International Journal of Analysis and Applications
Volume 17, Number 3 (2019), 464-478
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-17-2019-464

FRACTIONAL EXPONENTIALLY m-CONVEX FUNCTIONS AND INEQUALITIES

SAIMA RASHID[1,2,*], MUHAMMAD ASLAM NOOR[2] AND KHALIDA INYAT NOOR[2]

[1] *Government College University, Faisalabad, Pakistan*

[2] *COMSATS University Islamabad, Islamabad, Pakistan*

[*] *Corresponding author:* saimarashid@gcuf.edu.pk

Abstract. In this article, we introduce a new class of convex functions involving $m \in [0,1]$, which is called exponentially m-convex function. Some new Hermite-Hadamard inequalities for exponentially m-convex functions via Reimann-Liouville fractional integral are deduced. Several special cases are discussed. Results proved in this paper may stimulate further research in different areas of pure and applied sciences.

1. Introduction

Convex functions and their variant forms are being used to study a wide class of problems which arises in various branches of pure and applied sciences. This theory provides us a natural, unified and general framework to study a wide class of unrelated problems. For recent applications, generalizations and other aspects of convex functions and their variant forms, see [3–13, 15, 18–21, 24–27, 29–31] and the references therein.

An important class of convex functions, which is called exponential convex functions, was introduced and studied by Antczak [2], Dragomir et al [10] and Noor et al [19]. Alirezai and Mathar [1] have investigated their basic properties along with their potential applications in statistics and information theory,

Received 2019-02-19; accepted 2019-03-25; published 2019-05-01.

2000 *Mathematics Subject Classification.* 26D15, 26D10, 90C23.

Key words and phrases. convex function; exponential convex function; Reimann-Liouville Fractional integral inequalit; Holder's inequality and power-mean inequality.

see [1, 2, 14]. Awan et al [3] and Pecaric and Jaksetic [26] defined another kind of exponential convex functions and have shown that the class of exponential convex functions unifies various unrelated concepts. It has been shown [17]that the minimum of the differentiable exponentially convex functions on the convex sets can be characterized by an inequality, which is called the exponentially variational inequality. Exponentially variational inequalities can be viewed a natural generalization of the variational inequalities, see [32]. For the applications and numerical methods of variational inequalities, see Noor [16].

Toader [31] defined the m-convexity, an intermediate between the usual convexity and star shaped property. If $m = 0$, we have the concept of star shaped functions on $[a, b]$.
We would like to emphasize that exponentially convex functions and m-convex functions are two distinct classes of convex functions. It is natural to introduce a new class of convex functions, which unifies these concepts. Motivated by these facts, we introduce a new class of convex functions, which is called exponentially m-convex functions.

The advantages of fractional calculus have been described and pointed out in the last few decades by many authors. Fractional calculus is based on derivatives and integrals of fractional order, fractional differential equations and methods of their solution. The most celebrated inequality has been studied extensively since it is established, is the Hermite-Hadamard inequality not only established for classical integrals but also for fractional integrals, see [18, 20, 27, 29].

In this paper, we obtain some new Hermite-Hadamard type inequality for exponentially m-convex functions via Riemann-Liouville fractional integrals. Some special cases are also discussed which can be obtained from results. The ideas and techniques of this paper may motivate further research in this field.

2. Preliminaries

First of all, we recall the following basic concepts.

Definition 2.1. *[9, 31]. A set $K \subset \mathbb{R}$ is said to be a m-convex set with respect to a fixed constant $m \in [0, 1]$, if*

$$(1 - t)a + mtb \in K, \ \ \forall a, b \in K, t \in [0, 1].$$

The m-convex set contains the line segment between points a and mb for every pair of points a and b of K.

Toader [31] defined the notion of m-convex functions as follows.

Definition 2.2. *[31]. A function $f : K \subset \mathbb{R} \to \mathbb{R}$ is said to be a m-convex function, where $m \in [0, 1]$, if*

$$f((1-t)a + mtb) \leq (1-t)f(a) + mtf(b), \ \ \forall a, b \in K, t \in [0, 1].$$

Remark 2.1. *Clearly, a 1-convex function is a convex function in the ordinary sense. The 0-convex function are the starshaped functions. If we take $m = 1$, then we recapture the concept of convex functions and if we take $t = 1$, then*

$$f(mb) \leq mf(b) \ \ \forall b \in K.$$

This shows that the function f is sub-homogenous.

We now consider class of exponentially convex function, which are mainly due to Antczak [2], Dragomir [8] and Noor et al [20], respectively.

Definition 2.3. *[2, 8, 20]. A function $f : K \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be exponentially convex function,if*

$$e^{f((1-t)a+tb)} \leq [(1-t)e^{f(a)} + te^{f(b)}], \quad a, b \in K, \ t \in [0, 1], \tag{2.1}$$

where f is positive.

For the applications of exponentially convex functions in different field of statistics, information theory and mathematical sciences, see [1–3, 17] and the references therein.

We would like to point out that $u \in K$ is the minimum of the differentiable exponentially convex function f, if and only if, $u \in K$ satisfies the inequality

$$\langle f'(u)e^{f(u)}, v - u\rangle \geq 0, \quad \forall v \in K,$$

which is called the exponentially variational inequalities. For the more details, see Noor and Noor [17].

We now introduce a new concept of exponentially m-convex function.

Definition 2.4. *Let $f : K \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be an exponentially m-convex function, where $m \in (0, 1]$, if*

$$e^{f((1-t)a+mtb)} \leq [(1-t)e^{f(a)} + mte^{f(b)}], \quad a, b \in K, \ t \in [0, 1]. \tag{2.2}$$

For $t = \frac{1}{2}$, we have

$$e^{f(\frac{a+mb}{2})} \leq \frac{e^{f(a)} + me^{f(b)}}{2}, \quad \forall a, b \in K. \tag{2.3}$$

The function f is called exponentially Jensen m-convex function.

