



## COMPUTING SPECIAL SMARANDACHE CURVES ACCORDING TO DARBOUX FRAME IN EUCLIDEAN 4-SPACE

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**ABSTRACT.** In this paper, we study some special Smarandache curves and their differential geometric properties according to Darboux frame in Euclidean 4-space  $\mathbb{E}^4$ . Also, we compute some of these curves which lie fully on a hypersurface in  $\mathbb{E}^4$ . Moreover, we defray some computational examples in support our main results.

### 1. INTRODUCTION

The geometric modeling of free-form curves and surfaces is of central importance for sophisticated CAD/CAM systems. Among all space curves, Smarandache curves have special emplacement regarding their properties, because of this, they deserve especial attention in Euclidean geometry as well as in other geometries. It is known that Smarandache geometry is a geometry which has at least one Smarandache denied axiom [1]. An axiom is said to be Smarandache denied, if it behaves in at least two different ways within the same space. Smarandache geometries are connected with the theory of relativity and the parallel universes. Smarandache curves are the objects of Smarandache geometry. By definition, if the position

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Received 2019-03-27; accepted 2019-04-24; published 2019-07-01.

2010 *Mathematics Subject Classification.* 53A04, 53A07, 53A35.

*Key words and phrases.* Smarandache curves; hypersurfaces; Darboux frame; Euclidean 4-space.

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vector of a curve  $\delta$  is composed by Frenet frame's vectors of another curve  $\beta$ , then the curve  $\delta$  is called a Smarandache curve [2].

In *differential geometry*, frame fields constitute an important tool while studying curves and surfaces. The most familiar frame fields are Frenet frame along a space curve and Darboux frame along a surface curve [3]. It is analogous to the Frenet frame as applied to surface geometry. The Darboux frame exists at any non-umbilic point of a surface embedded in Euclidean space. In [4] M. Dülkü et al. extend the Darboux frame field into Euclidean 4-space  $\mathbb{E}^4$ .

In the light of the existing studies in the field of geometry, many interesting results on Smarandache curves have been obtained by many mathematicians, see for example, [2, 3, 5–12]. Turgut and Yilmaz [11] have introduced a particular circumstance of such curves, they entitled it Smarandache **TB**<sub>2</sub> curves in the space  $\mathbb{E}_4^1$ . They studied special Smarandache curves which are defined by the tangent and second binormal vector fields. In [8], the author has illustrated certain special Smarandache curves in the Euclidean 3-space.

Recently, H.S. Abdel-Aziz and M. Khalifa Saad [2, 5] have studied special Smarandache curves of an arbitrary curve such as **TN**, **TB** and **TNB** with respect to Frenet frame in the three-dimensional Galilean and pseudo-Galilean spaces. Also, in [6, 7] they have investigated Smarandache curves according to Darboux frame in the three-dimensional Minkowski space  $\mathbb{E}_1^3$ .

The main goal of this article is to investigate Smarandache curves in  $\mathbb{E}^4$  for a given curve with reference to its Darboux frame of first kind. The results presented in this paper generalize and refine some of the existing results in the literature and significant in mathematical modeling and other applications.

## 2. BASIC NOTIONS AND PROPERTIES

We now review some basic concepts on classical differential geometry of space curves and surfaces in Euclidean 4-space  $\mathbb{E}^4$  [4, 13–15]. Let  $\{e_i \mid i = 1, 2, 3, 4\}$  be the standard basis of  $\mathbb{E}^4$ , therefore  $\mathbb{E}^4 = \{X = \sum_{i=1}^4 x_i e_i \mid x_i \in \mathbb{R}\}$ . The scalar product and vector product of vectors  $X, Y, Z \in \mathbb{E}^4$  are respectively defined by

$$\langle X, Y \rangle = \sum_{i=1}^4 x_i y_i ,$$

$$X \times Y \times Z = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} .$$

**Definition 2.1.** Let  $f : D \subset \mathbb{E}^4 \rightarrow \mathbb{R}$  be a differentiable mapping of an open set  $D$ . Given  $c \in \mathbb{R}$ , we recall that the level set  $c$  of the  $f$  is the set defined as  $f^{-1}(c)$  which is the set of solutions in  $D$  of the equations  $f(x, y, z, w) = c$ .

**Proposition 2.1.** Consider that  $f : U \subset \mathbb{E}^4 \rightarrow \mathbb{R}$  is a differentiable function and  $c \in f(U)$  is a regular value of  $f$ , then  $f^{-1}(c)$  is a regular surface in  $\mathbb{E}^4$ . The implicit surface  $f$  is regular if  $\nabla f = (f_x, f_y, f_z, f_w) \neq 0$ . The unit surface normal vector of the implicit surface  $f$  is given by  $\mathbf{N} = \frac{\nabla f}{\|\nabla f\|}$ .

**Definition 2.2.** Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , if all second partial derivatives of  $f$  exist and are continuous on the domain of  $f$ , then the Hessian matrix  $H_f$  is a square  $n \times n$  matrix, usually defined as  $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ .

2.1. Curves on a hypersurface in  $\mathbb{E}^4$ .

Let  $r : I \subset \mathbb{R} \rightarrow M$  be a regular curve in  $\mathbb{E}^4$ , where  $\|r'\| = \|dr/du\| = 1, \forall u \in I$ , we will omit  $u$  for simplification. Then the Frenet frame is defined by

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}'_1 \\ \mathbf{b}'_2 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & \kappa_3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}, \left( ' = \frac{d}{du} \right) \tag{2.1}$$

where  $\mathbf{t}, \mathbf{n}, \mathbf{b}_1$  and  $\mathbf{b}_2$  are the tangent, principal normal, first binormal and the second binormal vector fields of  $r$ , respectively and the functions  $\kappa_i \mid i = 1, 2, 3$ , are the curvature functions of  $r$ .

