International Journal of Analysis and Applications
Volume 18, Number 1 (2020), 33-49
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-18-2020-33

AN EXTENDED S-ITERATION SCHEME FOR G-CONTRACTIVE TYPE MAPPINGS IN b-METRIC SPACES WITH GRAPH

NILAKSHI GOSWAMI[1], NEHJAMANG HAOKIP[1], VISHNU NARAYAN MISHRA[2,*]

[1] *Department of Mathematics, Gauhati University, Guwahati - 781014, India*

[2] *Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur 484 887, India*

[*] *Corresponding author:* vishnunarayanmishra@gmail.com

Abstract. In this paper, we introduce an extended S-iteration scheme for G-contractive type mappings and prove Δ-convergence as well as strong convergence in a nonempty closed and convex subset of a uniformly convex and complete b-metric space with a directed graph. We also give a numerical example in support of our result and compare the convergence rate between the studied iteration and the modified S-iteration.

1. Introduction

In 1922, Banach gave the proof of a fixed point result, which later on came to be known as the celebrated *Banach contraction principle*. He showed that a contraction mapping T on a complete metric space (X, d) has a unique fixed point. Moreover, for an arbitrary point x_0 in X, the sequence of Picard iterates given by the relation

$$x_n = Tx_{n-1} \qquad n = 1, 2, 3, \dots \tag{1.1}$$

converges to the unique fixed point. In the last few decades, many authors have extended this result by considering a more generalized space, altering the condition of the contraction or by considering different

Received 2019-05-01; accepted 2019-09-30; published 2020-01-02.

2010 *Mathematics Subject Classification.* Primary 47H10, Secondary 54E50.

Key words and phrases. b-metric space; extended S-iteration scheme; G-contractive type mapping; Δ-convergence; strong convergence.

iteration processes (one may refer to [6]– [8], [9], [10], [13], [14], [19], [21]– [24], [26], [27], [28], [34], [29]– [33] and the references therein).

The increasing interest in the study of iteration schemes is accelerated by the advancement in computational mathematics aided by computer programming. We list some of the prominent iteration schemes which are generalizations of (1.1).

For $x_0 \in X$, the iteration scheme given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \qquad n = 0, 1, 2, \ldots$$

where $\{\alpha_n\} \subset [0, 1]$ is called the *Mann iteration scheme* (refer to [20]).

For $x_0 \in X$, the iteration scheme given by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \qquad n = 0, 1, 2, \ldots \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ is called the *Ishikawa iteration scheme* (refer to [15]).

In 1976, Jungck [16] proved a common fixed point theorem for a pair of mappings S and T satisfying $d(Tx, Ty) \leq \alpha d(Sx, Sy)$ with $\alpha \in (0, 1)$ and $T(X) \subset S(X)$ in a complete metric space. For x_0 in a linear space X, the sequence $\{Sx_n\}$ defined by

$$Sx_{n+1} = \alpha_n Sx_n + (1 - \alpha_n)Tx_n, \qquad n = 0, 1, 2, \ldots$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ is called the *Jungck-Mann iteration scheme* (refer to [39]). If $\alpha_n = 0$, we get the *Jungck iteration scheme*.

For $x_0 \in X$, the sequence $\{Sx_n\}$ defined by

$$\begin{aligned} Sx_{n+1} &= (1 - \alpha_n)Sx_n + \alpha_n T y_n \\ Sy_n &= (1 - \beta_n)Sx_n + \beta_n T x_n, \qquad n = 0, 1, 2, \ldots \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ is called the *Jungck-Ishikawa iteration scheme* (refer to [25]).

In 2007, Agarwal et al. [2] introduced the *S-iteration scheme* and studied its convergence. For a convex subset K of a linear space X and a self mapping T on K, the iterative sequence $\{x_n\}$ of the S-iteration scheme is generated from $x_1 \in K$, and is defined by

$$\begin{aligned} x_{n+1} &= (1 - \lambda_n)Tx_n + \lambda_n T y_n, \\ y_n &= (1 - \mu_n)x_n + \mu_n T x_n, \qquad n \in \mathbb{N} \end{aligned}$$

where $\{\lambda_n\}$ and $\{\mu_n\}$ are real sequences in $(0, 1)$, satisfying

$$\sum_{n=1}^{\infty} \lambda_n \mu_n (1 - \mu_n) = \infty.$$

Recently, Suparatulatorn et al. [40] introduced the *modified S-iteration scheme* for G-nonexpansive mappings S_1 and S_2 in Banach spaces with graphs. Here, for $x_0 \in K$, $n \geq 0$

$$\begin{cases} x_{n+1} &= (1-\lambda_n)S_1x_n + \lambda_n S_2 y_n, \\ y_n &= (1-\mu_n)x_n + \mu_n S_1 x_n \end{cases} \tag{1.2}$$

where $\{\lambda_n\}$ and $\{\mu_n\}$ are sequences in $(0,1)$.

Motivated by [40], in this paper, we consider a convex b-metric space (X,d) with graph and define an *extended S-iteration scheme* for a triplet of three G-contractive type self mappings on a nonempty closed convex subset K of X. The convergence of this iteration scheme in comparison to the existing modified S-iteration scheme is also discussed with a numerical example.

In the following we reproduce the concepts of some of the terms used in this paper.

Definition 1.1. *[5], Let X be a non empty set and $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow [0,\infty)$ is called a b-metric if it satisfies the following properties.*

(1) $d(x,y) = 0$ *if and only if* $x = y$;

(2) $d(x,y) = d(y,x)$; *and*

(3) $d(x,z) \leq s[d(x,y) + d(y,z)]$, *for all* $x,y,z \in X$.

The pair (X,d) is called a b-metric space with coefficient s.

In 1970, Takahashi [41] introduced the following concept of convex structure in a metric space.

