2=k_3=1$ and $y=z=1$.

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