**Theorem 2.1.** [2] Let  $r : I \rightarrow \mathbb{E}^4$  be a regular curve. Then

$$\mathbf{t} = \frac{r'}{\|r'\|}, \mathbf{b}_2 = \frac{r' \times r'' \times r^{(3)}}{\|r' \times r'' \times r^{(3)}\|}, \mathbf{b}_1 = \frac{b_2 \times r' \times r''}{\|b_2 \times r' \times r''\|}, \mathbf{n} = \frac{b_1 \times b_2 \times r'}{\|b_1 \times b_2 \times r'\|}, \tag{2.2}$$

and

$$\kappa_1 = \frac{\langle n, r'' \rangle}{\|r'\|^2}, \kappa_2 = \frac{\langle b_1, r^{(3)} \rangle}{\|r'\|^3 \kappa_1}, \kappa_3 = \frac{\langle b_2, r^{(4)} \rangle}{\|r'\|^4 \kappa_1 \kappa_2}. \tag{2.3}$$

**Definition 2.3.** Let  $r$  be a regular curve in  $\mathbb{E}^4$ , which is parameterized by arc length and  $n$ -times continuously differentiable. Then  $r$  is called a Frenet curve, if at every point the vectors  $r', r'', \dots, r^{(n-1)}$  are linearly independent.

2.2. Darboux frame field of first kind in  $\mathbb{E}^4$ .

Let  $M$  be an oriented hypersurface in  $\mathbb{E}^4$  and  $r$  be a Frenet curve of class  $\mathcal{C}^n (n \geq 4)$  with arc-length parameter  $u$  lying on  $M$ . We denote the unit tangent and unit normal vector fields of the curve by  $\mathbf{T}$  and  $\mathbf{N}$ , and  $\mathbf{P}, \mathbf{U} \in T_u M$ .

**Definition 2.4.** Let  $r : I \rightarrow \mathbb{E}^4$  be a regular parameterized curve lying on  $M$  in  $\mathbb{E}^4$ , then the frame field  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  along  $r$  is called extended Darboux frame field of first kind if  $\{\mathbf{T}, \mathbf{N}, r''\}$  is linearly independent, therefore  $\mathbf{T} = \frac{r'}{\|r'\|}, \mathbf{N} = \frac{\nabla f}{\|\nabla f\|}, \mathbf{P} = \frac{r'' - (r'' \cdot \mathbf{N})\mathbf{N}}{\|r'' - (r'' \cdot \mathbf{N})\mathbf{N}\|}, \mathbf{U} = [\mathbf{T}, \mathbf{N}, \mathbf{E}]$

The derivatives of the Darboux frame field of first kind in  $\mathbb{E}^4$  are given by [16]

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{P}' \\ \mathbf{U}' \\ \mathbf{N}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g^1 & 0 & \kappa_n \\ -\kappa_g^1 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & \tau_g^2 \\ -\kappa_n & -\tau_g^1 & -\tau_g^2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{P} \\ \mathbf{U} \\ \mathbf{N} \end{pmatrix}, \tag{2.4}$$

where  $\kappa_n, \kappa_g^1, \kappa_g^2, \tau_g^1$  and  $\tau_g^2$  are real valued functions denote the normal curvature, geodesic curvatures and the geodesic torsions, respectively. These functions are given by

$$\begin{aligned} \kappa_n &= \langle \mathbf{T}', \mathbf{N} \rangle = \frac{1}{\|\nabla f\|} \langle r'', \nabla f \rangle = \frac{-1}{\|\nabla f\|} (r' H_f (r')^t), \\ \kappa_g^1 &= \langle \mathbf{T}', \mathbf{P} \rangle = \left( r'' (r')^t - \frac{1}{\|\nabla f\|^2} (r' H_f (r')^t)^2 \right)^{\frac{1}{2}}, \\ \kappa_g^2 &= \langle \mathbf{P}', \mathbf{U} \rangle = \frac{1}{(\kappa_g^1)^2 \|\nabla f\|} \left( m_1 + \frac{1}{\|\nabla f\|^2} (r' H_f (r')^t) m_2 \right), \\ \tau_g^1 &= \langle \mathbf{P}', \mathbf{N} \rangle = \frac{-1}{(\kappa_g^1)} \left( \frac{1}{\|\nabla f\|} (r' H_f (r'')^t) + \frac{1}{\|\nabla f\|^3} (r' H_f (\nabla f)^t) (r' H_f (r')^t) \right), \\ \tau_g^2 &= \langle \mathbf{U}', \mathbf{N} \rangle = \left( \frac{-1}{(\kappa_g^1) \|\nabla f\|^2} \right) m_3, \end{aligned} \tag{2.5}$$

where

$$m_1 = \begin{vmatrix} x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \\ x^{(3)} & y^{(3)} & z^{(3)} & w^{(3)} \\ f_x & f_y & f_z & f_w \end{vmatrix}, \quad m_2 = \begin{vmatrix} x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \\ a & b & c & d \\ f_x & f_y & f_z & f_w \end{vmatrix}, \quad m_3 = \begin{vmatrix} x' & y' & z' & w' \\ f_x & f_y & f_z & f_w \\ a & b & c & d \\ p & q & t & v \end{vmatrix},$$

and

$$\begin{aligned} [a, b, c, d] &= (\nabla f)' = r' H_f, \quad [p, q, t, v] = (\nabla f)'' = \alpha'' H_f + \alpha' \frac{dH_f'}{du}, \\ \frac{dH_f}{du} &= \left[ \frac{\partial H_f}{\partial x} (r')^t \dots \frac{\partial H_f}{\partial w} (r')^t \right]. \end{aligned}$$

**Remark 2.1.** [2] Let  $r : I \rightarrow \mathbb{E}^4$  be a regular parameterized curve in  $\mathbb{E}^4$ , then

- $r$  is called asymptotic curve if and only if  $\kappa_n = 0$ .
- $r$  is called a line of curvature if and only if  $\tau_g^1 = \tau_g^2 = 0$ .

### 3. FIRST KIND SMARANDACHE CURVES IN $\mathbb{E}^4$

Consider  $r = r(u)$  is a curve lying fully on an oriented hypersurface  $M$  in  $\mathbb{E}^4$ . Let  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  be a Darboux frame field of first kind along  $r(u)$  and  $\kappa_n, \kappa_g^1, \kappa_g^2, \tau_g^1, \tau_g^2$  are real valued functions in arc length parameter  $u$  of  $r$ . So, we have the following definition.