Definition 1.2. *[41] Let (X,d) be a metric space. A mapping $\mathcal{W} : X^2 \times [0,1] \longrightarrow X$ satisfying*

$$d\left(z, \mathcal{W}(x,y,t)\right) \leq td(z,x) + (1-t)d(z,y)$$

for all $x,y,z \in X$ and $t \in [0,1]$ is called a convex structure on X.

The above notion of convex structure can as well be extended naturally to b-metric spaces with the condition

$$sd\left(z, \mathcal{W}(x,y,t)\right) \leq td(z,x) + (1-t)d(z,y). \tag{1.3}$$

Kirk & Ray [17], in 1977, defined a metric space (X,d) to be *metrically convex* or simply *convex* if for every distinct elements x and y in X, there exists z in X, distinct from x and y such that

$$d(x,y) = d(x,z) + d(z,y).$$

A natural extension of this notion to b-metric spaces is by the equation

$$d(x,y) = s\left[d(x,z) + d(z,y)\right].$$

The mapping $\mathcal{W}$ satisfies $d(x,y) = s\Big\{d\left(x, \mathcal{W}(x,y,t)\right) + d\left(x, \mathcal{W}(x,y,t)\right)\Big\}$ for all $x, y \in X$ and $t \in [0,1]$ (refer to [41]). This can be seen in proving the following assertion as is done in [1] for metric spaces.

If (X,d) is a b-metric space on which a convex structure $\mathcal{W}$ is defined, then for all $x, y \in X$ and $t \in [0,1]$, $sd\left(x, \mathcal{W}(x,y,t)\right) = td(x,y)$ and $sd\left(y, \mathcal{W}(x,y,t)\right) = (1-t)d(x,y)$.
Let $\alpha = d\left(x, \mathcal{W}(x,y,t)\right)$, $\beta = d\left(y, \mathcal{W}(x,y,t)\right)$ and $\gamma = d(x,y)$. Then, from (1.3), we get

$$s\alpha \leq t\gamma \qquad \text{and} \qquad s\beta \leq (1-t)\gamma.$$

Now, by the triangle inequality of a b-metric, we have

$$\gamma \leq s(\alpha+\beta) \leq t\gamma + (1-t)\gamma = \gamma,$$

that is, $\gamma = s(\alpha+\beta)$. Now, if $s\alpha < t\gamma$, then $\gamma \leq s\alpha + s\beta < \gamma$, which is a contradiction. Hence $s\alpha = t\gamma$, and consequently, $s\beta = (1-t)\gamma$.

Definition 1.3. *[1] A b-metric space (X,d) on which a convex structure $\mathcal{W}$ is defined is called a convex b-metric space, denoted by $(X,\mathcal{W},d)$. A subset K of X is called convex if $\mathcal{W}(x,y,\lambda) \in K$ whenever $x, y \in K$ and $\lambda \in [0,1]$.*

Definition 1.4. *[38] A convex metric space $(X,\mathcal{W},d)$ is called uniformly convex if for every $\delta > 0$, $c > 0$ and $x, y, z \in X$, there exists $\mu > 0$ such that $d(z,x) \leq c$, $d(z,y) \leq c$ and $d(x,y) \geq c\delta$ implies*

$$d\left(z, \mathcal{W}\left(x, y; \tfrac{1}{2}\right)\right) \leq c(1-\mu) < c.$$

Hadamard manifolds and geodesic spaces are nonlinear examples of convex b-metric spaces while uniformly convex Banach b-metric spaces and CAT(0) spaces are examples of uniformly convex b-metric spaces [11].

Definition 1.5. *[36] Let K be a subset of a b-metric space (X,d) and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we take $r\left(x, \{x_n\}\right) = \lim_{n\to\infty} \sup d(x, x_n)$. Then*

(1) $r\left(\{x_n\}\right) = \inf\left\{r\left(x, \{x_n\}\right) : x \in K\right\}$ is said to be the asymptotic radius of $\{x_n\}$ with respect to $K \subseteq X$,

(2) For any $z \in K$, the set $A\left(\{x_n\}\right) = \left\{x \in X : r\left(x, \{x_n\}\right) \leq r\left(z, \{x_n\}\right)\right\}$ is said to be the asymptotic centre of $\{x_n\}$ with respect to $K \subseteq X$.

A sequence $\{x_n\}$ Δ-converges to x if $A\left(\{u_n\}\right) = \{x\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$, that is, x is the unique asymptotic centre for every subsequence $\{u_n\}$ of $\{x_n\}$. It is denoted by $\Delta - \lim_{n\to\infty} x_n = x$.

Equivalently, A sequence $\{x_n\}$ in X is said to Δ-converges to a point $x \in X$ if

$$\limsup_k d\big(x_{n_k}, x\big) \leq \limsup_k d\big(x_{n_k}, y\big)$$

for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and every $y \in X$ [18].

It can be seen that ordinary convergence implies Δ-convergence. However, the converse is not true.

Example 1.1. † *Let X be the set of positive rational numbers and define $d : X \times X \longrightarrow [0, \infty)$ by*

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

Then (X, d) is a b-metric space. Consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n}$, $n \in \mathbb{N}$. Then evidently, $\{x_n\}$ is not convergent but Δ-convergent to x for every $x \in X$. Because, given any x and y in X, we can choose k large enough with $x_{n_k} \neq x$ and $x_{n_k} \neq y$ so that

$$\limsup_k \Big(d\big(x_{n_k}, x\big) - d\big(x_{n_k}, y\big) \Big) \leq 1 - 1 = 0.$$

Definition 1.6. *[12] A subset K of a b-metric space (X, d) is said to be Chebychev if for every $x \in X$ there exists $y \in K$ such that $d(x, y) < d(k, x)$ for all $k \in K$ and $y \neq k$. If K is a Chebychev subset of a b-metric space X, then the nearest point projection $P : X \longrightarrow K$ is defined by sending x to y.*

As observed in [12], this notion of nearest point projection for Chebychev sets is in accordance with that of orthogonal projection onto a subspace of the Euclidean space. It was shown in [11] that every closed and convex subset of a uniformly convex b-metric space is Chebychev.