We now give the Definition of the fractional integral, which is mainly due to [27].

Definition 2.5. *[27]. Let $\alpha > 0$ with $n-1 < \alpha \le n$, $n \in \mathbb{N}$, and $1 < x < v$. The left- and right-hand side Riemann-Liouville fractional integrals of order α of function f are given by*

$$J^{\alpha}_{u+} f(x) = \frac{1}{\Gamma(\alpha)} \int_{u}^{x} (x-t)^{\alpha-1} f(t)dt,$$

and

$$J^{\alpha}_{v-} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{v} (t-x)^{\alpha-1} f(t)dt,$$

where $\Gamma(\alpha)$ is the gamma function.

We also made the convention $J^0_{u^+} = J^0_{v^-} = f(x)$.

We recall the special functions which are known as Gamma function,

$$\Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt.$$

For appropriate and suitable choice of m, one can obtain several new and known classes of exponentially convex functions as special cases. This shows that the concept of exponentially m-convex function is quite general and unifying one.

3. Main Results

In this section, we obtain Hermite-Hadamard type inequalities for exponentially m-convex function via Reimann-Liouville fractional integral.

Throughout this section, let $I = [a, mb]$ be an interval in real line. From now onward, we take $I = [a, mb]$, unless otherwise specified.

Theorem 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an exponentially convex function, where $m \in (0, 1]$. If $f \in L[a, mb]$, then*

$$\begin{aligned} e^{f(\frac{a+mb}{2})} &\le \frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}} \left\{ J^{\alpha}_{(a)_+} e^{f(mb)} + m^{\alpha+1} J^{\alpha}_{(b)_-} e^{f(\frac{a}{m})} \right\} \\ &\le \frac{\alpha[e^{f(a)} + m^2 e^{f(\frac{b}{m})}] + [me^{f(b)} + me^{f(\frac{a}{m})}]}{\alpha(\alpha+1)}. \end{aligned} \tag{3.1}$$

Proof. Let f be an exponentially m-convex function, from the inequality (2.2). Then, we have

$$e^{f(\frac{x+my}{2})} \leq \frac{e^{f(x)} + me^{f(y)}}{2}, \quad \forall x, y \in [a, mb].$$

Substituting $x = at + m(1-t)b$ and $y = (1-t)\frac{a}{m} + mt\frac{b}{m}$ for $t \in [0,1]$. Then

$$2e^{f(\frac{a+mb}{2})} \leq [e^{f(at+m(1-t)b)} + me^{f((1-t)\frac{a}{m}+mt\frac{b}{m})}].$$

Multiplying both sides of the above inequality with $t^{\alpha-1}$, and integrating over $[0,1]$, we have

$$\begin{aligned}\frac{2}{\alpha}e^{f(\frac{a+mb}{2})} &\leq \int_0^1 t^{\alpha-1}[e^{f(at+m(1-t)b)} + me^{f((1-t)\frac{a}{m}+mt\frac{b}{m})}]dt \\ &= \frac{1}{(mb-a)^\alpha}\left\{\int_a^{mb}(mb-u)^{\alpha-1}e^{f(u)}du + m^{\alpha+1}\int_{\frac{a}{m}}^{b}(v-\frac{a}{m})^{\alpha-1}e^{f(v)}dv\right\} \\ &= \frac{\Gamma(\alpha)}{(mb-a)^\alpha}\left\{J^\alpha_{(a)_+}e^{f(mb)} + m^{\alpha+1}J^\alpha_{(b)_-}e^{f(\frac{a}{m})}\right\},\end{aligned}$$

from which, one has

$$e^{f(\frac{a+mb}{2})} \leq \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left\{J^\alpha_{(a)_+}e^{f(mb)} + m^{\alpha+1}J^\alpha_{(b)_-}e^{f(\frac{a}{m})}\right\}. \tag{3.2}$$

On the other hand exponentially m-convexity of f gives

$$e^{f(at+m(1-t)b)} + me^{f((1-t)\frac{a}{m}+mt\frac{b}{m})} \leq t[e^{f(a)} + m^2e^{f(\frac{b}{m})} + (1-t)[me^{f(b)} + me^{f(\frac{a}{m})}].$$

. Multiplying both sides of the above inequality with $t^{\alpha-1}$, and integrating over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 t^{\alpha-1}e^{f(at+m(1-t)b)}dt + m\int_0^1 t^{\alpha-1}e^{f((1-t)\frac{a}{m}+mt\frac{b}{m})}dt \\ &\leq \int_0^1 t^\alpha[e^{f(a)} + m^2e^{f(\frac{b}{m})}]dt + \int_0^1 (t^{\alpha-1} - t^\alpha)[me^{f(b)} + me^{f(\frac{a}{m})}]dt.\end{aligned}$$

$$\frac{\Gamma(\alpha)}{(mb-a)^\alpha}\left\{J^\alpha_{(a)_+}e^{f(mb)} + m^{\alpha+1}J^\alpha_{(b)_-}e^{f(\frac{a}{m})}\right\} \leq \frac{e^{f(a)} + m^2e^{f(\frac{b}{m})}}{\alpha+1} + \frac{me^{f(b)} + me^{f(\frac{a}{m})}}{\alpha(\alpha+1)},$$

from which one has

$$\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left\{J^\alpha_{(a)_+}e^{f(mb)} + m^{\alpha+1}J^\alpha_{(b)_-}e^{f(\frac{a}{m})}\right\} \leq \frac{\alpha[e^{f(a)} + m^2e^{f(\frac{b}{m})}] + [me^{f(b)} + me^{f(\frac{a}{m})}]}{\alpha(\alpha+1)}. \tag{3.3}$$

