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GENERALIZED ULAM-HYERS STABILITY OF THE HARMONIC MEAN FUNCTIONAL EQUATION IN TWO VARIABLES

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ABSTRACT. In this paper, we find the solution and prove the generalized Ulam-Hyers stability of the harmonic mean functional equation in two variables. We also provide counterexamples for singular cases.

1. INTRODUCTION

In 1940, S.M. Ulam [47] raised the following question concerning the stability of group homomorphisms:

"Let G be a group and H be a metric group with metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G \to H$ satisfies

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $a : G \to H$ with $d(f(x), a(x)) < \epsilon$ for all $x \in G$?"

In 1941, D.H. Hyers [20] gave an answer to the Ulam's stability problem. He proved the following celebrated theorem.

Theorem 1.1. [D.H. Hyers] *Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \tag{1.1}$$

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for all $x, y \in X$. Then the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all $x \in X$ and $a : X \to Y$ is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \leq \epsilon \tag{1.3}$$

for all $x \in X$.

In 1950, Aoki [2] generalized the Hyers theorem for additive mappings. In 1978, Th.M. Rassias [43] provided a generalized version of the theorem of Hyers which permitted the cauchy difference to become unbounded. Th.M. Rassias proved the following theorem for sum of powers of norms.

Theorem 1.2. [Th.M. Rassias] *Let X and Y be two Banach spaces. Let $\theta \in [0, \infty)$ and let $p \in [0, 1)$. If a function $f : X \to Y$ satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.4}$$

for all $x, y \in X$. Then there exists a unique linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p \tag{1.5}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then the function T is linear.

The theorem of Th.M. Rassias was later extended for all $p \neq 1$. The stability phenomenon that was presented by Th.M. Rassias is called the generalized Hyers-Ulam stability. In 1982, J.M. Rassias [33] gave a further generalization of the result of D.H. Hyers and proved the following theorem using weaker conditions controlled by a product of powers of norms.

Theorem 1.3. [J.M. Rassias] *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon \|x\|^p \|y\|^p \tag{1.6}$$

for all $x, \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < \frac{1}{2}$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.7}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{2 - 2^{2p}} \|x\|^{2p} \tag{1.8}$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in t for each fixed x of X, then the function L is linear.

The above mentioned stability involving a product of powers of norms is called Ulam-Gavruta-Rassias stability by various authors ([4], [5], [6], [9], [11], [18], [19], [30], [31], [37], [38], [41], [45], [46]). Very recently, J.M. Rassias introduced mixed type product sum of powers of norms in [38]. The investigation of stability of functional equations involving with the mixed type product sum of powers of norms is known as J.M. Rassias stability. Several functional equations and its J.M. Rassias stability were investigated by many mathematicians ([5], [6], [18], [19], [39], [41], [45]).

Theorem 1.4. **[42]** *Let $(E, \perp)$ be an orthogonality normed space with norm $\|.\|_E$ and $(F, \|.\|_F)$ be a Banach space. Let $f : E \to F$ be a mapping which satisfies the inequality*

$$\begin{aligned}\left\|f(mx+y) + f(mx-y) - 2f(x+y) - 2f(x-y) - 2(m^2-2)f(x) + 2f(y)\right\|_F \\ \leq \epsilon \left\{ \|x\|_E^p \|y\|_E^p + \left(\|x\|_E^{2p} + \|y\|_E^{2p} \right) \right\}\end{aligned}$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon, p > 0$ and either $m > 1; p < 1$ or $m < 1; p > 1$ with $m \neq 0$;$m \neq \pm 1$; $m \neq \pm\sqrt{2}$ and $-1 \neq |m|^{p-1} < 1$. Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}}$$

exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler-Lagrange quadratic mapping such that

$$\|f(x) - Q(x)\|_F \leq \frac{\epsilon}{2\,|m^2 - m^{2p}|} \|x\|_E^{2p}$$

for all $x \in E$.

In the last two decades, several form of quadratic, cubic and quartic functional equations and its Hyers-Ulam-Rassias stability were discussed by various authors ([3], [7], [12], [21], [24], [27], [29], [32], [34], [35]). Recently, the mixed type of functional equations that is having additive and quadratic, quadratic and quartic, additive and cubic, additive, quadratic, cubic and quartic property were investigated in the literature ([8], [13], [14], [15], [16], [17], [18], [22], [26], [28], [36], [41], [42], [44]). Very recently, we find that the functional equations are dealt in various spaces like fuzzy normed spaces, random normed spaces, quasi-Banach spaces, quasi-normed linear spaces by various authors, one can refer to ([10], [15], [16], [17], [18], [23], [25], [26], [42], [44]). But, not many papers are available in the reciprocal functional equation (RFE) and Reciprocal difference functional equation (RDFE). K. Ravi and B.V. Senthil Kumar [39] introduced and investigated a new 2-dimensional reciprocal functional equation (RFE)

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)} \tag{1.9}$$

and obtained some interesting results on the Ulam-Gavruta-Rassias stability for the equation (1.9). In 2009, J.M. Rassias introduced the Reciprocal Difference Functional equation (RDF equation)

$$r\left(\frac{x+y}{2}\right) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \tag{1.10}$$

and the Reciprocal Adjoint Functional equation (RAF equation)

$$r\left(\frac{x+y}{2}\right) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)}. \tag{1.11}$$

and discussed Ulam stability problem as well as the extended Ulam stability problem for the equation (1.10) and (1.11) in [40].