Lemma 1.1. *[12] Let K be a nonempty, closed and convex subset of a complete uniformly convex metric space $(X, \mathcal{W}, d)$. Then every bounded sequence $\{x_n\}$ in K has a unique asymptotic centre in K.*

Lemma 1.2. *[12] Let K be a nonempty, closed and convex subset of a complete uniformly convex metric space $(X, \mathcal{W}, d)$. Let $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 1.3. *[12] Let $(X, \mathcal{W}, d)$ be a uniformly convex metric space and $\{\alpha_n\}$ a sequence in $[b, c] \subset (0, 1)$. Suppose that the sequences $\{x_n\}$ and $\{y_n\}$ in X are such that*

$$\lim_{n\to\infty} \sup d(x_n, w) \leq c, \quad \lim_{n\to\infty} \sup d(y_n, w) \leq c$$

and

$$\lim_{n\to\infty} \sup d\left(\mathcal{W}(x_n, y_n; \alpha_n), w\right) \leq c$$

for some $w \in X$ and some $c \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

†This example is adapted from https://math.stackexchange.com/a/3370319/445538

The proofs of the above lemmas are independent of the property – the triangle inequality of the metric d. Thus these results also holds true for the corresponding b-metric spaces.

Let K be a non-empty subset of a b-metric space (X,d) and Δ be the diagonal of the Cartesian product $K \times K$, i.e., $\Delta = \{(x,x) : x \in K\}$. Let G be a reflexive digraph (directed graph) with the set $V(G)$ of its vertices coinciding with K, and the set $E(G)$ of its edges containing all loops, i.e., $E(G) \supseteq \Delta$. Assuming G has no parallel edges, we identify the graph G with the pair $(V(G), E(G))$.

Definition 1.7. *[40] The conversion of a graph G is the graph obtained from G by reversing the direction of the edges, denoted by G^{-1}, that is,*

$$E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in G\}.$$

A directed graph/digraph $G = (V(G), E(G))$ is said to be transitive if for every $x, y, z \in V(G)$ with $(x,y), (y,z) \in E(G)$, we have $(x,z) \in E(G)$.

Definition 1.8. *Let (X,d) be a b-metric space with coefficient $s \geq 1$. A mapping $S : X \longrightarrow X$ is said to be a G-contractive type mapping if S is edge-preserving ($(x,y) \in E(G)$ implies $(Sx, Sy) \in E(G)$) and*

$$d\,(Sx, Sy) \leq k\Big(d(x,y) + d(y, Sy)\Big) \tag{1.4}$$

for some $k < 1$ and for all $x, y \in X$.

Taking $y = Sx$ in the above relation, we get

$$d(Sx, S^2x) \leq k\Big(d(x, Sx) + d(Sx, S^2x)\Big)$$

that is,

$$d(Sx, S^2x) \leq \frac{k}{1-k} d(x, Sx) = k' d(x, Sx)$$

for some $k' > 0$. If $y = w_0 \in F$, then we have

$$d\,(Sx, w_0) \leq k d(x, w_0),$$

where $F = F(S)$ is the set of fixed points of S.

2. Main results

For a b-metric space (X,d) with graph and a non-empty closed convex subset K of X, we introduce the *extended S-iteration scheme* given below. For $x_0 \in K$,

$$
\begin{cases}
x_{n+1} & = \mathcal{W}\left(S_2S_1x_n, S_3S_1y_n, \alpha_n\right), \\
y_n & = \mathcal{W}\left(S_1x_n, S_3z_n, \beta_n\right) \\
z_n & = \mathcal{W}\left(S_2S_1x_n, S_3S_1x_n, \gamma_n\right)
\end{cases}
\tag{2.1}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ and $S_i : X \longrightarrow X$ is a G-contractive type mapping on K for $i = 1,2,3$.

The existence of a fixed point for contractive type mappings given by (1.4) is known from various existing literatures (for example, refer to [11]). In this section, we prove a result on Δ-convergence and strong convergence of the iteration scheme given by (2.1) in a closed convex subset of a uniformly convex b-metric space.

Let $F = \bigcap_{i=1}^{3} F(S_i)$, where $F(S_i)$ are the sets of fixed points of S_i.

Lemma 2.1. *Let $w_0 \in F$ be such that $(x_0, w_0), (w_0, x_0) \in E(G)$. Then (x_n, w_0), (w_0, x_n), (y_n, w_0), (w_0, y_n), (z_n, w_0), (w_0, z_n), (x_n, y_n), (y_n, z_n) and (x_n, x_{n+1}) are in $E(G)$.*

Proof. We will prove by induction. Since S_1, S_2 and S_3 are edge-preserving and $E(G)$ is convex, using (2.1) we have

$$(x_0, w_0) \in E(G) \quad \Longrightarrow \quad (\mathcal{W}(S_2S_1x_0, S_3S_1x_0; \gamma_0), w_0) = (z_0, w_0) \in E(G)$$

and

$$(y_0, w_0) = (\mathcal{W}(S_1x_0, z_0; \beta_0), w_0) \in E(G).$$

Since $(x_0, w_0), (y_0, w_0) \in E(G)$, we have

$$(x_1, w_0) = (\mathcal{W}(S_2S_1x_0, S_3S_1y_0; \alpha_0), w_0) \in E(G)$$

and therefore,

$$(z_1, w_0) = (\mathcal{W}(S_2S_1x_1, S_3S_1x_1; \gamma_0), w_0) \in E(G).$$

Similarly, $(y_1, w_0) = (\mathcal{W}(S_1x_1, z_1; \beta_1), w_0) \in E(G)$.