Combining inequality (3.2) and inequality (3.3), we get (3.4). □

Corollary 3.1. *If we choose $m = 1$ in Theorem 3.1, then we have a new result*

$$e^{f(\frac{a+b}{2})} \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left\{J^{\alpha}_{(a)_+}e^{f(b)} + J^{\alpha}_{(b)_-}e^{f(a)}\right\} \leq \frac{[e^{f(a)} + e^{f(b)}]}{\alpha}.$$

Corollary 3.2. *If we choose $m = 1$ and $\alpha = 1$ in Theorem 3.1, then we have a new result*

$$2e^{f(\frac{a+b}{2})} \leq \frac{1}{b-a}\int_a^b e^{f(x)}dx \leq 2[e^{f(a)} + e^{f(b)}].$$

Theorem 3.2. *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$ be an exponentially m-convex function, where $m \in (0, 1]$. If $fg \in L[a, mb]$, then*

$$\frac{\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left\{J^{\alpha}_{mb_-}e^{f(a)} + J^{\alpha}_{a_+}e^{g(mb)}\right\} \leq \frac{e^{f(a)} + me^{g(b)} + \alpha(me^{f(b)} + e^{g(a)})}{\alpha(\alpha+1)}.$$

Proof. Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$ be an exponentially m-convex function. Then

$$e^{f(a(1-t)+mtb)} \leq (1-t)e^{f(a)} + tme^{f(b)}, \quad a, b \in [a, mb], \ \ t \in [0, 1],$$

$$e^{g(at+m(1-t)b)} \leq te^{g(a)} + m(1-t)e^{g(b)}, \quad a, b \in [a, mb], \ \ t \in [0, 1].$$

Adding both sides of the above inequalities, we have

$$e^{f(a(1-t)+mtb)} + e^{g(at+m(1-t)b)} \leq (1-t)[e^{f(a)} + me^{g(b)}] + t[me^{f(b)} + e^{g(a)}].$$

Multiplying both sides of the above inequality with $t^{\alpha-1}$, and integrating over $[0, 1]$, we have

$$\int_0^1 t^{\alpha-1}[e^{f(a(1-t)+mtb)} + e^{g(at+m(1-t)b)}]dt$$
$$\leq \int_0^1 (t^{\alpha-1} - t^{\alpha})[e^{f(a)} + me^{g(b)}]dt + \int_0^1 t^{\alpha}[me^{f(b)} + e^{g(a)}]dt.$$

$$\frac{\Gamma(\alpha)}{(mb-a)^{\alpha}}\left\{\int_a^{mb}(u-a)^{\alpha-1}e^{f(u)}du + \int_a^{mb}(mb-v)^{\alpha-1}e^{g(v)}dv\right\}$$
$$\leq \frac{e^{f(a)} + me^{g(b)}}{\alpha(\alpha+1)} + \frac{\alpha(me^{f(b)} + e^{g(a)})}{\alpha(\alpha+1)},$$

from which, we have

$$\frac{\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left\{J^{\alpha}_{mb_-}e^{f(a)}+J^{\alpha}_{a_+}e^{g(mb)}\right\}\le\frac{e^{f(a)}+me^{g(b)}+\alpha(me^{f(b)}+e^{g(a)})}{\alpha(\alpha+1)},$$

which is the required result. □

Corollary 3.3. *If we choose $m=1$ in Theorem 3.2, then we have a new result*

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left\{J^{\alpha}_{b_-}e^{f(a)}+J^{\alpha}_{a_+}e^{g(b)}\right\}\le\frac{e^{f(a)}+e^{g(b)}+\alpha(e^{f(b)}+e^{g(a)})}{\alpha(\alpha+1)}.$$

Corollary 3.4. *If we choose $m=1$ and $\alpha=1$ in Theorem 3.2, then we have a new result*

$$\frac{1}{b-a}\int_a^b[e^{g(x)}+e^{f(x)}]dx\le\frac{e^{f(a)}+e^{f(b)}+e^{g(a)}+e^{g(b)}}{2}.$$

We now derive Hermite-Hadamard type inequalities for m-convex functions using Reimann-Liouville fractional integral.

Theorem 3.3. *Let $\alpha>0$ and $f:I\subset\mathbb{R}\to\mathbb{R}$ be an exponentially convex function, where $m\in(0,1]$. If $f\in L[a,mb]$, then*

$$\begin{aligned}e^{f(\frac{a+mb}{2})}&\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})^+}e^{f(mb)}+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})^-}e^{f(\frac{a}{m})}\right]\\&\le\frac{\alpha}{4(\alpha+1)}\left[e^{f(a)}-m^2e^{f(\frac{a}{m^2})}\right]+\frac{m}{2}\left[e^{f(b)}+me^{f(\frac{a}{m^2})}\right].\end{aligned}$$

Proof. Let f be an exponentially m-convex function. Then, from the inequality (2.3), we have

$$e^{f(\frac{x+my}{2})}\le\frac{e^{f(x)}+me^{f(y)}}{2},\quad x,y\in I.$$

Substituting $x=\frac{t}{2}a+m\frac{2-t}{2}b,\ \ y=\frac{2-t}{2m}a+\frac{t}{2}b$ for $t\in[0,1]$. Then

$$2e^{f(\frac{a+mb}{2})}\le e^{f(\frac{t}{2}a+m\frac{2-t}{2}b)}+me^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}.$$

Multiplying both sides of the above inequality with $t^{\alpha-1}$, and integrating over $[0,1]$, we have

$$\frac{2}{\alpha}e^{f(\frac{a+mb}{2})} \leq \int_0^1 t^{\alpha-1}e^{f(\frac{t}{2}a+m\frac{2-t}{2}b)}dt + m\int_0^1 t^{\alpha-1}e^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}dt$$

$$= \frac{2}{a-mb}\int_{mb}^{\frac{a+mb}{2}}\left(\frac{2(mb-u)}{mb-a}\right)^{\alpha-1}e^{f(u)}du + \frac{2m^2}{mb-a}\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(\frac{2(v-\frac{a}{m})}{b-\frac{a}{m}}\right)^{\alpha-1}e^{f(v)}dv$$

$$= \frac{2^{\alpha}\Gamma(\alpha)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})^+}e^{f(mb)} + m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})^-}e^{f(\frac{a}{m})}\right].$$