In this paper, we obtain the solution and generalized Ulam-Hyers stability of the harmonic mean functional equation in two variables of the form

$$H\Big(\frac{xu}{x+u}, \frac{yv}{y+v}\Big) = \frac{H(x,y)H(u,v)}{H(x,y)+H(u,v)} \tag{1.12}$$

which is originating from the harmonic mean of two positive real numbers x and y. The function $H(x,y) = \frac{2xy}{x+y}$ is a solution of the functional equation (1.12). The above function $H(x,y)$ represents the harmonic mean of x and y. We also provide counterexamples for singular cases.

Before we proceed to the main theorem, we present some definitions, which will be useful to prove our theorems.

Definition 1.5. The harmonic mean of two positive real numbers x and y is the reciprocal of arithmetic mean of the reciprocals of x and y. Thus, harmonic mean of x and y is

$$\text{Harmonic mean} = \frac{2}{\frac{1}{x}+\frac{1}{y}} = \frac{2xy}{x+y}.$$

Definition 1.6. A mapping $a : X \to Y$ between Banach spaces is called additive if a satisfies the functional equation

$$a(x+y) = a(x) + a(y). \tag{1.13}$$

The function $a(x) = cx$ is a solution of the functional equation (1.13).

Definition 1.7. A function $r : X \to Y$ between sets of non-zero real numbers is called reciprocal if r satisfies the functional equation (1.9). The function $r(x) = \frac{c}{x}$ is a solution of the functional equation (1.9).

Definition 1.8. Let X be set of positive real numbers. A function $H : X \times X \to \mathbb{R}$ be such that $H(x,y) = \frac{2xy}{x+y}$ is called harmonic mean function if it satisfies the functional equation (1.12). The functional equation (1.12) is called harmonic mean functional equation.

Throughout this paper, let X be set of positive real numbers.

2. RELATION BETWEEN (1.9) AND (1.12)

Theorem 2.1. *Let $r : X \to \mathbb{R}$ be a mapping satisfying $r(x) = \frac{c}{x}$ where $c \in \mathbb{R} - \{0\}$. If $H : X \times X \to \mathbb{R}$ is a mapping given by*

$$H(x,y) = \frac{2}{r(x)+r(y)} \tag{2.1}$$

for all $x, y \in X$, then H satisfies (1.12).

Proof. From (2.1), we have

$$\begin{aligned} H\Big(\frac{xu}{x+u}, \frac{yv}{y+v}\Big) &= \frac{2}{r\Big(\frac{xu}{x+u}\Big) + r\Big(\frac{yv}{y+v}\Big)} \\ &= \frac{2}{c\Big[\frac{x+u}{xu}\Big] + c\Big[\frac{y+v}{yv}\Big]} \\ &= \frac{2/c}{\frac{1}{x}+\frac{1}{u}+\frac{1}{y}+\frac{1}{v}} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \frac{H(x,y)H(u,v)}{H(x,y)+H(u,v)} &= \frac{\frac{2}{r(x)+r(y)}\,\frac{2}{r(u)+r(v)}}{\frac{2}{r(x)+r(y)}+\frac{2}{r(u)+r(v)}} \\ &= \frac{2}{r(x)+r(u)+r(y)+r(v)} \\ &= \frac{2/c}{\frac{1}{x}+\frac{1}{u}+\frac{1}{y}+\frac{1}{v}}. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we see that H satisfies (1.12). □

3. SOLUTION OF FUNCTIONAL EQUATION (1.12)

Theorem 3.1. *A mapping $H : X \times X \to \mathbb{R}$ satisfies (1.12) if and only if there exists an additive mapping $A : X \to \mathbb{R}$ such that*

$$H(x,y) = \frac{2cA(x)A(y)}{A(x)+A(y)}$$

for all $x, y \in X$, where $c \in \mathbb{R} - \{0\}$.