Now we assume that $(x_k, w_0) \in E(G)$ for some positive integer k. Then by the same argument as before,

$$(z_k, w_0) = (\mathcal{W}(S_2S_1x_k, S_3S_1x_k; \gamma_k), w_0) \in E(G),$$

$$(y_k, w_0) = (\mathcal{W}(S_1x_k, z_k; \beta_k), w_0) \in E(G) \qquad \text{and}$$

$$(x_{k+1}, w_0) = (\mathcal{W}(S_2S_1x_k, S_3S_1y_k; \alpha_k), w_0) \in E(G).$$

This implies $(x_{k+1}, w_0) \in E(G)$ which in turn gives (y_{k+1}, w_0) and (z_{k+1}, w_0) are in $E(G)$. Therefore, (x_{n+1}, w_0), (y_{n+1}, w_0) and (z_{n+1}, w_0) are in $E(G)$ for all $n \in \mathbb{N}$.

In a similar way, we can show that (w_0, x_n), (w_0, y_n) and (w_0, z_n) are in $E(G)$ if $(w_0, x_0) \in E(G)$.

Finally, the transitivity of $E(G)$ implies (x_n, y_n), (y_n, z_n) and (x_n, x_{n+1}) are also in $E(G)$. □

Lemma 2.2. *If (X, d) is a convex b-metric space and $(x_0, w_0), (w_0, x_0) \in E(G)$ for arbitrary $x_0 \in X$ and $w_0 \in F$, then $d(x_{n+1}, w_0) \le kd(x_n, w_0)$ for all n and hence*

$$\lim_{n\to\infty} d(x_n, w_0) = 0.$$

Proof. If $w_0 \in F$, then by Lemma 2.1, $(x_n, w_0), (y_n, w_0), (z_n, w_0) \in E(G)$. Let $x'_n = S_1 x_n$. Since S_1, S_2 and S_3 are G-contractive type mappings, we have

$$d\left(Ax, Aw_0\right) \le k\Big(d(x, w_0) + d(w_0, Aw_0)\Big) = kd(x, w_0)$$

for some $k < 1$ and $A = S_1$, S_2 or S_3. Now,

$$\begin{aligned}
d(x_{n+1}, w_0) &= d\left(\mathcal{W}(S_2S_1x_n, S_3S_1y_n, \alpha_n), w_0\right) \\
&\le \alpha_n d(S_2S_1x_n, w_0) + (1-\alpha_n)d(S_3S_1y_n, w_0) \\
&\le k\alpha_n d(S_1x_n, w_0) + k(1-\alpha_n)d(S_1y_n, w_0) \\
&= k\alpha_n d(S_1x_n, w_0) + k(1-\alpha_n)d\left(\mathcal{W}(S_1x_n, z_n; \beta_n), w_0\right) \\
&\le k\alpha_n d(S_1x_n, w_0) + k(1-\alpha_n)\beta_n d(S_1x_n, w_0) \\
&\qquad + k(1-\alpha_n)(1-\beta_n)d(\mathcal{W}(S_2S_1x_n, S_3S_1x_n; \gamma_n), w_0) \\
&\le k\left(\alpha_n + \beta_n - \alpha_n\beta_n\right) d(S_1x_n, w_0) + k(1-\alpha_n)(1-\beta_n) \\
&\qquad \Big(\gamma_n d(S_2S_1x_n, w_0) + (1-\gamma_n)d(S_3S_1x_n, w_0)\Big) \\
&\le k\left(\alpha_n + \beta_n - \alpha_n\beta_n\right) d(S_1x_n, w_0) + k(1-\alpha_n)(1-\beta_n) \\
&\qquad \Big(\gamma_n kd(S_1x_n, w_0) + (1-\gamma_n)kd(S_1x_n, w_0)\Big) \\
&\le k\Big(\alpha_n + \beta_n - \alpha_n\beta_n + (1-\alpha_n)(1-\beta_n)\Big)d(S_1x_n, w_0) \\
&= kd(S_1x_n, w_0) \le kd(x_n, w_0) \qquad \text{for all} \quad n \in \mathbb{N}.
\end{aligned}$$

Thus the sequence $\{d(x_n, w_0)\}$ of positive numbers is monotonically decreasing and hence $\lim_{n\to\infty} d(x_n, w_0)$ exists. In fact, since $d(x_{n+1}, w_0) \le kd(x_n, w_0)$ for all $n \ge 0$, we have

$$d(x_{n+1}, w_0) \le k^n d(x_0, w_0).$$

This proves the assertion. □

Lemma 2.3. *If X is a convex b-metric space and $(x_0, w_0), (w_0, x_0) \in E(G)$ for arbitrary $x_0 \in X$ and $w_0 \in F$, then*

$$\lim_{n\to\infty} d(S_1x_n, x_n) = \lim_{n\to\infty} d(S_2x_n, x_n) = \lim_{n\to\infty} d(S_3x_n, x_n) = 0.$$

Proof. From Lemma 2.2, we see that

$$\lim_{n\to\infty} d(x_n, w_0) = 0. \tag{2.2}$$

Then using (1.4) we have

$$\begin{aligned} d(Ax_n, x_n) &\le sd(Ax_n, w_0) + sd(w_0, x_n) \\ &\le sk\Big(d(x_n, w_0) + d(w_0, Aw_0)\Big) + sd(x_n, w_0) \\ &\le s(k+1)d(x_n, w_0) \end{aligned}$$

where $A = S_1$, S_2 or S_3. In the limiting case, we have

$$\lim_{n\to\infty} d(Ax_n, x_n) = 0.$$

□

We now prove a result on Δ-convergence in convex b-metric spaces following the method used in [12].