Thus

$$\begin{aligned}
e^{f(\frac{a+mb}{2})} &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})^+}e^{f(mb)} + m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})^-}e^{f(\frac{a}{m})}\right] \\
&= \frac{\alpha}{2}\int_0^1 t^{\alpha-1}\left\{e^{f(\frac{t}{2}a+m\frac{2-t}{2}b)} + me^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}\right\}dt \\
&\leq \frac{\alpha}{2}\int_0^1 t^{\alpha-1}\left\{\left[\frac{t}{2}e^{f(a)} + \frac{m(2-t)}{2}e^{f(b)}\right] + m\left[m\frac{2-t}{2}e^{f(\frac{a}{m^2})} + \frac{t}{2}e^{f(b)}\right]\right\}dt \\
&= \frac{\alpha}{4}\left[e^{f(a)} - m^2e^{f(\frac{a}{m^2})}\right]\int_0^1 t^{\alpha}dt + \frac{m\alpha}{2}\left[e^{f(b)} + me^{f(\frac{a}{m^2})}\int_0^1 t^{\alpha-1}dt\right] \\
&= \frac{\alpha}{4(\alpha+1)}\left[e^{f(a)} - m^2e^{f(\frac{a}{m^2})}\right] + \frac{m}{2}\left[e^{f(b)} + me^{f(\frac{a}{m^2})}\right],
\end{aligned}$$

which is the required result. □

Lemma 3.1. *Let $\alpha > 0$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable exponentially m-convex function on the interior I° of I, where $m \in (0,1]$. If $|f'| \in L[[a,mb]$, is a m-convex function, then*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})^+}e^{f(mb)} + m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})^-}e^{f(\frac{a}{m})}\right] - \frac{1}{2}\left[e^{f(\frac{a+mb}{2})} + me^{f((\frac{a+mb}{2m})}\right]$$

$$= \frac{mb-a}{4}\left[\int_0^1 t^{\alpha}e^{f(\frac{t}{2}a+m\frac{2-t}{2}b)}f'(\frac{t}{2}a+m\frac{2-t}{2}b)dt\right.$$

$$\left.-\int_0^1 t^{\alpha}e^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}f'(\frac{2-t}{2m}a+\frac{t}{2}b)dt\right]. \tag{3.4}$$

Proof. It suffices to note that

$$\begin{aligned}
&\frac{mb-a}{4}\Big[\int_0^1 t^{\alpha}e^{f(\frac{t}{2}a+m\frac{2-t}{2}b)}f'(\frac{t}{2}a+m\frac{2-t}{2}b)dt\Big]\\
&=\frac{mb-a}{4}\Big[\frac{2}{mb-a}e^{f(\frac{a+mb}{2})}-\frac{2\alpha}{(a-mb)}\int_{mb}^{\frac{a+mb}{2}}\Big(\frac{2(mb-x)}{mb-a}\Big)^{\alpha-1}\frac{2e^{f(x)dx}}{a-mb}\Big]\\
&=\frac{mb-a}{4}\Big[-\frac{2}{mb-a}e^{f(\frac{a+mb}{2})}+\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(mb-a)^{\alpha+1}}J^{\alpha}_{(\frac{a+mb}{2})^-}e^{f(mb)}\Big]. \qquad (3.5)
\end{aligned}$$

Similarly, one can have

$$\begin{aligned}
&-\frac{mb-a}{4}\Big[\int_0^1 t^{\alpha}e^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}f'(\frac{2-t}{2m}a+\frac{t}{2}b)dt\Big]\\
&=-\frac{mb-a}{4}\Big[\frac{2m}{mb-a}e^{f(\frac{a+mb}{2m})}-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(mb-a)^{\alpha+1}}J^{\alpha}_{(\frac{a+mb}{2m})^+}e^{f(\frac{a}{m})}\Big]. \qquad (3.6)
\end{aligned}$$

Adding (3.5) and (3.6), gives (3.4). □

Theorem 3.4. *Let $\alpha>0$ and $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable exponentially m-convex function on the interior I° of I, where $m\in(0,1]$. If $|f'|\in L[[a,mb]$, is a m-convex function, then*

$$\begin{aligned}
&\Big|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\big[J^{\alpha}_{(\frac{a+mb}{2})^+}e^{f(mb)}+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})^-}e^{f(\frac{a}{m})}\big]\\
&-\frac{1}{2}\Big[e^{f(\frac{a+mb}{2})}+me^{f((\frac{a+mb}{2m})}\Big]\Big|\\
&\leq\frac{mb-a}{4}\Big\{\frac{1}{4(\alpha+3)}\big\{\big|e^{f(a)}f'(a)\big|+\big|e^{f(b)}f'(b)\big|\big\}+\frac{m^2(\alpha^2+7\alpha+14)}{4(\alpha+1)(\alpha+2)(\alpha+3)}\\
&\big\{\big|e^{f(b)}f'(b)\big|+\big|e^{f(\frac{a}{m^2})}f'(\frac{a}{m^2})\big|\big\}+\frac{m(\alpha+4)}{((\alpha+2)(\alpha+3))}\big\{\Delta_1(a,b)+\Delta_2(\frac{a}{m^2},b)\big\}\Big\},
\end{aligned}$$

where

$$\Delta_1(a,b)=\big|e^{f(a)}f'(b)\big|+\big|e^{f(b)}f'(a)\big|,$$

and

$$\Delta_2(\frac{a}{m^2},b)=\big|e^{f(\frac{a}{m^2})}f'(b)\big|+\big|e^{f(b)}f'(\frac{a}{m^2})\big|.$$