Proof. Let $H(x,y)$ be a solution of (1.12). Then by Theorem 2.1, H will be of the form (2.1). That is, $H(x,y) = \frac{2}{r(x)+r(y)}$, for all $x,y \in X$. In particular, $H(x,x) = \frac{2}{r(x)+r(x)} = \frac{x}{c}$, for all $x \in X$. Now, we define a mapping $A : X \to \mathbb{R}$ be such that $A(x) = H(x,x) = \frac{x}{c}$, for all $x \in X$. Then, clearly A is an additive mapping. Further,

$$\begin{aligned} \frac{2cA(x)A(y)}{A(x)+A(y)} &= \frac{2c\frac{x}{c}\frac{y}{c}}{\frac{x}{c}+\frac{y}{c}} \\ &= \frac{2xy}{x+y} \\ &= H(x,y), \text{ for all } x,y \in X. \end{aligned}$$

Conversely, if there exists an additive mapping $A : X \to \mathbb{R}$ such that $H(x,y) = \frac{2cA(x)A(y)}{A(x)+A(y)}$, for all $x, y \in X$, then

$$H\Big(\frac{xu}{x+u}, \frac{yv}{y+v}\Big) = \frac{2cA\Big(\frac{xu}{x+u}\Big)A\Big(\frac{yv}{y+v}\Big)}{A\Big(\frac{xu}{x+u}\Big) + A\Big(\frac{yv}{y+v}\Big)}$$

$$\begin{aligned}
&= \frac{2\frac{xy}{x+y}\frac{uv}{u+v}}{\frac{xy}{x+y} + \frac{uv}{u+v}} \\
&= \frac{2\Big(c\frac{\frac{x}{c}\frac{y}{c}}{\frac{x}{c}+\frac{y}{c}}\Big)\Big(c\frac{\frac{u}{c}\frac{v}{c}}{\frac{u}{c}+\frac{v}{c}}\Big)}{\Big(c\frac{\frac{x}{c}\frac{y}{c}}{\frac{x}{c}+\frac{y}{c}}\Big) + \Big(c\frac{\frac{u}{c}\frac{v}{c}}{\frac{u}{c}+\frac{v}{c}}\Big)} \\
&= \frac{\Big[\frac{2cA(x)A(y)}{A(x)+A(y)}\Big]\Big[\frac{2cA(u)A(v)}{A(u)+A(v)}\Big]}{\Big[\frac{2cA(x)A(y)}{A(x)+A(y)}\Big] + \Big[\frac{2cA(u)A(v)}{A(u)+A(v)}\Big]} \\
&= \frac{H(x,y)H(u,v)}{H(x,y)+H(u,v)}
\end{aligned}$$

for all $x, u, y, v \in X$. □

4. GENERALIZED ULAM-HYERS STABILITY OF FUNCTIONAL EQUATION (1.12)

Theorem 4.1. *Let $H : X \times X \to \mathbb{R}$ be a mapping for which there exists a function $\varphi : X \times X \times X \times X \to \mathbb{R}$ with the condition*

$$\lim_{n\to\infty} 2^n \varphi(2^{-n}x, 2^{-n}x, 2^{-n}y, 2^{-n}y) = 0 \tag{4.1}$$

such that the functional inequality

$$\Big| H\Big(\frac{xu}{x+u}, \frac{yv}{y+v}\Big) - \frac{H(x,y)H(u,v)}{H(x,y)+H(u,v)} \Big| \le \varphi(x,u,y,v) \tag{4.2}$$

holds for all $x, u, y, v \in X$. Then there exists a unique harmonic mean mapping $M : X \times X \to \mathbb{R}$ satisfying the functional equation (1.12) and

$$|M(x,y) - H(x,y)| \le \sum_{i=0}^{\infty} 2^{i+1}\varphi(2^{-i}x, 2^{-i}x, 2^{-i}y, 2^{-i}y) \tag{4.3}$$

for all $x, y \in X$. The mapping $M(x,y)$ is defined by

$$M(x,y) = \lim_{n\to\infty} 2^n H(2^{-n}x, 2^{-n}y)$$

for all $x, y \in X$.

Proof. Replacing (x,u,y,v) by (x,x,y,y) in (4.2) and multiplying by 2, we obtain

$$\left|2H\left(\frac{x}{2},\frac{y}{2}\right)-H(x,y)\right|\leq 2\varphi(x,x,y,y). \tag{4.4}$$

Now, replacing (x,y) by $(\frac{x}{2},\frac{y}{2})$ in (4.4), multiplying by 2 and summing the resulting inequality with (4.4), we arrive

$$|2^2H\left(2^{-2}x,2^{-2}y\right)-H(x,y)|\leq 2\sum_{i=0}^{1}2^i\varphi(2^{-i}x,2^{-i}x,2^{-i}y,2^{-i}y).$$

Using induction on a positive integer n, we get

$$\begin{aligned}|2^nH\left(2^{-n}x,2^{-n}y\right)-H(x,y)| &\leq 2\sum_{i=0}^{n-1}2^i\varphi(2^{-i}x,2^{-i}x,2^{-i}y,2^{-i}y)\\ &\leq 2\sum_{i=0}^{\infty}2^i\varphi(2^{-i}x,2^{-i}x,2^{-i}y,2^{-i}y) \qquad (4.5)\end{aligned}$$

for all $x,y\in X$. In order to prove the convergence of the sequence $\{2^nH(2^{-n}x,2^{-n}y)\}$, replacing (x,y) by $(2^{-p}x,2^{-p}y)$ in (4.5) and multiplying by 2^p, we find that for $n>p>0$

$$\begin{aligned}&|2^{n+p}H(2^{-n-p}x,2^{-n-p}y)-2^pH(2^{-p}x,2^{-p}y)|\\ &\qquad = 2^p|2^nH(2^{-n-p}x,2^{-n-p}y)-H(2^{-p}x,2^{-p}y)|\\ &\qquad \leq 2\sum_{i=0}^{\infty}2^{p+i}\varphi(2^{-p-i}x,2^{-p-i}x,2^{-p-i}y,2^{-p-i}y)\\ &\qquad \to 0 \text{ as } p\to\infty.\end{aligned}$$