Theorem 2.1. *Let K be a nonempty closed convex subset of a uniformly convex and complete b-metric space X with a continuous convex structure $\mathcal{W}$ and, $S_1, S_2, S_3 : K \longrightarrow K$ be continuous G-contractive type mappings on K with $F \neq \emptyset$. If $(x_0, w_0), (w_0, x_0) \in E(G)$ for arbitrary $x_0 \in K$ and some $w_0 \in F$, then the sequence $\{x'_n = S_1x_n\}$ given by (2.1) Δ-converges to an element of F.*

Proof. In Lemma 2.2, it is shown that $\lim_{n\to\infty} d(x_n, w_0)$ exists, which in turn shows that the sequence $\{x_n\}$ is bounded. Therefore by Lemma 1.1, $A(\{x_n\}) = \{x\}$. Let $\{v_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. As in Theorem 2.4 of [12], we can show that $v \in K$. By Lemma 2.3,

$$\lim_{n\to\infty} d(S_1v_n, v_n) = \lim_{n\to\infty} d(S_2v_n, v_n) = \lim_{n\to\infty} d(S_3v_n, v_n) = 0.$$

Define $u_m = T^m v$ ($T = S_i$, $i = 1, 2$ or 3) and we observe that

$$\begin{aligned} d(u_m, v_n) &\le sd(T^m v, T^m v_n) + \sum_{j=1}^{m} s^j d\left(T^{m-j}v_n, T^{m-j+1}v_n\right) \\ &\le skd\left(T^{m-1}v, T^{m-1}v_n\right) + skd\left(T^{m-1}v_n, T^m v_n\right) \\ &\quad + \sum_{j=1}^{m} s^j d\left(T^{m-j}v_n, T^{m-j+1}v_n\right) \end{aligned}$$

$$
\begin{aligned}
&\leq sk^2 d\left(T^{m-2}v, T^{m-2}v_n\right) + skd\left(T^{m-1}v_n, T^m v_n\right) \\
&\quad + sk^2 d\left(T^{m-2}v_n, T^{m-1}v_n\right) + \sum_{j=1}^{m} s^j d\left(T^{m-j}v_n, T^{m-j+1}v_n\right) \\
&\qquad\qquad \vdots \\
&\leq sk^m d\left(v, v_n\right) + \sum_{j=1}^{m} (sk^j + s^j) d\left(T^{m-j}v_n, T^{m-j+1}v_n\right) \\
&\leq d\left(v, v_n\right) + d\left(Tv_n, v_n\right) \sum_{j=1}^{m} (sk^j + s^j) k_1^j
\end{aligned}
$$

where $k_1 = \frac{k}{1-k} > 0$. Hence,

$$
r\left(u_m, \{v_n\}\right) \leq \lim_{n\to\infty} \sup d\left(u_m, v_n\right) \leq \lim_{n\to\infty} \sup d\left(v, v_n\right) \leq r\left(v, \{v_n\}\right),
$$

which shows that

$$
\left|r\left(u_m, \{v_n\}\right) - r\left(v, \{v_n\}\right)\right| \longrightarrow 0 \quad \text{as} \quad m \to \infty.
$$

Now, from Lemma 1.2, $\lim_{m\to\infty} u_m = \lim_{m\to\infty} T^m v = v$.

K being closed, $\lim_{m\to\infty} T^m v = v \in K$, and $\lim_{m\to\infty} T^{m+1} v = Tv$, which implies

$$
Tv = v.
$$

Therefore, by Lemma 2.2 (since $v \in F$) $\lim_{n\to\infty} d\left(v, x_n\right)$ exists.

Now, as in Theorem 2.4 of [12], it directly follows that $x = v$.

Thus, x is the unique asymptotic centre for any subsequence $\{v_n\}$ of $\{x_n\}$, showing that $\{x_n\}$ Δ-converges to x. □

In [37], Shahzad & Al-Dubiban stated a condition called Condition (B) and proved a strong convergence theorem for nonexpansive mappings in Banach spaces. We restate the condition in a b-metric setting and prove a strong convergence theorem.

The mappings $S_1, S_2, S_3 : K \longrightarrow K$ with $F = F(S_1) \cap F(S_2) \cap F(S_3) \neq \emptyset$ are said to satisfy *Condition (B)* if there is a non decreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$ and $f(x) > 0$ for all $x > 0$ such that for all $x \in K$,

$$
\max\left\{d(S_1x, x), d(S_2x, x), d(S_3x, x)\right\} \geq f\left(d(x, F)\right).
$$

Theorem 2.2. *Let K be a nonempty closed convex subset of a uniformly convex and complete b-metric space X with continuous convex structure $\mathcal{W}$ and, $S_1, S_2, S_3 : K \longrightarrow K$ be G-contractive type mappings on K satisfying $F \neq \emptyset$. Let $(x_0, w_0), (w_0, x_0) \in E(G)$ for arbitrary $x_0 \in X$ and $w_0 \in F$. If S_1, S_2 and S_3 satisfy condition (B), then the sequence $\{x_n\}$ given by (2.1) converges strongly to an element of F.*

Proof. Let $w \in F$. From Lemma 2.2, we get that $\{x_n\}$ is a bounded seequence and hence $\lim_{n\to\infty} d(x_n, w)$ exists. Also, we have

$$d(x_{n+1}, w) < d(x_n, w) \qquad \text{for all} \ \ n \geq 1,$$

from which we get that

$$d(x_{n+1}, F) \leq d(x_n, F) \qquad \text{for all} \ \ n \geq 1.$$

By the same argument as in Lemma 2.2, we conclude that $\lim_{n\to\infty} d(x_n, F)$ exists.

By Lemma 2.3, we have $\lim_{n\to\infty} d(S_i x_n, x_n) = 0$, where $i = 1, 2, 3$. Since S_1, S_2 and S_3 satisfy conddition (B), we get

$$\lim_{n\to\infty} f\left(d(x_n, F)\right) = 0$$

and hence

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

So, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{w_k\}$ in F satisfying

$$d\left(x_{n_k}, w_k\right) \leq 2^{-k}.$$

Setting $n_{k+1} = n_k + j$ for some $j \geq 1$, we have

$$d\left(x_{n_{k+1}}, w_k\right) \leq d\left(x_{n_k+j-1}, w_k\right) \leq d\left(x_{n_k}, w_k\right) \leq \frac{1}{2^k},$$

using which we get

$$\begin{aligned} d(w_{k+1}, w_k) &\leq s\Big(d\left(w_{k+1}, x_{n_{k+1}}\right) + d\left(x_{n_{k+1}}, w_k\right)\Big) \\ &\leq s\left(\frac{1}{2^{k+1}} + \frac{1}{2^k}\right) = \frac{s}{2^{k+1}}. \end{aligned}$$

Thus $\{w_k\}$ is a Cauchy sequence in F. Since F is closed, there exists $w^* \in F$ such that $\lim_{k\to\infty} w_k = w^*$. Thus, $\lim_{k\to\infty} x_{n_k} = w^*$.
As $\lim_{n\to\infty} d\left(x_n, w^*\right)$ exists and equals 0 by Lemma 2.2, the result follows. □

3. Numerical example

In this section, we present an example with its numerical experiment in support of our results. We also make a comparison of the rate of convergence of the iteration scheme (2.1) to that of the one given in [40].