Proof. Using Lemma 3.1and exponentially m-convexity function of f, we have

$$\begin{aligned}
&\Big|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\big[J^{\alpha}_{(\frac{a+mb}{2})^{+}}e^{f(mb)}+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})^{-}}e^{f(\frac{a}{m})}\big]\\
&-\frac{1}{2}\Big[e^{f(\frac{a+mb}{2})}+me^{f((\frac{a+mb}{2m})}\Big]\Big|\\
&\leq\frac{mb-a}{4}\Big[\int_{0}^{1}t^{\alpha}\Big\{\Big|e^{f(\frac{t}{2}a+m\frac{2-t}{2}b)}f'(\frac{t}{2}a+m\frac{2-t}{2}b)\Big|\\
&+\Big|e^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}f'(\frac{2-t}{2m}a+\frac{t}{2}b)\Big|\Big\}dt\Big]. \qquad (3.7)
\end{aligned}$$

Using m-convexity of f, we have

$$\begin{aligned}
&\Big|e^{f(\frac{t}{2}a+m\frac{2-t}{2}b)}f'(\frac{t}{2}a+m\frac{2-t}{2}b)\Big|+\Big|e^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}f'(\frac{2-t}{2m}a+\frac{t}{2}b)\Big|\\
&\leq\Big\{\big[\frac{t}{2}|e^{f(a)}|+\frac{m(2-t)}{2}|e^{f(b)}|\big]\big[\frac{t}{2}|f'(a)|+\frac{m(2-t)}{2}|f'(b)|\big]\Big\}\\
&\quad\Big\{\big[m\frac{2-t}{2}|e^{f(\frac{a}{m^2})}|+\frac{t}{2}|e^{f(b)}|\big]\big[\frac{m(2-t)}{2}|f'\frac{a}{m^2}|+\frac{t}{2}|f'(b)|\big]\Big\}\\
&=\frac{t^2}{4}\big|e^{f(a)}f'(a)\big|+\frac{m^2(2-t)^2}{4}\big|e^{f(b)}f'(b)\big|+\frac{m(2-t)t}{4}\big[\big|e^{f(a)}f'(b)\big|+\big|e^{f(b)}f'(a)\big|\big]\\
&+\frac{m^2(2-t)^2}{4}\big|e^{f(\frac{a}{m^2})}f'(\frac{a}{m^2})\big|+\frac{t^2}{4}\big|e^{f(b)}f'(b)\big|+\frac{m(2-t)t}{4}\big[\big|e^{f(\frac{a}{m^2})}f'(b)\big|\\
&+\big|e^{f(b)}f'(\frac{a}{m^2})\big|\big]\\
&=\frac{t^2}{4}\big\{\big|e^{f(a)}f'(a)\big|+\big|e^{f(b)}f'(b)\big|\big\}+\frac{m^2(2-t)^2}{4}\big\{\big|e^{f(b)}f'(b)\big|+\big|e^{f(\frac{a}{m^2})}f'(\frac{a}{m^2})\big|\big\}\\
&+\frac{m(2-t)t}{4}\big\{\Delta_1(a,b)+\Delta_2(\frac{a}{m^2},b)\big\}. \qquad (3.8)
\end{aligned}$$

Thus

$$\begin{aligned}
&\Big|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\big[J^{\alpha}_{(\frac{a+mb}{2})^{+}}e^{f(mb)}+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})^{-}}e^{f(\frac{a}{m})}\big]\\
&-\frac{1}{2}\Big[e^{f(\frac{a+mb}{2})}+me^{f((\frac{a+mb}{2m})}\Big]\Big|\\
&\leq\frac{mb-a}{4}\Big[\int_{0}^{1}t^{\alpha}\Big\{\frac{t^2}{4}\big\{\big|e^{f(a)}f'(a)\big|+\big|e^{f(b)}f'(b)\big|\big\}+\frac{m^2(2-t)^2}{4}\big\{\big|e^{f(b)}f'(b)\big|\\
&+\big|e^{f(\frac{a}{m^2})}f'(\frac{a}{m^2})\big|\big\}+\frac{m(2-t)t}{4}\big\{\Delta_1(a,b)+\Delta_2(\frac{a}{m^2},b)\big\}\Big\}dt\Big]\\
&=\frac{mb-a}{4}\Big\{\frac{1}{4(\alpha+3)}\big\{\big|e^{f(a)}f'(a)\big|+\big|e^{f(b)}f'(b)\big|\big\}+\frac{m^2(\alpha^2+7\alpha+14)}{4(\alpha+1)(\alpha+2)(\alpha+3)}\\
&\big\{\big|e^{f(b)}f'(b)\big|+\big|e^{f(\frac{a}{m^2})}f'(\frac{a}{m^2})\big|\big\}+\frac{m(\alpha+4)}{((\alpha+2)(\alpha+3))}\big\{\Delta_1(a,b)+\Delta_2(\frac{a}{m^2},b)\big\}\Big\},
\end{aligned}$$

which is the required result. □

Corollary 3.5. *[21]. If we choose $m = 1$ and $\alpha = 1$, in Theorem 3.4, then*

$$\left|e^{f(\frac{a+b}{2})} - \frac{1}{b-a}\int_a^b e^{f(x)}dx\right|$$
$$\leq \frac{b-a}{4}\left[\frac{7\{\left|e^{f(a)}f'(a)\right| + \left|e^{f(b)}f'(b)\right|\} + 10[\Delta_1(a,b) + \Delta_2(a,b)]}{24}\right].$$