This shows that the sequence $\{2^nH(2^{-n}x,2^{-n}y)\}$ is a Cauchy sequence in $(\mathbb{R},|.|)$. Since $(\mathbb{R},|.|)$ is complete, the sequence $\{2^nH(2^{-n}x,2^{-n}y)\}$ converges for all $x,y\in X$. Thus, we can define a function $M:X\times X\to\mathbb{R}$ by $M(x,y)=\lim\limits_{n\to\infty}2^nH(2^{-n}x,2^{-n}y)$. Allow $n\to\infty$ in (4.5), we arrive (4.3). To show that M satisfies (1.12), replacing (x,u,y,v) by $(2^{-n}x,2^{-n}u,2^{-n}y,2^{-n}v)$ in (4.2) and multiplying by 2^n, we obtain

$$\begin{aligned}&2^n\left|H\left(\frac{2^{-n}x2^{-n}u}{2^{-n}x+2^{-n}u},\frac{2^{-n}y2^{-n}v}{2^{-n}y+2^{-n}v}\right)-\frac{H(2^{-n}x,2^{-n}y)H(2^{-n}u,2^{-n}v)}{H(2^{-n}x,2^{-n}y)+H(2^{-n}u,2^{-n}v)}\right|\\ &\quad =2^n\left|H\left(2^{-n}\left(\frac{xu}{x+u}\right),2^{-n}\left(\frac{yv}{y+v}\right)\right)-\frac{H(2^{-n}x,2^{-n}y)H(2^{-n}u,2^{-n}v)}{H(2^{-n}x,2^{-n}y)+H(2^{-n}u,2^{-n}v)}\right|\\ &\quad \leq 2^n\varphi(2^{-n}x,2^{-n}u,2^{-n}y,2^{-n}v). \qquad (4.6)\end{aligned}$$

Allow $n \to \infty$ in (4.6), we see that M satisfies (1.12) for all $x, u, y, v \in X$. To prove M is unique mapping satisfying (1.12), let $N : X \times X \to \mathbb{R}$ be another mapping which satisfies (1.12) and the inequality (4.3). Clearly N and M satisfy (1.12) and using (4.3), we arrive

$$\begin{aligned}
&|N(x,y) - M(x,y)| \\
&\quad = 2^n |N(2^{-n}x, 2^{-n}y) - M(2^{-n}x, 2^{-n}y)| \\
&\quad \leq 2^n \Big(|N(2^{-n}x, 2^{-n}y) - H(2^{-n}x, 2^{-n}y)| + |H(2^{-n}x, 2^{-n}y) - M(2^{-n}x, 2^{-n}y)| \Big) \\
&\quad \leq 4 \sum_{i=0}^{\infty} 2^{n+i} \varphi(2^{-n-i}x, 2^{-n-i}x, 2^{-n-i}y, 2^{-n-i}y)
\end{aligned} \tag{4.7}$$

for all $x, y \in X$. Allow $n \to \infty$ in (4.7) and using (4.1), we find that M is unique. This completes the proof of Theorem 4.1. □

Theorem 4.2. *Let $H : X \times X \to \mathbb{R}$ be a mapping satisfying (4.2) for all $x, u, y, v \in X$. Then there exists a unique harmonic mapping $M : X \times X \to \mathbb{R}$ satisfying the functional equation (1.12) and*

$$|H(x,y) - M(x,y)| \leq \sum_{i=0}^{\infty} 2^{-i} \varphi(2^{i+1}x, 2^{i+1}x, 2^{i+1}y, 2^{i+1}y) \tag{4.8}$$

for all $x, y \in X$. The mapping $M(x,y)$ is defined by

$$M(x,y) = \lim_{n \to \infty} 2^{-n} H(2^n x, 2^n y)$$

for all $x, y \in X$, where $\varphi : X \times X \times X \times X \to \mathbb{R}$ is a function.