**Definition 3.1.** [11] A regular curve  $\alpha(s(u))$  in  $\mathbb{E}^4$ , whose position vector is obtained by extended Darboux frame vectors of another regular curve  $r(u)$  is called Smarandache curve .

In the following we continue our studies of special Smarandache curves that we started in [2, 5, 6]. Here we investigate some special Smarandache curves of first kind called **TP**, **TU**, **PU** and **PN** ( the other special Smarandache curves can be computed in the same manner) and then obtain some of their differential geometric properties which represent the main results.

Let  $r(u) = (x(u), y(u), z(u), w(u))$  be a curve of class  $C^n (n \geq 4)$  lying on  $M$ . Then, by using proposition 2.1, the unit normal vector field along  $r$  is given by

$$\bar{\mathbf{N}} = \frac{\nabla f}{\|\nabla f\|}. \tag{3.1}$$

3.1. **TP-Smarandache curves.**

**Definition 3.2.** Let  $M$  be an oriented hypersurface in  $\mathbb{E}^4$  and the Frenet curve  $r = r(u)$  lying fully on  $M$  with Darboux frame  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  and non-zero curvatures  $\kappa_n, \kappa_g^1, \kappa_g^2, \tau_g^1$  and  $\tau_g^2$ . Then the **TP-Smarandache** curve of  $r$  is defined as

$$\alpha(s) = \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{P}). \tag{3.2}$$

**Theorem 3.1.** Let  $r = r(u)$  be a Frenet curve lying on a hypersurface  $M$  in  $\mathbb{E}^4$  with Darboux frame  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  and non-zero constant curvatures  $\kappa_n, \kappa_g^1, \kappa_g^2, \tau_g^1$  and  $\tau_g^2$ . Then the curvature functions of the **TP-Smarandache** curve of  $r$  satisfy the following equations:

$$\begin{aligned} \bar{\kappa}_n &= \frac{1}{\lambda_1 \|\nabla f\|} \begin{pmatrix} (\kappa_g^1(-\kappa_n + \tau_g^1) + \kappa_g^2 \tau_g^2) f_w - ((\kappa_g^1)^2 + \\ \kappa_n(\kappa_n + \tau_g^1)) f_x - ((\kappa_g^1)^2 + (\kappa_g^2)^2 + \\ \tau_g^1(\kappa_n + \tau_g^1)) f_y + (\kappa_g^1 \kappa_g^2 - (\kappa_n + \tau_g^1) \tau_g^2) f_z \end{pmatrix}, \\ \bar{\kappa}_g^1 &= \frac{1}{\lambda_1 \lambda_6} \begin{pmatrix} (\kappa_g^1(-\kappa_n + \tau_g^1) + \kappa_g^2 \tau_g^2) \lambda_5 + ((\kappa_g^1)^2 + \\ \kappa_n(\kappa_n + \tau_g^1)) \lambda_2 + ((\kappa_g^1)^2 + (\kappa_g^2)^2 + \\ \tau_g^1(\kappa_n + \tau_g^1)) \lambda_3 + (\kappa_g^1 \kappa_g^2 - (\kappa_n + \tau_g^1) \tau_g^2) \lambda_4 \end{pmatrix}, \\ \bar{\kappa}_g^2 &= \frac{1}{\lambda_6^2 \lambda_{11}} \begin{pmatrix} -(\lambda_2 \lambda_7 + \lambda_3 \lambda_8 + \lambda_4 \lambda_9 + \lambda_5 \lambda_{10}) \lambda'_6 + \\ \lambda_6((\kappa_g^1 \lambda_2 - \kappa_g^2 \lambda_4) \lambda_8 - \lambda_5(\kappa_n \lambda_7 + \tau_g^1 \lambda_8 + \\ \tau_g^2 \lambda_9) + \lambda_3(-\kappa_g^1 \lambda_7 + \kappa_g^2 \lambda_9 + \tau_g^1 \lambda_{10}) + \lambda_7 \lambda'_2 + \\ \lambda_8 \lambda'_3 + \lambda_9 \lambda'_4 + \lambda_{10}(\kappa_n \lambda_2 + \tau_g^2 \lambda_4 + \lambda'_5)) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \bar{\tau}_g^1 &= \frac{1}{\lambda_6^2 \|\nabla f\|} \begin{pmatrix} -(f_x \lambda_2 + f_y \lambda_3 + f_z \lambda_4 + f_w \lambda_5) \lambda'_6 + \\ \lambda_6 (-\kappa_g^1 f_x \lambda_3 + \kappa_g^2 f_z \lambda_3 - \kappa_n f_x \lambda_5 - \\ \tau_g^2 f_z \lambda_5 + f_x \lambda'_2 + f_y (\kappa_g^1 \lambda_2 - \kappa_g^2 \lambda_4 - \tau_g^1 \lambda_5 + \\ \lambda'_3) + f_z \lambda'_4 + f_w (\kappa_n \lambda_2 + \tau_g^1 \lambda_3 + \tau_g^2 \lambda_4 + \lambda'_5)) \end{pmatrix}, \\ \bar{\tau}_g^2 &= \frac{1}{\lambda_{11}^2 \|\nabla f\|} \begin{pmatrix} -(f_x \lambda_7 + f_y \lambda_8 + f_z \lambda_9 + f_w \lambda_{10}) \lambda'_{11} + \\ \lambda_{11} (-\kappa_g^1 f_x \lambda_8 + \kappa_g^2 f_z \lambda_8 - \kappa_n f_x \lambda_{10} - \\ \tau_g^2 f_z \lambda_{10} + f_x \lambda'_7 + f_y (\kappa_g^1 \lambda_7 - \kappa_g^2 \lambda_9 - \tau_g^1 \lambda_{10} + \\ \lambda'_8) + f_z \lambda'_9 + f_w (\kappa_n \lambda_7 + \tau_g^1 \lambda_8 + \tau_g^2 \lambda_9 + \lambda'_{10})) \end{pmatrix}. \end{aligned} \tag{3.3}$$