In 1976, Rhoades [35] gave a comparison between two iterations $\{x_n\}$ and $\{z_n\}$, both converging to a fixed point p of a mapping $T : K \longrightarrow K$ by saying $\{x_n\}$ converge faster than $\{z_n\}$ if

$$d(x_n, p) \leq d(z_n, p), \qquad n \geq 1,$$

where K is a non-empty closed and convex subset of a complete metric space.

In numerical analysis, the order of convergence of a real sequence $\{\alpha_n\}$ converging to α is studied using the well known method mentioned below (refer to [4]).

Let $\{\alpha_n\}$ be a real sequence which converges to α with $\alpha_n \neq \alpha$ for all $n \in \mathbb{N}$. If

$$\lim_{n\to\infty} \frac{d(\alpha_{n+1}, \alpha)}{[d(\alpha_n, \alpha)]^{\mu}} = \lambda$$

for some positive constants λ and μ, then $\{\alpha_n\}$ is said to *converge to α of the order μ*, with *asymptotic error constant* λ. For $\lambda < 1$, if $\mu = 1$ the sequence is *linearly convergent* and if $\mu = 2$, the sequence is *quadratically convergent*.

In 2002, using the above method of comparison, Berinde [3] compared the rate of convergence between two iteration schemes as given below.

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive real numbers converging to α and β, respectively. Suppose that

$$\lim_{n\to\infty} \frac{d(\alpha_n, \alpha)}{d(\beta_n, \beta)} = l.$$

(i). If $l = 0$, then the sequence $\{\alpha_n\}$ is said to converge to α faster than that of the sequence $\{\beta_n\}$ to $\{\beta\}$.

(ii). If $0 < l < \infty$, then the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are said to have the same rate of convergence.

For a nonempty convex subset K of a complete b-metric space X with a self map $T : K \longrightarrow K$, if $\{x_n\}$ and $\{u_n\}$ are two iterations both of which converge to a fixed point p of T, then $\{x_n\}$ converges faster than $\{u_n\}$ to p if

$$\lim_{n\to\infty} \frac{d(x_n, p)}{d(u_n, p)} = 0.$$

We are now in a position to give an example for our main results and compare the rate of convergence of the studied iteration scheme against the modified S-iteration scheme. In the cases when the b-metric d is induced by the norm $\|.\|_X$, the mapping $\mathcal{W} : X^2 \times [0, 1] \longrightarrow X$ such that

$$\mathcal{W}(x, y, t) = (1 - t)x + ty$$

defines a convex structure on X. The iteration (2.1) then takes the following form. For $x_0 \in K$,

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)S_2S_1x_n + \alpha_nS_3S_1y_n, \\ y_n &= (1 - \beta_n)S_1x_n + \beta_nS_3z_n, \\ z_n &= (1 - \gamma_n)S_2S_1x_n + \gamma_nS_3S_1x_n \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$ and $S_i : X \longrightarrow X$ is a G-contractive type mapping on K for $i = 1, 2, 3$.

Example 3.1. *Let $X = \mathbb{R}$ and $K = [0, 2]$. Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = K$ and $(x, y) \in E(G)$ if and only if $1 \le x, y \le \frac{7}{4}$ and $x, y \in \mathbb{Q}$. We consider the mappings $P, Q, R : K \longrightarrow K$ given by*

$$Px = x^{\log x}$$

$$Qx = \frac{1}{3}\arcsin(x-1) + 1$$

and $$Rx = \sqrt{x}$$

for all $x \in K$. To show that P, Q and R are G-contractive type mappings, it is enough to show that they are G-contraction mappings on $[0, 2]$.

Now, to show that $Px = x^{\log x}$ is a contraction mapping on $[1, 2]$, we note that by Mean value theorem,

$$\frac{|P(x) - P(y)|}{|x - y|} \le c$$

where $c = \max\{P'x : x \in [1, 2]\}$ with $P'x = 2x^{\log x - 1}\log x$.
Since $P'(x) < \frac{7}{10}$ for all $1 \le x \le 2$, we see that P is a contraction on $[1, 2]$ and hence a G-contraction mapping.

Similarly, $Q'x = \frac{1}{3}\frac{1}{\sqrt{1-(x-1)^2}} \le \frac{4}{3\sqrt{7}}$ for all $1 \le x \le \frac{7}{4}$ and $R'x = \frac{1}{2}\frac{1}{\sqrt{x}} \le \frac{1}{2}$ for all $x \ge 1$ implies that Q and R are G-contraction mappings on $[0, 2]$. Their common fixed point here being $x = 1$. Consider the real sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $[0, 1]$, where

$$a_n = \frac{n+1}{5n+3}, \quad b_n = \frac{n+4}{10n+7} \quad \text{and} \quad c_n = \frac{n+2}{2n+3}.$$

Let $\{x_n\}$ be a sequence generated by the extended S-iteration (2.1) with $x_0 = 1.5$ and $S_1 = P$, $S_2 = Q$, $S_3 = R$ and $\alpha_n = a_n$, $\beta_n = b_n$, $\gamma_n = c_n$ as defined above. Let $\{u_n\}$ be a sequence generated by the modified S-iteration (1.2) with $u_0 = 1.5$ and using $S_1 = P$, $S_2 = Q$ with $\lambda_n = a_n$ and $\mu_n = b_n$. The numerical observations for the error estimates and the rate of convergence for these two iteration schemes are shown in Tables 1 & 2 below.