Proof. The proof is obtained by replacing (x, u, y, v) by (x, x, y, y) in (4.2) and proceeding further by similar arguments as in Theorem 4.1. □

Corollary 4.3. *Let $k_1 \geq 0$ be fixed and $a > 1$ or $a < 1$. If a mapping $H : X \times X \to \mathbb{R}$ satisfies the inequality*

$$\left| H\Big(\frac{xu}{x+u}, \frac{yv}{y+v}\Big) - \frac{H(x,y)H(u,v)}{H(x,y) + H(u,v)} \right| \leq k_1 \Big(x^a + u^a + y^a + v^a \Big)$$

for all $x, u, y, v \in X$, then there exists a unique harmonic mean mapping $M : X \times X \to \mathbb{R}$ satisfying the functional equation (1.12) and

$$|M(x,y) - H(x,y)| \leq \begin{cases} 4k_1 \Big(\frac{2^a}{2^a - 2}\Big) \Big(x^a + y^a\Big) & \text{for } a > 1 \\ 4k_1 \Big(\frac{2^a}{2 - 2^a}\Big) \Big(x^a + y^a\Big) & \text{for } a < 1 \end{cases}$$

for all $x, y \in X$.

Proof. Choosing $\varphi(x,u,y,v) = k_1\Big(x^a + u^a + y^a + v^a\Big)$, for all $x,u,y,v \in X$ in Theorem 4.1, we arrive

$$|M(x,y) - H(x,y)| \leq 4k_1\Big(\frac{2^a}{2^a-2}\Big)\Big(x^a + y^a\Big), \text{ for all } x,y \in X \text{ and } a > 1$$

and using Theorem 4.2, we arrive

$$|M(x,y) - H(x,y)| \leq 4k_1\Big(\frac{2^a}{2-2^a}\Big)\Big(x^a + y^a\Big), \text{ for all } x,y \in X \text{ and } a < 1.$$

□

Corollary 4.4. *Let $H : X \times X \to \mathbb{R}$ be a mapping and there exist real numbers $p, q : \sigma = p + q > \frac{1}{2}$ or $p, q : \sigma = p + q < \frac{1}{2}$. If there exists k_2 such that*

$$\Big|H\Big(\frac{xu}{x+u}, \frac{yv}{y+v}\Big) - \frac{H(x,y)H(u,v)}{H(x,y)+H(u,v)}\Big| \leq k_2 x^p u^p y^q v^q$$

for all $x,u,y,v \in X$, then there exists a unique harmonic mean mapping $M : X \times X \to \mathbb{R}$ satisfying the functional equation (1.12) and

$$|M(x,y) - H(x,y)| \leq \begin{cases} 2k_2\Big(\frac{2^{2\sigma}}{2^{2\sigma}-2}\Big)x^{2p}y^{2q} & \text{for } \sigma > \frac{1}{2} \\ 2k_2\Big(\frac{2^{2\sigma}}{2-2^{2\sigma}}\Big)x^{2p}y^{2q} & \text{for } \sigma < \frac{1}{2} \end{cases}$$

for all $x, y \in X$.

Proof. Taking $\varphi(x,u,y,v) = k_2 x^p u^p y^q v^q$, for all $x,u,y,v \in X$ in Theorem 4.1, we arrive

$$|M(x,y) - H(x,y)| \leq 2k_2\Big(\frac{2^{2\sigma}}{2^{2\sigma}-2}\Big)x^{2p}y^{2q}, \quad \text{for all } x,y \in X \text{ and } \sigma > \frac{1}{2}$$

and using Theorem 4.2, we arrive

$$|M(x,y) - H(x,y)| \leq 2k_2\Big(\frac{2^{2\sigma}}{2-2^{2\sigma}}\Big)x^{2p}y^{2q}, \quad \text{for all } x,y \in X \text{ and } \sigma < \frac{1}{2}.$$

□

Corollary 4.5. *Let $k_3 > 0$ and $r > \frac{1}{4}$ or $r < \frac{1}{4}$ be real numbers, and $H : X \times X \to \mathbb{R}$ be a mapping satisfying the functional inequality*

$$\Big|H\Big(\frac{xu}{x+u}, \frac{yv}{y+v}\Big) - \frac{H(x,y)H(u,v)}{H(x,y)+H(u,v)}\Big|$$
$$\leq k_3\Big(x^r u^r y^r v^r + \Big(x^{4r} + u^{4r} + y^{4r} + v^{4r}\Big)\Big)$$

for all $x, u, y, v \in X$. Then there exists a unique harmonic mean mapping $M : X \times X \to \mathbb{R}$ satisfying the functional equation (1.12) and

$$|M(x,y) - H(x,y)| \leq \begin{cases} 2k_3\Big(\frac{2^{4r}}{2^{4r}-2}\Big)\Big[x^{2r}y^{2r} + 2\Big(x^{4r}+y^{4r}\Big)\Big] & \text{for } r > \frac{1}{4} \\ 2k_3\Big(\frac{2^{4r}}{2-2^{4r}}\Big)\Big[x^{2r}y^{2r} + 2\Big(x^{4r}+y^{4r}\Big)\Big] & \text{for } r < \frac{1}{4} \end{cases}$$

for all $x, y \in X$.