*Proof.* Because  $\alpha = \alpha(s)$  is a **TP**- Smarandache curve reference to Frenet curve  $r$ , then differentiating Eq. (3.2), we get

$$\alpha' = \frac{1}{\sqrt{2}} (-\kappa_g^1 \mathbf{T} + \kappa_g^1 \mathbf{P} + \kappa_g^2 \mathbf{U} + (\tau_g^1 + \kappa_n) \mathbf{N}). \tag{3.4}$$

Again, differentiating Eq. (3.4), we obtain

$$\alpha'' = \frac{1}{\sqrt{2}} \begin{pmatrix} (-\kappa_g^1)^2 - \kappa_n (\kappa_n + \tau_g^1) \mathbf{T} - ((\kappa_g^1)^2 + (\kappa_g^2)^2 + \tau_g^1 (\kappa_n + \tau_g^1)) \mathbf{P} \\ + (\kappa_g^1 \kappa_g^2 - (\kappa_n + \tau_g^1) \tau_g^2) \mathbf{U} + (-\kappa_g^1 \kappa_n + \kappa_g^1 \tau_g^1 + \kappa_g^2 \tau_g^2) \mathbf{N} \end{pmatrix}.$$

Also, Eq. (3.4) leads to

$$\bar{\mathbf{T}} = \frac{-\kappa_g^1 \mathbf{T} + \kappa_g^1 \mathbf{P} + \kappa_g^2 \mathbf{U} + (\tau_g^1 + \kappa_n) \mathbf{N}}{\lambda_1},$$

where

$$\lambda_1 = \sqrt{2(\kappa_g^1)^2 + (\kappa_g^2)^2 + (\tau_g^1 + \kappa_n)^2},$$

and we get

$$\bar{\mathbf{N}} = \frac{f_x \mathbf{T} + f_y \mathbf{P} + f_z \mathbf{U} + f_w \mathbf{N}}{\|\nabla f\|}.$$

On the other hand, we have

$$\bar{\mathbf{P}} = \frac{\lambda_2 \mathbf{T} + \lambda_3 \mathbf{P} + \lambda_4 \mathbf{U} + \lambda_5 \mathbf{N}}{\lambda_6},$$

where

$$\begin{aligned} \lambda_2 &= \frac{1}{\sqrt{2}} ((\kappa_g^1)^2 + (\kappa_n)^2 + \kappa_n \tau_g^1) + \frac{\sigma_1}{\|\nabla f\|} f_x, \\ \lambda_3 &= \frac{1}{\sqrt{2}} ((\kappa_g^1)^2 + (\kappa_g^2)^2 + \kappa_n \tau_g^1 + (\tau_g^1)^2) + \frac{\sigma_1}{\|\nabla f\|} f_y, \\ \lambda_4 &= \frac{1}{\sqrt{2}} (-\kappa_g^1 \kappa_g^2 + \kappa_n \tau_g^2 + \tau_g^1 \tau_g^2) + \frac{\sigma_1}{\|\nabla f\|} f_z, \\ \lambda_5 &= \frac{1}{\sqrt{2}} (\kappa_g^1 \kappa_n - \kappa_g^1 \tau_g^1 - \kappa_g^2 \tau_g^2) + \frac{\sigma_1}{\|\nabla f\|} f_w, \\ \lambda_6 &= \sqrt{(\lambda_2)^2 + (\lambda_3)^2 + (\lambda_4)^2 + (\lambda_5)^2}, \end{aligned}$$

$$\sigma_1 = \langle \alpha'', \mathbf{N} \rangle = \frac{1}{\sqrt{2}\|\nabla f\|} \begin{pmatrix} -(\kappa_g^1)^2 - \kappa_n(\kappa_n + \tau_g^1))f_x - ((\kappa_g^1)^2 + (\kappa_g^2)^2 + \\ \tau_g^1(\kappa_n + \tau_g^1))f_y + (\kappa_g^1\kappa_g^2 - (\kappa_n + \tau_g^1)\tau_g^2)f_z + \\ (-\kappa_g^1\kappa_n + \kappa_g^1\tau_g^1 + \kappa_g^2\tau_g^2)f_w \end{pmatrix}.$$

Also, we get

$$\bar{\mathbf{U}} = \frac{\lambda_7\mathbf{T} + \lambda_8\mathbf{P} + \lambda_9\mathbf{U} + \lambda_{10}\mathbf{N}}{\lambda_{11}},$$

where,

$$\begin{aligned} \lambda_7 &= \begin{pmatrix} \kappa_g^2 f_w \lambda_3 - \kappa_n f_z \lambda_3 - \tau_g^1 f_z \lambda_3 - \kappa_g^1 f_w \lambda_4 + \\ \kappa_n f_y \lambda_4 + \tau_g^1 f_y \lambda_4 - \kappa_g^2 f_y \lambda_5 + \kappa_g^1 f_z \lambda_5 \end{pmatrix}, \\ \lambda_8 &= \begin{pmatrix} -\kappa_g^2 f_w \lambda_2 + \kappa_n f_z \lambda_2 + \tau_g^1 f_z \lambda_2 - \kappa_g^1 f_w \lambda_4 - \\ \kappa_n f_x \lambda_4 - \tau_g^1 f_x \lambda_4 + \kappa_g^2 f_x \lambda_5 + \kappa_g^1 f_z \lambda_5 \end{pmatrix}, \\ \lambda_9 &= \begin{pmatrix} \kappa_g^1 f_w \lambda_2 - \kappa_n f_y \lambda_2 - \tau_g^1 f_y \lambda_2 + \kappa_g^1 f_w \lambda_3 + \\ \kappa_n f_x \lambda_3 + \tau_g^1 f_x \lambda_3 - \kappa_g^1 f_x \lambda_5 - \kappa_g^1 f_y \lambda_5 \end{pmatrix}, \\ \lambda_{10} &= \left( \kappa_g^2 f_y \lambda_2 - \kappa_g^1 f_z \lambda_2 - \kappa_g^2 f_x \lambda_3 - \kappa_g^1 f_z \lambda_3 + \kappa_g^1 f_x \lambda_4 + \kappa_g^1 f_y \lambda_4 \right), \\ \lambda_{11} &= \lambda_1 \lambda_6 \|\nabla f\|. \end{aligned}$$