Theorem 3.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable exponentially m-convex function on the interior I° of I, where $m \in (0, 1]$. If $|f'| \in L[[a, mb]$, is a m-convex function on I and $p^{-1} + q^{-1} = 1$, where $q > 1$, then we have*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^\alpha}\left[J^\alpha_{(\frac{a+mb}{2})^+}e^{f(mb)} + m^{\alpha+1}J^\alpha_{(\frac{a+mb}{2m})^-}e^{f(\frac{a}{m})}\right]\right.$$
$$\left.-\frac{1}{2}\left[e^{f(\frac{a+mb}{2})} + me^{f((\frac{a+mb}{2m})}\right]\right|$$
$$\leq \frac{mb-a}{4(\alpha+1)^{\frac{1}{p}}}\left[\left\{\frac{1}{4(\alpha+3)}|e^{f(a)}f'(a)|^q + \frac{m^2(\alpha^2+7\alpha+14)}{4(\alpha+1)(\alpha+2)(\alpha+3)}|e^{f(b)}f'(b)|^q\right.\right.$$
$$\left.+\frac{m(\alpha+4)}{(\alpha+2)(\alpha+3)}\Delta_3(a,b)\right\}^{\frac{1}{q}} + \left\{\frac{m^2(\alpha^2+7\alpha+14)}{4(\alpha+1)(\alpha+2)(\alpha+3)}|e^{f(\frac{a}{m^2})}f'(\frac{a}{m^2})|^q\right.$$
$$\left.\left.+\frac{1}{4(\alpha+3)}|e^{f(b)}f'(b)|^q + \frac{m(\alpha+4)}{(\alpha+2)(\alpha+3)}\Delta_4(\frac{a}{m^2}, b)\right\}^{\frac{1}{q}}\right],$$

where

$$\Delta_3(a,b) = |e^{f(a)}f'(a)|^q + |e^{f(b)}f'(b)|^q$$

and

$$\Delta_4(\frac{a}{m^2}, b) = |e^{f(\frac{a}{m^2})}f'(\frac{a}{m^2})|^q + |e^{f(b)}f'(b)|^q.$$

Proof. Using Lemma 3.1 and the power mean inequality, we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^\alpha}\left[J^\alpha_{(\frac{a+mb}{2})^+}e^{f(mb)} + m^{\alpha+1}J^\alpha_{(\frac{a+mb}{2m})^-}e^{f(\frac{a}{m})}\right]\right.$$
$$\left.-\frac{1}{2}\left[e^{f(\frac{a+mb}{2})} + me^{f((\frac{a+mb}{2m})}\right]\right|$$

$$\leq \frac{mb-a}{4}\Big[\int_0^1 t^\alpha \Big\{\Big|e^{f(\frac{t}{2}a+m\frac{2-t}{2}b)} f'(\frac{t}{2}a+m\frac{2-t}{2}b)\Big|$$

$$+\Big|e^{f(\frac{2-t}{2m}a+\frac{t}{2}b)} f'(\frac{2-t}{2m}a+\frac{t}{2}b)\Big|\Big\}dt\Big]$$

$$= \frac{mb-a}{4}\Big(\frac{1}{(\alpha+1)}\Big)^{\frac{1}{p}}\Big[\Big\{\int_0^1 t^\alpha \Big\{\Big|e^{f(\frac{t}{2}a+m\frac{2-t}{2}b)} f'(\frac{t}{2}a+m\frac{2-t}{2}b)\Big|^q dt\Big\}^{\frac{1}{q}}$$

$$+\Big\{\int_0^1 t^\alpha \Big|e^{f(\frac{2-t}{2m}a+\frac{t}{2}b)} f'(\frac{2-t}{2m}a+\frac{t}{2}b)\Big|^q dt\Big\}^{\frac{1}{q}}\Big]$$

$$= \frac{mb-a}{4}\Big(\frac{1}{(\alpha+1)}\Big)^{\frac{1}{p}}\Big[\Big\{\int_0^1 t^\alpha \big[\frac{t^2}{4}|e^{f(a)}f'(a)|^q + \frac{m^2(2-t)^2}{4}|e^{f(b)}f'(b)|^q$$

$$+\frac{mt(2-t)}{4}\Delta_3(a,b)\big]\Big\}^{\frac{1}{q}} + \Big\{\int_0^1 t^\alpha \big[\frac{t^2}{4}|e^{f(b)}f'(b)|^q + \frac{m^2(2-t)^2}{4}|e^{f(\frac{a}{m^2})}f'(\frac{a}{m^2})|^q$$

$$+\frac{mt(2-t)}{4}\Delta_4(\frac{a}{m^2},b)\big]\Big\}^{\frac{1}{q}}\Big]$$

$$= \frac{mb-a}{4(\alpha+1)^{\frac{1}{p}}}\Big[\Big\{\frac{1}{4(\alpha+3)}|e^{f(a)}f'(a)|^q + \frac{m^2(\alpha^2+7\alpha+14)}{4(\alpha+1)(\alpha+2)(\alpha+3)}|e^{f(b)}f'(b)|^q$$

$$+\frac{m(\alpha+4)}{(\alpha+2)(\alpha+3)}\Delta_3(a,b)\Big\}^{\frac{1}{q}} + \Big\{\frac{m^2(\alpha^2+7\alpha+14)}{4(\alpha+1)(\alpha+2)(\alpha+3)}|e^{f(\frac{a}{m^2})}f'(\frac{a}{m^2})|^q$$

$$+\frac{1}{4(\alpha+3)}|e^{f(b)}f'(b)|^q + \frac{m(\alpha+4)}{(\alpha+2)(\alpha+3)}\Delta_4(\frac{a}{m^2},b)\Big\}^{\frac{1}{q}}\Big],$$

which completes the proof. □

Corollary 3.6. *[21]. If we choose $m=1$ and $\alpha=1$, in Theorem 3.5, then*

$$\Big|e^{f(\frac{a+b}{2})} - \frac{1}{b-a}\int_a^b e^{f(x)}dx\Big|$$

$$\leq \frac{b-a}{4(2)^{\frac{1}{p}}}\Big[\Big\{\frac{3|e^{f(a)}f'(a)|^q + 11|e^{f(b)}f'(b)|^q + 20\Delta_3(a,b)}{48}\Big\}^{\frac{1}{q}}$$

$$+\Big\{\frac{11|e^{f(a)}f'(a)|^q + 3|e^{f(b)}f'(b)|^q + 20\Delta_4(a,b)}{48}\Big\}^{\frac{1}{q}}\Big],$$