Proof. Choosing $\varphi(x,u,y,v) = k_3\Big(x^r u^r y^r v^r + \Big(x^{4r} + u^{4r} + y^{4r} + v^{4r}\Big)\Big)$, for all $x, u, y, v \in X$ in Theorem 4.1, we arrive

$$|M(x,y) - H(x,y)|$$
$$\leq 2k_3\Big(\frac{2^{4r}}{2^{4r}-2}\Big)\Big[x^{2r}y^{2r} + 2\Big(x^{4r}+y^{4r}\Big)\Big] \text{ for all } x, y \in X \text{ and } r > \frac{1}{4}$$

and using Theorem 4.2, we arrive

$$|M(x,y) - H(x,y)|$$
$$\leq 2k_3\Big(\frac{2^{4r}}{2-2^{4r}}\Big)\Big[x^{2r}y^{2r} + 2\Big(x^{4r}+y^{4r}\Big)\Big] \text{ for all } x, y \in X \text{ and } r < \frac{1}{4}.$$

□

Corollary 4.6. *Let $k_4 > 0$ and $p > \frac{1}{2}$ or $p < \frac{1}{2}$ be real numbers, and $H : X \times X \to \mathbb{R}$ be a mapping satisfying the functional inequality*

$$\left| H\left(\frac{xu}{x+u}, \frac{yv}{y+v}\right) - \frac{H(x,y)H(u,v)}{H(x,y)+H(u,v)} \right| \leq k_4\left(\frac{x^p u^p y^p v^p}{x^p u^p + y^p v^p}\right)$$

for all $x, u, y, v \in X$. Then there exists a unique harmonic mean mapping $M : X \times X \to \mathbb{R}$ satisfying the functional equation (1.12) and

$$|M(x,y) - H(x,y)| \leq \begin{cases} 2k_4\left(\frac{2^{2p}}{2^{2p}-2}\right)\left(\frac{x^{2p}y^{2p}}{x^{2p}+y^{2p}}\right) & \text{for } p > \frac{1}{2} \\ 2k_4\left(\frac{2^{2p}}{2-2^{2p}}\right)\left(\frac{x^{2p}y^{2p}}{x|^{2p}+y^{2p}}\right) & \text{for } p < \frac{1}{2} \end{cases}$$

for all $x, y \in X$.

Proof. Choosing $\varphi(x,u,y,v) = k_4\left(\frac{x^p u^p y^p v^p}{x^p u^p + y^p v^p}\right)$, for all $x, u, y, v \in X$ in Theorem 4.1, we arrive

$$|M(x,y) - H(x,y)| \leq 2k_4\left(\frac{2^{2p}}{2^{2p}-2}\right)\left(\frac{x^{2p}y^{2p}}{x^{2p}+y^{2p}}\right) \text{ for all } x, y \in X \text{ and } p > \frac{1}{2}$$

and using Theorem 4.2, we arrive

$$|M(x,y) - H(x,y)| \leq 2k_4\left(\frac{2^{2p}}{2-2^{2p}}\right)\left(\frac{x^{2p}y^{2p}}{x^{2p}+y^{2p}}\right) \text{ for all } x, y \in X \text{ and } p < \frac{1}{2}.$$

$\square$

5. COUNTER-EXAMPLES

Now, we give an example to show that the functional equation (1.12) is not stable for $p = \frac{1}{2}$ in Corollary 4.6.

Let us define a function $H : X \times X \to \mathbb{R}$ be

$$H(x, y) = \sum_{n=0}^{\infty} \frac{\phi(2^n x, 2^n y)}{2^n},$$

for all $x, y \in X$ where the function $\phi : X \times X \to \mathbb{R}$ is given by

$$\phi(x, y) = \begin{cases} \frac{\alpha x y}{x+y} & \text{for } x, y \in (0, 1) \\ \alpha & \text{otherwise} \end{cases}$$

with $\alpha > 0$ is a constant. Then the function H satisfies the inequality

$$\left| H\left(\frac{xu}{x+u}, \frac{yv}{y+v}\right) - \frac{H(x, y)H(u, v)}{H(x, y) + H(u, v)} \right| \leq 6\alpha \left(\frac{x^{\frac{1}{2}} u^{\frac{1}{2}} y^{\frac{1}{2}} v^{\frac{1}{2}}}{x^{\frac{1}{2}} u^{\frac{1}{2}} + y^{\frac{1}{2}} v^{\frac{1}{2}}} \right) \tag{5.1}$$

for all $x, u, y, v \in X$. Then there does not exist a harmonic mapping $M : X \times X \to \mathbb{R}$ and a constant $\lambda > 0$ such that

$$|H(x, y) - M(x, y)| \leq \lambda \left(\frac{xy}{x+y} \right) \tag{5.2}$$

for all $x, y \in X$.