n	*Modified S-iteration*		*Extended S-iteration*	
	u_n	$\|u_n - u_{n-1}\|$	x_n	$\|x_n - x_{n-1}\|$
1	1.15489	0.345105	1.04139	0.458612
2	1.02536	0.129532	1.00041	0.0409766
3	1.00201	0.023352	1.00000	0.000411667
4	1.00012	0.00188701	1.00000	4.34451×10^{-8}
5	1.00001	0.000116509	1.00000	4.44089×10^{-16}
6	1.00000	7.01765×10^{-6}	1.00000	0.00000
7	1.00000	4.22157×10^{-7}	1.00000	0.00000

TABLE 1. Numerical errors of modified S-iteration and extended S-iteration schemes

n	*Modified S-*	*Extended S-*	*Rate of Convergence*		
	u_n	x_n	$\|u_n - 1\|$	$\|x_n - 1\|$	$\frac{\|x_n-1\|}{\|u_n-1\|}$
1	1.15489	1.04139	0.154895	0.0413883	0.267203
2	1.02536	1.00041	0.025363	0.000411711	0.0162327
3	1.00201	1.00000	0.00201099	4.34451×10^{-8}	0.0000216038
4	1.00012	1.00000	0.000123976	4.44089×10^{-16}	3.58206×10^{-12}
5	1.00001	1.00000	7.46682×10^{-6}	0.00000	0.000000
6	1.00000	1.00000	4.49177×10^{-7}	0.00000	0.000000
7	1.00000	1.00000	2.70198×10^{-8}	0.00000	0.000000

TABLE 2. Rate of Convergence

From Tables 1 & 2, it is evident that the sequence of iterates $\{u_n\}$ *and* $\{x_n\}$ *both converge to* $1 \in F$*. We also observe that* $|x_n - 1| \leq |u_n - 1|$ *and* $\lim_{n\to\infty} \frac{|x_n-1|}{|u_n-1|} = 0$*, so the seqeuence of iterates* $\{x_n\}$ *converges faster than* $\{u_n\}$*, generated by the Modified* S*-iteration (see Figure 1).*

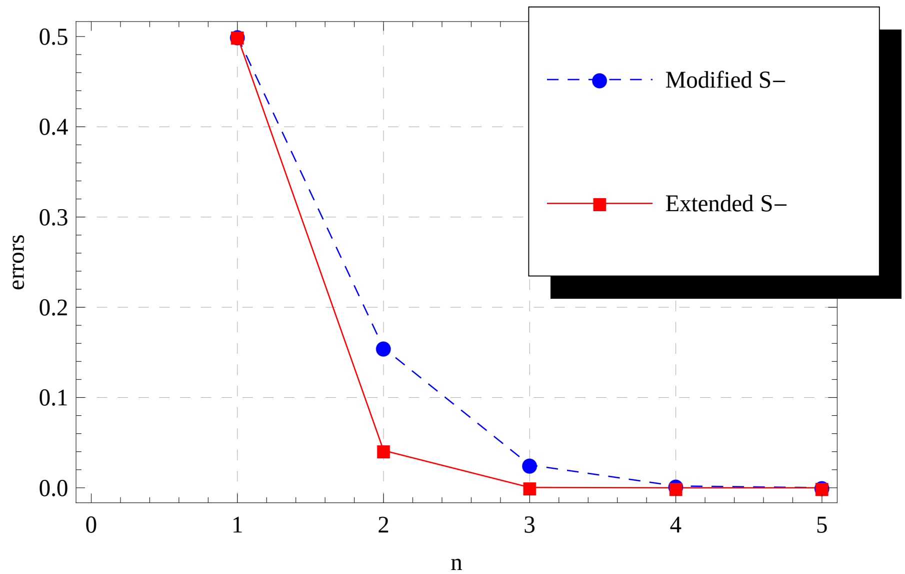

Figure 1. *Comparison of error estimates of the Modified* S*-iteration and the Extended* S*-iteration schemes*

Conclusion

In this paper, we have introduced an extended S-iteration scheme for G-contractive type mappings and proved Δ-convergence as well as strong convergence in a nonempty closed and convex subset of a uniformly convex and complete b-metric space with a directed graph. An example is also given to compare the convergence rate between the studied iteration and the modified S-iteration. The iteration considered in this paper may as well be studied for other contractive type mappings and its rate of convergence can be compared with existing iteration schemes. The application of our results for solving constrained optimization problem is also a scope for further study.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A. A. Abdelhakim, A convexity of functions on convex metric spaces of Takahashi and applications, J. Egypt. Math. Soc. 24 (2016), 348–354.

[2] R. P. Agarwal, D. ORegan and D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex. Anal. 8 (2007), 61–79.

[3] V. Berinde, Iterative Approximation of Fixed Points, Editura Efemeride, Baia Mare, 2002.

[4] R. L. Burden and J. D. Faires, Numerical Analysis, Brooks/Cole Cengage Learning, 9th edition, Boston, 2010.

[5] S. Czerwik, Contraction mappings in b-metric spaces, Acta. Math. Inform. Univ. Ostra. 1 (1993), 5–11.

[6] D. Das and N. Goswami, Some fixed point theorems on the sum and product of operators in tensor product spaces, Int. J. Pure Appl. Math. 109 (2016), no.3, 651–663.

[7] D. Das, N. Goswami and V.N. Mishra, Some results on fixed point theorems in Banach algebras, Int. J. Anal. Appl. 13 (2017), no. 1, 32–40.

[8] D. Das, N. Goswami and V.N. Mishra, Some results on the projective cone normed tensor product spaces over Banach algebras, (online available) Bol. Soc. Paran. Mat. (3s.) 38 (2020), no. 1, 197–221.

[9] Deepmala, A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications, Ph.D. Thesis (2014), Pt. Ravishankar Shukla University, Raipur 492 010, Chhatisgarh, India.