In the light of the above calculations, the curvature functions  $\bar{\kappa}_n, \bar{\kappa}_g^1, \bar{\kappa}_g^2, \bar{\tau}_g^1$  and  $\bar{\tau}_g^2$  of  $\alpha$  are computed as in Eqs. (3.3). □

**Corollary 3.1.** *If  $r$  is an asymptotic curve. Then, the following equations hold:*

$$\begin{aligned} \bar{\mathbf{T}} &= \frac{-\kappa_g^1\mathbf{T} + \kappa_g^1\mathbf{P} + \kappa_g^2\mathbf{U} + \tau_g^1\mathbf{N}}{\lambda_1}, \quad \bar{\mathbf{N}} = \frac{f_x\mathbf{T} + f_y\mathbf{P} + f_z\mathbf{U} + f_w\mathbf{N}}{\|\nabla f\|}, \\ \bar{\mathbf{P}} &= \frac{\lambda_{13}\mathbf{T} + \lambda_{14}\mathbf{P} + \lambda_{15}\mathbf{U} + \lambda_{16}\mathbf{N}}{\lambda_{17}}, \quad \bar{\mathbf{U}} = \frac{\lambda_{18}\mathbf{T} + \lambda_{19}\mathbf{P} + \lambda_{20}\mathbf{U} + \lambda_{21}\mathbf{N}}{\lambda_{22}}, \end{aligned}$$

and then the curvature functions are computed as follows

$$\begin{aligned} \bar{\kappa}_n &= \frac{(\kappa_g^1\tau_g^1 + \kappa_g^2\tau_g^2)f_w - (\kappa_g^1)^2 f_x - ((\kappa_g^1)^2 + (\kappa_g^2)^2 + (\tau_g^1)^2)f_y + (\kappa_g^1\kappa_g^2 - \tau_g^1\tau_g^2)f_z}{\lambda_{12}\|\nabla f\|}, \\ \bar{\kappa}_g^1 &= \frac{(\kappa_g^1\tau_g^1 + \kappa_g^2\tau_g^2)\lambda_{16} - (\kappa_g^1)^2\lambda_{13} - ((\kappa_g^1)^2 + (\kappa_g^2)^2 + (\tau_g^1)^2)\lambda_{14} + (\kappa_g^1\kappa_g^2 - \tau_g^1\tau_g^2)\lambda_{15}}{\lambda_{12}\lambda_{17}}, \\ \bar{\kappa}_g^2 &= \frac{1}{\lambda_{22}\lambda_{17}^2} \begin{pmatrix} \lambda_{17}(\kappa_g^1\lambda_{13}\lambda_{19} - \kappa_g^2\lambda_{15}\lambda_{19} - \tau_g^1\lambda_{16}\lambda_{19} - \\ \tau_g^2\lambda_{16}\lambda_{20} + \lambda_{14}(-\kappa_g^1\lambda_{18} + \kappa_g^2\lambda_{20}) + \lambda_{18}\lambda'_{13} + \\ \lambda_{19}\lambda'_{14} + \lambda_{20}\lambda'_{15} + \lambda_{21}(\tau_g^1\lambda_{14} + \tau_g^2\lambda_{15} + \\ \lambda'_{16})) - (\lambda_{21}\lambda_{16} + \lambda_{13}\lambda_{18} + \lambda_{14}\lambda_{19} + \lambda_{15}\lambda_{20})\lambda'_{17} \end{pmatrix}, \end{aligned}$$

$$\bar{\tau}_g^1 = \frac{1}{\lambda_{17}^2 \|\nabla f\|} \begin{pmatrix} -f_x(\kappa_g^1 \lambda_{14} \lambda_{17} - \lambda_{17} \lambda'_{13} + \lambda_{13} \lambda'_{17}) + f_y(\lambda_{17}(\kappa_g^1 \lambda_{13} - \kappa_g^2 \lambda_{15} - \tau_g^1 \lambda_{16} + \lambda'_{14}) - \lambda_{14} \lambda'_{17}) + \\ f_z(\lambda_{17}(\kappa_g^2 \lambda_{14} - \tau_g^2 \lambda_{16} + \lambda'_{15}) - \lambda_{15} \lambda'_{17}) + \\ f_w(\lambda_{17}(\tau_g^1 \lambda_{14} + \tau_g^2 \lambda_{15} + \lambda'_{16}) - \lambda_{16} \lambda'_{17}) \end{pmatrix},$$

$$\bar{\tau}_g^2 = \frac{1}{\lambda_{22}^2 \|\nabla f\|} \begin{pmatrix} -(\lambda_{21} f_w + \lambda_{18} f_x + \lambda_{19} f_y + \lambda_{20} f_z) \lambda'_{22} + \lambda_{22} ((-\tau_g^1 \lambda_{21} + \kappa_g^1 \lambda_{18}) f_y + \lambda_{20} (\tau_g^2 f_w - \kappa_g^2 f_y) + \lambda_{19} (\tau_g^1 f_w - \kappa_g^1 f_x + \kappa_g^2 f_z) + f_w \lambda'_{21} + f_x \lambda'_{18} + f_y \lambda'_{19} + f_z (-\tau_g^2 \lambda_{21} + \lambda'_{20})) \end{pmatrix},$$