Theorem 3.6. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be differentiable exponentially m-convex function on the interior I° of I, where $m \in (0,1]$. If $|f'| \in L[[a,mb]$, is a m-convex function on I and $p^{-1}+q^{-1}=1$, where $q \geq 1$, then we* *have*

$$\begin{aligned}&\Big|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\big[J^{\alpha}_{(\frac{a+mb}{2})^{+}}e^{f(mb)}+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})^{-}}e^{f(\frac{a}{m})}\big]\\&-\frac{1}{2}\Big[e^{f(\frac{a+mb}{2})}+me^{f((\frac{a+mb}{2m})}\Big]\Big|\\&\leq\frac{mb-a}{4(p\alpha+1)^{\frac{1}{p}}}\Big[\Big\{\frac{|e^{f(a)}f'(a)|^{q}+7m^{2}|e^{f(b)}f'(b)|^{q}+2m\Delta_{3}(a,b)}{12}\Big\}^{\frac{1}{q}}\\&+\Big\{\frac{7m^{2}|e^{f(\frac{a}{m^{2}})}f'(\frac{a}{m^{2}})|^{q}+|e^{f(b)}f'(b)|^{q}+2m\Delta_{4}(\frac{a}{m^{2}},b)}{12}\Big\}^{\frac{1}{q}}\Big].\end{aligned}$$

Proof. Using Lemma3.1 and the Holder's inequality, we have

$$\begin{aligned}&\Big|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\big[J^{\alpha}_{(\frac{a+mb}{2})^{+}}e^{f(mb)}+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})^{-}}e^{f(\frac{a}{m})}\big]\\&-\frac{1}{2}\Big[e^{f(\frac{a+mb}{2})}+me^{f((\frac{a+mb}{2m})}\Big]\Big|\\&\leq\frac{mb-a}{4}\Big[\Big(\int_{0}^{1}t^{p\alpha}dt\Big)^{\frac{1}{p}}\Big\{\int_{0}^{1}\Big|e^{f(\frac{t}{2}a+m\frac{2-t}{2}b)}f'(\frac{t}{2}a+m\frac{2-t}{2}b)\Big|^{q}dt\Big\}^{\frac{1}{q}}\\&+\Big(\int_{0}^{1}t^{p\alpha}dt\Big)^{\frac{1}{p}}\Big\{\int_{0}^{1}\Big|e^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}f'(\frac{2-t}{2m}a+\frac{t}{2}b)\Big|^{q}dt\Big\}^{\frac{1}{q}}\Big]\end{aligned}$$

$$\begin{aligned}&\leq\frac{mb-a}{4(p\alpha+1)^{\frac{1}{p}}}\Big[\Big\{\int_{0}^{1}\big[\frac{t^{2}}{4}|e^{f(a)}f'(a)|^{q}+\frac{m^{2}(2-t)^{2}}{4}|e^{f(b)}f'(b)|^{q}+\frac{mt(2-t)}{4}\Delta_{3}(a,b)\big]dt\Big\}^{\frac{1}{q}}\\&+\Big\{\big[\frac{m^{2}(2-t)^{2}}{4}|e^{f(\frac{a}{m^{2}})}f'(\frac{a}{m^{2}})|^{q}+\frac{t^{2}}{4}|e^{f(b)}f'(b)|^{q}+\frac{mt(2-t)}{4}\Delta_{4}(\frac{a}{m^{2}},b)\big]\Big\}^{\frac{1}{q}}\Big]\\&=\frac{mb-a}{4(p\alpha+1)^{\frac{1}{p}}}\Big[\Big\{\frac{|e^{f(a)}f'(a)|^{q}+7m^{2}|e^{f(b)}f'(b)|^{q}+2m\Delta_{3}(a,b)}{12}\Big\}^{\frac{1}{q}}\\&+\Big\{\frac{7m^{2}|e^{f(\frac{a}{m^{2}})}f'(\frac{a}{m^{2}})|^{q}+|e^{f(b)}f'(b)|^{q}+2m\Delta_{4}(\frac{a}{m^{2}},b)}{12}\Big\}^{\frac{1}{q}}\Big],\end{aligned}$$

which completes the proof. □

Corollary 3.7. *[21]. If we choose $m = 1$ and $\alpha = 1$, in Theorem 3.6, then*

$$\begin{aligned}\Big|e^{f(\frac{a+b}{2})}-\frac{1}{b-a}\int_{a}^{b}e^{f(x)}dx\Big|&\leq\frac{b-a}{4(p+1)^{\frac{1}{p}}}\Big[\Big\{\frac{|e^{f(a)}f'(a)|^{q}+7|e^{f(b)}f'(b)|^{q}+2\Delta_{3}(a,b)}{12}\Big\}^{\frac{1}{q}}\\&+\Big\{\frac{7|e^{f(a)}f'(a)|^{q}+|e^{f(b)}f'(b)|^{q}+2\Delta_{4}(a,b)}{12}\Big\}^{\frac{1}{q}}\Big].\end{aligned}$$

4. Conclusions

In this paper, we have introduced and studied a new class of exponentially convex functions involving the parameter m. We have obtained several new Hermite-Hadamard inequalities via Riemann-Liouville fractional integrals. It is shown that previously known results can be obtained as special cases from our results. It is shown that the class of exponentially m-convex functions is quite general, flexible and unifying one.