[10] Deepmala and H. K. Pathak, A study on some problems on existence of solutions for nonlinear functional-integral equations, Acta Math. Sci. 33 B(5) (2013), 1305–1313.

[11] H. Fukhar-ud-din, Existence and approximation of fixed points in convex metric spaces, Carpathian J. Math. 30 (2014), 175–185.

[12] H. Fukhar-ud-din, One step iterative scheme for a pair of nonexpansive mappings in a convex metric space, Hacet. J. Math. Stat. 44 (2015), 1023–1031.

[13] N. Goswami, N. Haokip and V. N. Mishra, F-contractive type mappings in b-metric spaces and some related fixed point results, Fixed Point Theory and Applications 2019 (2019), 13.

[14] N. Haokip and N. Goswami, Some fixed point theorems for generalized Kannan type mappings in b-metric spaces, Proyecciones (Antofagasta) 38 (4) (2019), 763-782.

[15] S. Ishikawa, Fixed Point by a New Iteration Method, Proc. Amer. Math. Soc. 44 (1974), 147–150.

[16] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly 83 (1976), 261–263.

[17] W. A. Kirk and W. O. Ray, A note on Lipschitzian mappings in convex metric spaces, Canad. Math. Bull. 20 (1977), 463–466.

[18] T. C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc. 60 (1976), 179 182.

[19] X. Liu, M. Zhou, L.N. Mishra, V.N. Mishra and B. Damjanović, Common fixed point theorem of six self-mappings in Menger spaces using (CLR_{ST}) property, Open Math. 16 (2018), 1423–1434.

[20] W. R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 44 (1953), 506–510.

[21] L. N. Mishra, H.M. Srivastava and M. Sen, On existence results for some nonlinear functional-integral equations in Banach algebra with applications, Int. J. Anal. Appl., 11 (2016), 1–10.

[22] L. N. Mishra, K. Jyoti, A. Rani and Vandana, Fixed point theorems with digital contractions image processing, Nonlinear Sci. Lett. A, 9 (2018), 104–115.

[23] L. N. Mishra, S. K. Tiwari, V. N. Mishra and I. A. Khan, Unique Fixed Point Theorems for Generalized Contractive Mappings in Partial Metric Spaces, J. Funct. Spaces, 2015 (2015), Article ID 960827, 8 pages.

[24] L. N. Mishra, S. K. Tiwari and V. N. Mishra, Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces, J. Appl. Anal. Comput. 5 (2015), 600–612.

[25] M. O. Olatinwo and C. O. Imoru, Some convergence results for the Jungck-Mann and Jungck-Ishikawa processes in the class of generalized Zamfirescu operators, Acta Math. Univ. Comenianae LXXVII (2008), 299–304.

[26] H. K. Pathak and Deepmala, Common fixed point theorems for PD-operator pairs under Relaxed conditions with applications, J. Comput. Appl. Math. 239 (2013), 103–113.

[27] B. Patir, N. Goswami and L. N. Mishra, Fixed point theorems in fuzzy metric spaces for mappings with some contractive type conditions, Korean J. Math. 26 (2018), 307–326.

[28] B. Patir, N. Goswami and V. N. Mishra, Some results on fixed point theory for a class of generalized nonexpansive mappings, Fixed Point Theory Appl. 2018 (2018), 19.

[29] T. Rasham, A. Shoaib, C. Park and M. Arshad, Fixed Point Results for a Pair of Multi Dominated Mappings on a Smallest Subset in K-Sequentially Dislocated quasi Metric Space with Application, J. Comput. Anal. Appl., 25 (5) (2018), 975–986.

[30] T. Rasham, A. Shoaib, N. Hussain, B. S. Alamri and M. Arshad, Multivalued Fixed Point Results in Dislocated b-Metric Spaces with Application to the System of Nonlinear Integral Equations, Symmetry, 11 (1) (2019), 40.

[31] T. Rasham, A. Shoaib, B. S. Alamri and M. Arshad, Fixed Point Results for Multivalued Contractive Mappings Endowed With Graphic Structure, J. Math. 2018 (2018) Article ID 5816364, 8 pages.

[32] T. Rasham, A. Shoaib, B. S. Alamri and M. Arshad, Multivalued fixed point results for new generalized F-Dominted contractive mappings on dislocated metric space with application, J. Funct. Spaces, 2018 (2018), Article ID 4808764, 12 pages.

[33] T. Rasham, A. Shoaib, B. A. S. Alamri, A. Asif and M. Arshad, Fixed Point Results for α^*-ψ-Dominated Multivalued Contractive Mappings Endowed with Graphic Structure, Mathematics 7 (2019), 307.

[34] A. Razani and M. Bagherboum, Convergence and stability of Jungck-type iterative procedures in convex b-metric spaces, Fixed Point Theory Appl. 2013 (2013), 331.

[35] B. E. Rhoades, Comments on two fixed point iteration method, J. Math. Anal. Appl. 56 (1976), 741–750.

[36] G. S. Saluja, Strong and Δ-convergence of modified two-step iterations for nearly asymptotically nonexpansive mappings in hyperbolic spaces, Int. J. Anal. Appl. 8 (2015), 39–52.

[37] S. Shahzad and R. Al-Dubiban, Approximating common fixed points of nonexpansive mappings in Banach spaces, Georgian Math. J. 13 (2006), 529–537.

[38] T. Shimizu and W. Takahashi, Fixed points of multivalued mappings in certain convex metric spaces, Topol. Methods Nonlinear Anal. 8 (1996), 197–203.

[39] S. L. Singh, C. Bhatnagar and S. N. Mishra, Stability of Jungck-Type Iterative Procedures, Int. J. Math. Math. Sci. 19 (2005), 3035–3043.

[40] S. Suparatulatorn, W. Cholamjiak and S. Suantai, A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs, Numer. Algorithms 77 (2017), 479–490.

[41] W. Takahashi, A convexity in metric spaces and nonexpansive mappings, Kodai Math. Sem. Rep. 22 (1970), 142–149.