where

$$\begin{aligned} \lambda_{12} &= \sqrt{2(\kappa_g^1)^2 + (\kappa_g^2)^2 + (\tau_g^1)^2}, \\ \lambda_{13} &= \sqrt{2}(\kappa_g^1)^2 \|\nabla f\| + 2\sigma_1 f_x, \\ \lambda_{14} &= \sqrt{2}((\kappa_g^1)^2 + (\kappa_g^2)^2 + (\tau_g^1)^2) \|\nabla f\| + 2\sigma_1 f_y, \\ \lambda_{15} &= \sqrt{2}(-\kappa_g^1 \kappa_g^2 + \tau_g^1 \tau_g^2) \|\nabla f\| + 2\sigma_1 f_z, \\ \lambda_{16} &= \sqrt{2}(-\kappa_g^1 \tau_g^1 - \kappa_g^2 \tau_g^2) \|\nabla f\| + 2\sigma_1 f_w, \\ \lambda_{17} &= \sqrt{(\lambda_{13})^2 + (\lambda_{14})^2 + (\lambda_{15})^2 + (\lambda_{16})^2}, \\ \lambda_{18} &= \kappa_g^2 f_w \lambda_{14} - \tau_g^1 f_z \lambda_{14} - \kappa_g^1 f_w \lambda_{15} + \tau_g^1 f_y \lambda_{15} - \kappa_g^2 f_y \lambda_{16} + \kappa_g^1 f_z \lambda_{16}, \\ \lambda_{19} &= -\kappa_g^2 f_w \lambda_{13} + \tau_g^1 f_z \lambda_{13} - \kappa_g^1 f_w \lambda_{15} - \tau_g^1 f_x \lambda_{15} + \kappa_g^2 f_x \lambda_{16} + \kappa_g^1 f_z \lambda_{16}, \\ \lambda_{20} &= k_g^1 \lambda_{13} f_w + k_g^1 \lambda_{14} f_w + \tau_g^1 \lambda_{14} f_x - k_g^1 \lambda_{16} f_x - \tau_g^1 \lambda_{13} f_y - k_g^1 \lambda_{16} f_y, \\ \lambda_{21} &= \kappa_g^2 f_y \lambda_{13} - \kappa_g^1 f_z \lambda_{13} - \kappa_g^2 f_x \lambda_{14} - \kappa_g^1 f_z \lambda_{14} + \kappa_g^1 f_x \lambda_{15} + \kappa_g^1 f_y \lambda_{15}, \\ \lambda_{22} &= \lambda_{12} \lambda_{17} \|\nabla f\|, \\ \sigma_1 &= \frac{1}{\sqrt{2} \|\nabla f\|} \begin{pmatrix} -(\kappa_g^1)^2 f_x - ((\kappa_g^1)^2 + (\kappa_g^2)^2 + (\tau_g^1)^2) f_y + \\ (\kappa_g^1 \kappa_g^2 - \tau_g^1 \tau_g^2) f_z + (\kappa_g^1 \tau_g^1 + \kappa_g^2 \tau_g^2) f_w \end{pmatrix}. \end{aligned}$$

**Corollary 3.2.** *If  $r$  is a line of curvature. Then, the following equations hold:*

$$\begin{aligned} \bar{\mathbf{T}} &= \frac{-\kappa_g^1 \mathbf{T} + \kappa_g^1 \mathbf{P} + \kappa_g^2 \mathbf{U} + \kappa_n \mathbf{N}}{\lambda_1}, \quad \bar{\mathbf{N}} = \frac{f_x \mathbf{T} + f_y \mathbf{P} + f_z \mathbf{U} + f_w \mathbf{N}}{\|\nabla f\|}, \\ \bar{\mathbf{P}} &= \frac{\lambda_{24} \mathbf{T} + \lambda_{25} \mathbf{P} + \lambda_{26} \mathbf{U} + \lambda_{27} \mathbf{N}}{\lambda_{28}}, \quad \bar{\mathbf{U}} = \frac{\lambda_{29} \mathbf{T} + \lambda_{30} \mathbf{P} + \lambda_{31} \mathbf{U} + \lambda_{32} \mathbf{N}}{\lambda_{33}}, \end{aligned}$$

and then the curvature functions are computed as follows

$$\begin{aligned} \bar{\kappa}_n &= -\frac{\kappa_g^1 \kappa_n f_w + ((\kappa_g^1)^2 + \kappa_n^2) f_x + ((\kappa_g^1)^2 + (\kappa_g^2)^2) f_y - \kappa_g^1 \kappa_g^2 f_z}{\lambda_{23} \|\nabla f\|}, \\ \bar{\kappa}_g^1 &= -\frac{\kappa_g^1 \kappa_n \lambda_{27} + ((\kappa_g^1)^2 + \kappa_n^2) \lambda_{24} + ((\kappa_g^1)^2 + (\kappa_g^2)^2) \lambda_{25} - \kappa_g^1 \kappa_g^2 \lambda_{26}}{\lambda_{23} \lambda_{28}}, \end{aligned}$$



$$\begin{aligned} \bar{\kappa}_g^2 &= \frac{1}{\lambda_{33}\lambda_{28}^2} \left( \begin{array}{l} \lambda_{28}(\lambda_{25}(-\kappa_g^1\lambda_{29} + \kappa_g^2\lambda_{31}) + \lambda_{29}(-\kappa_n\lambda_{27} + \lambda'_{24}) + \\ \lambda_{30}(\kappa_g^1\lambda_{24} - \kappa_g^2\lambda_{26} + \lambda'_{25}) + \lambda_{31}\lambda'_{26} + \lambda_{32}(\kappa_n\lambda_{24} + \\ \lambda'_{27})) - (\lambda_{32}\lambda_{27} + \lambda_{24}\lambda_{29} + \lambda_{25}\lambda_{30} + \lambda_{26}\lambda_{31})\lambda'_{28} \end{array} \right), \\ \bar{\tau}_g^1 &= \frac{1}{\lambda_{28}^2\|\nabla f\|} \left( \begin{array}{l} -f_x(\lambda_{28}(\kappa_g^1\lambda_{25} + \kappa_n\lambda_{27} - \lambda'_{24}) + \lambda_{24}\lambda'_{28}) + \\ f_y(\lambda_{28}(\kappa_g^1\lambda_{24} - \kappa_g^2\lambda_{26} + \lambda'_{25}) - \lambda_{25}\lambda'_{28}) + \\ f_z(\lambda_{28}(\kappa_g^2\lambda_{25} + \lambda'_{26}) - \lambda_{26}\lambda'_{28}) + \\ f_w(\lambda_{28}(\kappa_n\lambda_{24} + \lambda'_{27}) - \lambda_{27}\lambda'_{28}) \end{array} \right), \\ \bar{\tau}_g^2 &= \frac{1}{\lambda_{33}^2\|\nabla f\|} \left( \begin{array}{l} -(\lambda_{32}f_w + \lambda_{29}f_x + \lambda_{30}f_y + \lambda_{31}f_z)\lambda'_{33} + \\ \lambda_{33}(-\kappa_n\lambda_{32}f_x - \kappa_g^1\lambda_{30}f_x + \\ \lambda_{29}(\kappa_n f_w + \kappa_g^1 f_y) + \kappa_g^2\lambda_{30}f_z + f_w\lambda'_{32} + \\ f_x\lambda'_{29} + f_y(-\kappa_g^2\lambda_{31} + \lambda'_{30}) + f_z\lambda'_{31}) \end{array} \right), \end{aligned}$$