Acknowledgments: The authors would like to thank the Rector, COMSATS University Islamabad, Islamaabd, Pakistan for providing excellent research and academic environments.

References

[1] G. Alirezaei and R. Mathar, On exponentially concave functions and their impact in information theory, Information Theory and Applications Workshop, San Diego, California, USA, 2018.

[2] T. Antczak, (p, r)-invex sets and functions, J. Math. Anal. Appl. 263(2001), 355-379.

[3] M. U. Awan, M. A. Noor and K. I. Noor, Hermite-Hadamard inequalities for exponentiaaly convex functions, Appl. Math. Inform. Sci. 2(12)(2018), 405-409.

[4] M. K. Bakula, M. E. Ozdemir and J. Pecaric, Hadamard type inequalities for m-convex and $(\alpha; m)$-convex functions, J. Inequal. Pure Appl. Math. 9(4)(2008), Art. ID 96.

[5] I. A. Baloch and I. Iscan, Some Hermite-Hadamard type inequalities for harmonically $(s; m)$-convex functions in second sense, arXiv:1604.08445v1 [math.CA], 2016.

[6] M. K. Bacul, J. Pecaric and M. Ribicic, Companion inequalities to Jensen's inequality for m-convex and (α, m) convex functions, J. Inequal. Pure. Appl. Math. 7(5)(2006), Art. ID 194.

[7] M. Braccamonte, J. Gimenez, N. Merentes and M. Vivas, Fejer type inequalities for m-convex functions, Publicaciones en Ciencias y Tecnología, 10(1)(2016), 7-11.

[8] S. S. Dragomir and I. Gomm, Some Hermite-Hadamard type inequalities for functions whose exponentials are convex, Stud. Univ. Babes-Bolyai Math. 60(4)(2015), 527-534.

[9] S. S. Dragomir and G. Toader, Some inequalities for m-convex functions, Stud. Univ. Babes-Bolyai Math. 38 (1993), 21-28.

[10] S. S. Dragomir, On some new inequalities of Hermite-Hadamard type for m - convex functions, Tamkang J. Math. 33(1)(2002), 45-55.

[11] L. Fejéer, Uber die fourierreihen, II, Math Naturwise. Anz Ungar. Akad. Wiss. (24)(1906), 369-390.

[12] J. Hadamard, Etude sur les proprietes des fonctions entieres e.t en particulier dune fonction consideree par Riemann. J. Math. Pure Appl. (58)(1893), 171-215.

[13] C. Y. He, Y. Wang, B. Y.Xi and F. Qi, Hermite-Hadamard type inequalities for $(\alpha; m)$-HA and strongly $(\alpha; m)$-HA convex functions, J. Nonlinear Sci. Appl. (10)(2017), 205-214.

[14] M. Mahdavi, Exploiting Smoothness in Statistical Learning, Sequential Prediction, and Stochastic Optimization. East Lansing, MI, USA: Michigan State University, (2014).

[15] C. P. Niculescu and L. E. Persson, Convex Functions and Their Applications. Springer-Verlag, New York, (2018).

[16] M. A. Noor, Some developments in general variational inequalities, Appl. Math. Comput. 152(2004), 199-277.

[17] M. A. Noor and K. I. Noor, Exponentially convex functions, Preprint.

[18] M. A. Noor, K. I. Noor and M. U. Awan, Fractional Hermite-Hadamard inequalities for convex functions and applications, Tbilisi J. Math. 8(2)(2015), 103-113.

[19] M. A. Noor, K. I. Noor, and S. Rashid, Exponential r-convex functions and inequalities, Preprint.

[20] M. A. Noor, K. I. Noor and S. Rashid, Fractal exponential convex functions and inequalities, Preprint.

[21] S. Rashid, M. A. Noor and K. I. Noor, Modified exponential convex functions and inequalities, Open Acess J. Math. Theor. Phy. 2(2)(2019), 45-51.

[22] M. E. Ozdemir, M. Avci. and H. Kavurmaci, Hermite-Hadamard type inequalities via (α, m)-convexity, J. Comput. Math. Appl. 61(2011), 2614-2620.

[23] S. Pal and T. K. L. Wong, Exponentially concave functions and a new information geometry, Ann. Probab. 46(2)(2018), 1070-1113.

[24] J. Park, New Ostrowski-like type inequalities for differentable (s, m)-convex mappings, Int. J. Pure Appl. Math. 78 (8) (2012), 1077-1089.

[25] J. E. Pecaric, F. Proschan and Y. L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academics Press, New York, (1992).

[26] J. Pecaric and J. Jaksetic, Exponential onvexity, Euler-Radau expansions and stolarsky means, Rad Hrvat. Matematicke Znanosti, 515(2013), 81-94

[27] I. Podlubny, Fractional Differential Equations: Mathematics in Science and Engineering, Academic Press, San Diego, (1999).

[28] M. Rostamian, S. S. Dragomir and M. D. L. Sen, Estimation type results related to Fejér inequality with applications, J. Inequal. Appl, 2018 (2018), Art. ID 85.

[29] E. Set, A. O. Akdemir and I. Mumcu, The Hermite-Hadamard type inequality and its extensions for conformable fractional integrals of any order $\alpha > 0$, Preprint.

[30] G. Toader, Some generalizations of the convexity, Proc. Colloq. Approx. Opt. Cluj-Napoca (Romania), University of Cluj-Napoca, 1984, 329-338.

[31] G. Toader, The order of starlike convex function, Bull. Appl. Comp. Math. 85(1998), 347-350.

[32] G. Stampacchia, Formes bilineaires coercivvities sur les ensembles convexes, C. R. Acad. Sci. Paris, 258(1964), 4413-4416.