Proof. $|H(x, y)| \leq \sum_{n=0}^{\infty} \frac{|\phi(2^n x, 2^n y)|}{|2^n|} \leq \sum_{n=0}^{\infty} \frac{\alpha}{2^n} = \alpha \left(1 - \frac{1}{2}\right)^{-1} = 2\alpha$. Hence H is bounded by 2α. If $\frac{x^{\frac{1}{2}} u^{\frac{1}{2}} y^{\frac{1}{2}} v^{\frac{1}{2}}}{x^{\frac{1}{2}} u^{\frac{1}{2}} + y^{\frac{1}{2}} v^{\frac{1}{2}}} \geq 1$, then the left hand side of the inequality (5.1) is less than 6α. Now, consider the case: $0 < \frac{x^{\frac{1}{2}} u^{\frac{1}{2}} y^{\frac{1}{2}} v^{\frac{1}{2}}}{x^{\frac{1}{2}} u^{\frac{1}{2}} + y^{\frac{1}{2}} v^{\frac{1}{2}}} < 1$. Then there exists a $k \in \mathbb{N}$ such that

$$\frac{1}{2^{k+1}} < \frac{x^{\frac{1}{2}} u^{\frac{1}{2}} y^{\frac{1}{2}} v^{\frac{1}{2}}}{x^{\frac{1}{2}} u^{\frac{1}{2}} + y^{\frac{1}{2}} v^{\frac{1}{2}}} < \frac{1}{2^k}. \tag{5.3}$$

Hence $\frac{x^{\frac{1}{2}} u^{\frac{1}{2}} y^{\frac{1}{2}} v^{\frac{1}{2}}}{x^{\frac{1}{2}} u^{\frac{1}{2}} + y^{\frac{1}{2}} v^{\frac{1}{2}}} < \frac{1}{2^k}$ implies

$$\frac{2^k x^{\frac{1}{2}} u^{\frac{1}{2}} y^{\frac{1}{2}} v^{\frac{1}{2}}}{x^{\frac{1}{2}} u^{\frac{1}{2}} + y^{\frac{1}{2}} v^{\frac{1}{2}}} < 1$$

or

$$\frac{2^{\frac{k}{2}} x^{\frac{1}{2}} 2^{\frac{k}{2}} u^{\frac{1}{2}} 2^{\frac{k}{2}} y^{\frac{1}{2}} 2^{\frac{k}{2}} v^{\frac{1}{2}}}{2^{\frac{k}{2}} x^{\frac{1}{2}} 2^{\frac{k}{2}} u^{\frac{1}{2}} + 2^{\frac{k}{2}} y^{\frac{1}{2}} 2^{\frac{k}{2}} v^{\frac{1}{2}}} < 1$$

or

$$2^{\frac{k}{2}} x^{\frac{1}{2}} 2^{\frac{k}{2}} u^{\frac{1}{2}} 2^{\frac{k}{2}} y^{\frac{1}{2}} 2^{\frac{k}{2}} v^{\frac{1}{2}} < 2^{\frac{k}{2}} x^{\frac{1}{2}} 2^{\frac{k}{2}} u^{\frac{1}{2}} + 2^{\frac{k}{2}} y^{\frac{1}{2}} 2^{\frac{k}{2}} v^{\frac{1}{2}}$$