where

$$\begin{aligned} \lambda_{23} &= \sqrt{2(\kappa_g^1)^2 + (\kappa_g^2)^2 + (\kappa_n)^2}, \quad \lambda_{24} = \sqrt{2((\kappa_g^1)^2 + \kappa_n^2)\|\nabla f\| + 2\sigma_1 f_x}, \\ \lambda_{25} &= \sqrt{2((\kappa_g^1)^2 + (\kappa_g^2)^2)\|\nabla f\| + 2\sigma_1 f_y}, \quad \lambda_{26} = -\sqrt{2\kappa_g^1\kappa_g^2\|\nabla f\| + 2\sigma_1 f_z}, \\ \lambda_{27} &= \sqrt{2\kappa_g^1\kappa_n\|\nabla f\| + 2\sigma_1 f_w}, \quad \lambda_{28} = \sqrt{(\lambda_{24})^2 + (\lambda_{25})^2 + (\lambda_{26})^2 + (\lambda_{27})^2}, \\ \lambda_{29} &= \kappa_g^2 f_w \lambda_{25} - \kappa_n f_z \lambda_{25} - \kappa_g^1 f_w \lambda_{26} + \kappa_n f_y \lambda_{26} - \kappa_g^2 f_y \lambda_{27} + \kappa_g^1 f_z \lambda_{27}, \\ \lambda_{30} &= -\kappa_g^2 f_w \lambda_{24} + \kappa_n f_z \lambda_{24} - \kappa_g^1 f_w \lambda_{26} - \kappa_n f_x \lambda_{26} + \kappa_g^2 f_x \lambda_{27} + \kappa_g^1 f_z \lambda_{27}, \\ \lambda_{31} &= \kappa_g^1 f_w \lambda_{24} - \kappa_n f_y \lambda_{24} + \kappa_g^1 f_w \lambda_{25} + \kappa_n f_x \lambda_{25} - \kappa_g^1 f_x \lambda_{27} - \kappa_g^1 f_y \lambda_{27}, \\ \lambda_{32} &= \kappa_g^2 f_y \lambda_{24} - \kappa_g^1 f_z \lambda_{24} - \kappa_g^2 f_x \lambda_{25} - \kappa_g^1 f_z \lambda_{25} + \kappa_g^1 f_x \lambda_{26} + \kappa_g^1 f_y \lambda_{26}, \\ \lambda_{33} &= \lambda_{23} \lambda_{28} \|\nabla f\|, \\ \sigma_1 &= \frac{1}{\sqrt{2}\|\nabla f\|} (-\kappa_g^1\kappa_n f_w - ((\kappa_g^1)^2 + \kappa_n^2) f_x - ((\kappa_g^1)^2 + (\kappa_g^2)^2) f_y + \kappa_g^1\kappa_g^2 f_z). \end{aligned}$$

### 3.2. TU-Smarandache curves.

**Definition 3.3.** Let  $M$  be an oriented hypersurface in  $\mathbb{E}^4$  and the Frenet curve  $r = r(u)$  lying fully on  $M$  with Darboux frame  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  and non-zero curvatures  $\kappa_n, \kappa_g^1, \kappa_g^2, \tau_g^1$  and  $\tau_g^2$ . Then the **TU-Smarandache** curve of  $r$  is defined as

$$\beta(u) = \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{U}). \tag{3.5}$$

**Theorem 3.2.** Let  $r = r(u)$  be a Frenet curve lying on a hypersurface  $M$  in  $\mathbb{E}^4$  with Darboux frame  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  and non-zero constant curvatures;  $\kappa_n, \kappa_g^1, \kappa_g^2, \tau_g^1$  and  $\tau_g^2$ . Then the curvature functions of the

**TU**– Smarandache curve of  $r$  satisfy the following equations:

$$\begin{aligned}
 \bar{\kappa}_n &= \frac{1}{\mu_1 \|\nabla f\|} \left( \begin{array}{c} (\kappa_g^1 - \kappa_g^2) \tau_g^1 f_w - (\kappa_g^1 (\kappa_g^1 - \kappa_g^2) + \kappa_n (\kappa_n + \tau_g^2)) f_x + \\ (\kappa_g^1 - \kappa_g^2) \kappa_g^2 f_y - (\kappa_n + \tau_g^2) (\tau_g^1 f_y + \tau_g^2 f_z) \end{array} \right), \\
 \bar{\kappa}_g^1 &= \frac{1}{\mu_1 \mu_6} \left( \begin{array}{c} (\kappa_g^1 - \kappa_g^2) \tau_g^1 \mu_5 - (\kappa_g^1 (\kappa_g^1 - \kappa_g^2) + \kappa_n (\kappa_n + \tau_g^2)) \mu_2 + \\ (\kappa_g^1 - \kappa_g^2) \kappa_g^2 \mu_3 - (\kappa_n + \tau_g^2) (\tau_g^1 f_y + \tau_g^2 \mu_4) \end{array} \right), \\
 \bar{\kappa}_g^2 &= \frac{1}{\mu_6^2 \mu_{11}} \left( \begin{array}{c} -(\mu_2 \mu_7 + \mu_3 \mu_8 + \mu_4 \mu_9 + \mu_5 \mu_{10}) \mu'_6 + \mu_6 ((\kappa_g^1 \mu_2 - \\ \kappa_g^2 \mu_4) \mu_8 - \mu_5 (\kappa_n \mu_7 + \tau_g^1 \mu_8 + \tau_g^2 \mu_9) + \mu_3 (-\kappa_g^1 \mu_7 + \\ \kappa_g^2 \mu_9 + \tau_g^1 \mu_{10}) + \mu_7 \mu'_2 + \mu_8 \mu'_3 + \\ \mu_9 \mu'_4 + \mu_{10} (\kappa_n \mu_2 + \tau_g^2 \mu_4 + \mu'_5)) \end{array} \right), \\
 \bar{\tau}_g^1 &= \frac{1}{\mu_6^2 \|\nabla f\|} \left( \begin{array}{c} -(f_x \mu_2 + f_y \mu_3 + f_z \mu_4 + f_w \mu_5) \mu'_6 + \mu_6 (f_x \mu'_2 \\ -\kappa_g^1 f_x \mu_3 + \kappa_g^2 f_z \mu_3 - \kappa_n f_x \mu_5 - \tau_g^2 f_z \mu_5 + \\ f_y (\kappa_g^1 \mu_2 - \kappa_g^2 \mu_4 - \tau_g^1 \mu_5 + \mu'_3) + f_z \mu'_4 + \\ f_w (\kappa_n \mu_2 + \tau_g^1 \mu_3 + \tau_g^2 \mu_4 + \mu'_5)) \end{array} \right), \\
 \bar{\tau}_g^2 &= \frac{1}{\mu_{11}^2 \|\nabla f\|} \left( \begin{array}{c} -(f_x \mu_7 + f_y \mu_8 + f_z \mu_9 + f_w \mu_{10}) \mu'_{11} + \mu_{11} (f_x \mu'_7 \\ -\kappa_g^1 f_x \mu_8 + \kappa_g^2 f_z \mu_8 - \kappa_n f_x \mu_{10} - \tau_g^2 f_z \mu_{10} + \\ f_y (\kappa_g^1 \mu_7 - \kappa_g^2 \mu_9 - \tau_g^1 \mu_{10} + \mu'_8) + \\ f_z \mu'_9 + f_w (\kappa_n \mu_7 + \tau_g^1 \mu_8 + \tau_g^2 \mu_9 + \mu'_{10})) \end{array} \right). \tag{3.6}
 \end{aligned}$$