or
$$1 < \frac{2^{\frac{k}{2}}x^{\frac{1}{2}}2^{\frac{k}{2}}u^{\frac{1}{2}}+2^{\frac{k}{2}}y^{\frac{1}{2}}2^{\frac{k}{2}}v^{\frac{1}{2}}}{2^{\frac{k}{2}}x^{\frac{1}{2}}2^{\frac{k}{2}}u^{\frac{1}{2}}2^{\frac{k}{2}}y^{\frac{1}{2}}2^{\frac{k}{2}}v^{\frac{1}{2}}}$$
or
$$1 < \frac{1}{2^{\frac{k}{2}}x^{\frac{1}{2}}}\frac{1}{2^{\frac{k}{2}}u^{\frac{1}{2}}} + \frac{1}{2^{\frac{k}{2}}y^{\frac{1}{2}}}\frac{1}{2^{\frac{k}{2}}v^{\frac{1}{2}}}.$$
Therefore,
$$\frac{1}{2^{\frac{k}{2}}x^{\frac{1}{2}}}\frac{1}{2^{\frac{k}{2}}u^{\frac{1}{2}}} + \frac{1}{2^{\frac{k}{2}}y^{\frac{1}{2}}}\frac{1}{2^{\frac{k}{2}}v^{\frac{1}{2}}} > 1.$$
Hence
$$\frac{1}{2^{\frac{k}{2}}x^{\frac{1}{2}}} > 1, \frac{1}{2^{\frac{k}{2}}u^{\frac{1}{2}}} > 1, \frac{1}{2^{\frac{k}{2}}y^{\frac{1}{2}}} > 1, \frac{1}{2^{\frac{k}{2}}v^{\frac{1}{2}}} > 1$$
which implies
$$2^{\frac{k}{2}}x^{\frac{1}{2}} < 1, 2^{\frac{k}{2}}u^{\frac{1}{2}} < 1, 2^{\frac{k}{2}}y^{\frac{1}{2}} < 1, 2^{\frac{k}{2}}v^{\frac{1}{2}} < 1$$
so that
$$2^{\frac{k-1}{2}}x^{\frac{1}{2}} < \frac{1}{\sqrt{2}}, 2^{\frac{k-1}{2}}u^{\frac{1}{2}} < \frac{1}{\sqrt{2}}, 2^{\frac{k-1}{2}}y^{\frac{1}{2}} < \frac{1}{\sqrt{2}}, 2^{\frac{k-1}{2}}v^{\frac{1}{2}} < \frac{1}{\sqrt{2}},$$
$$2^{\frac{k-1}{2}}\left(\frac{xu}{x+u}\right)^{\frac{1}{2}} < \frac{1}{\sqrt{2}}, 2^{\frac{k-1}{2}}\left(\frac{yv}{y+v}\right)^{\frac{1}{2}} < \frac{1}{\sqrt{2}}$$
and consequently
$$2^{k-1}(x), 2^{k-1}(u), 2^{k-1}(y), 2^{k-1}(v), 2^{k-1}\left(\frac{xu}{x+u}\right), 2^{k-1}\left(\frac{yv}{y+v}\right) \in (0,1).$$
Therefore, for each $n = 0, 1, 2, \ldots, k-1$, we have
$$2^{n}(x), 2^{n}(u), 2^{n}(y), 2^{n}(v), 2^{n}\left(\frac{xu}{x+u}\right), 2^{n}\left(\frac{yv}{y+v}\right) \in (0,1)$$
and
$$\phi\left(2^n\left(\frac{xu}{x+u}\right), 2^n\left(\frac{yv}{y+v}\right)\right) - \frac{\phi\left(2^n(x), 2^n(y)\right)\phi\left(2^n(u), 2^n(v)\right)}{\phi\left(2^n(x), 2^n(y)\right) + \phi\left(2^n(u), 2^n(v)\right)} = 0$$
for $n = 0, 1, 2, \ldots, k-1$. Using (5.3) and definition of H, we obtain
$$\begin{aligned}
&\left|H\left(\frac{xu}{x+u}, \frac{yv}{y+v}\right) - \frac{H(x,y)H(u,v)}{H(x,y)+H(u,v)}\right| \\
&\quad \leq \sum_{n=0}^{\infty}\frac{1}{2^n}\left|\phi\left(2^n\left(\frac{xu}{x+u}\right), 2^n\left(\frac{yv}{y+v}\right)\right) - \frac{\phi\left(2^n(x), 2^n(y)\right)\phi\left(2^n(u), 2^n(v)\right)}{\phi\left(2^n(x), 2^n(y)\right) + \phi\left(2^n(u), 2^n(v)\right)}\right| \\
&\quad \leq \sum_{n=k}^{\infty}\frac{1}{2^n}\left|\phi\left(2^n\left(\frac{xu}{x+u}\right), 2^n\left(\frac{yv}{y+v}\right)\right) - \frac{\phi\left(2^n(x), 2^n(y)\right)\phi\left(2^n(u), 2^n(v)\right)}{\phi\left(2^n(x), 2^n(y)\right) + \phi\left(2^n(u), 2^n(v)\right)}\right| \\
&\quad \leq \sum_{n=k}^{\infty}\frac{1}{2^n}\frac{3}{2}\alpha \\
&\quad \leq \frac{6\alpha}{2^{k+1}} \leq 6\alpha\left(\frac{x^{\frac{1}{2}}u^{\frac{1}{2}}y^{\frac{1}{2}}v^{\frac{1}{2}}}{x^{\frac{1}{2}}u^{\frac{1}{2}} + y^{\frac{1}{2}}v^{\frac{1}{2}}}\right),
\end{aligned}$$

that is the inequality (5.1) holds true.

We claim that the harmonic mean functional equation (1.12) is not stable for $p = \frac{1}{2}$ in Corollary 4.6.

Assume that there exists a harmonic mean mapping $M : X \times X \to \mathbb{R}$ satisfying (5.2). Therefore, we have

$$|H(x,y)| \leq (\lambda + 2)\left(\frac{xy}{x+y}\right). \tag{5.4}$$

But we can choose a positive integer m with $m\alpha > \lambda + 2$. If $x \in \left(0, 2^{1-m}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, 2, \ldots, m-1$. For this x, we get

$$\begin{aligned} H(x,y) &= \sum_{n=0}^{\infty} \frac{\phi(2^n x, 2^n y)}{2^n} \\ &\geq \sum_{n=0}^{m-1} \frac{\alpha\left(\frac{2^n xy}{xy}\right)}{2^n} = m\alpha\left(\frac{xy}{x+y}\right) > (\lambda+2)\left(\frac{xy}{x+y}\right) \end{aligned}$$

which in comparison with (5.4) is a contradiction. Therefore, the harmonic mean functional equation (1.12) is not stable if $p = \frac{1}{2}$ in Corollary 4.6. □

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