*Proof.* Since  $\beta = \beta(s)$  is a **TU**– Smarandache curve reference to Frenet curve  $r$ . Then, by differentiating Eq. (3.5), we get

$$\beta' = \frac{1}{\sqrt{2}} ((\kappa_g^1 - \kappa_g^2) \mathbf{P} + (\kappa_n + \tau_g^2) \mathbf{U}), \tag{3.7}$$

Again, by differentiating Eq. (3.7), we have

$$\beta'' = \frac{1}{\sqrt{2}} \left( \begin{array}{c} (-\kappa_g^1 (\kappa_g^1 - \kappa_g^2) - \kappa_n (\kappa_n + \tau_g^2)) \mathbf{T} - \tau_g^1 (\kappa_n + \tau_g^2) \mathbf{P} \\ + ((\kappa_g^1 - \kappa_g^2) \kappa_g^2 - (\kappa_n + \tau_g^2) \tau_g^2) \mathbf{U} + (\kappa_g^1 - \kappa_g^2) \tau_g^1 \mathbf{N} \end{array} \right).$$

Also, from Eq. (3.7), we obtain

$$\bar{\mathbf{T}} = \frac{(\kappa_g^1 - \kappa_g^2) \mathbf{P} + (\kappa_n + \tau_g^2) \mathbf{U}}{\mu_1},$$

where

$$\mu_1 = \sqrt{2(\kappa_g^1 - \kappa_g^2)^2 + (\kappa_n + \tau_g^2)^2},$$

and we have

$$\begin{aligned}
 \bar{\mathbf{N}} &= \frac{f_x \mathbf{T} + f_y \mathbf{P} + f_z \mathbf{U} + f_w \mathbf{N}}{\|\nabla f\|}, \\
 \bar{\mathbf{P}} &= \frac{\mu_2 \mathbf{T} + \mu_3 \mathbf{P} + \mu_4 \mathbf{U} + \mu_5 \mathbf{N}}{\mu_6},
 \end{aligned}$$

where

$$\begin{aligned} \mu_2 &= \frac{1}{\sqrt{2}} \left( (\kappa_g^1)^2 - \kappa_g^1 \kappa_g^2 + \kappa_n (\kappa_n + \tau_g^2) \right) + \frac{\sigma_2}{\|\nabla f\|} f_x, \\ \mu_3 &= \frac{1}{\sqrt{2}} \left( \tau_g^1 (\kappa_n + \tau_g^2) \right) + \frac{\sigma_2}{\|\nabla f\|} f_y, \\ \mu_4 &= \frac{1}{\sqrt{2}} \left( (\kappa_g^2 (-\kappa_g^1 + \kappa_g^2) + \tau_g^2 (\kappa_n + \tau_g^2)) \right) + \frac{\sigma_2}{\|\nabla f\|} f_z, \\ \mu_5 &= \frac{1}{\sqrt{2}} \left( (-\kappa_g^1 + \kappa_g^2) \tau_g^1 \right) + \frac{\sigma_2}{\|\nabla f\|} f_w, \end{aligned}$$

$$\mu_6 = \sqrt{(\mu_2)^2 + (\mu_3)^2 + (\mu_4)^2 + (\mu_5)^2}$$

$$\sigma_2 = \langle \beta'', \mathbf{N} \rangle = \frac{1}{\sqrt{2} \|\nabla f\|} \left( \begin{aligned} &(-\kappa_g^1)^2 + \kappa_g^1 \kappa_g^2 - \kappa_n (\kappa_n + \tau_g^2) f_x + \tau_g^1 (\kappa_n + \tau_g^2) f_y \\ &+ ((\kappa_g^1 - \kappa_g^2) \kappa_g^2 - \tau_g^2 (\kappa_n + \tau_g^2)) f_z + (\kappa_g^1 - \kappa_g^2) \tau_g^1 f_w \end{aligned} \right).$$

Also, we get

$$\bar{\mathbf{U}} = \frac{\mu_7 \mathbf{T} + \mu_8 \mathbf{P} + \mu_3 \mathbf{U} + \mu_{10} \mathbf{N}}{\delta},$$

where

$$\begin{aligned} \mu_7 &= -\kappa_n f_z \mu_3 - \tau_g^2 f_z \mu_3 - \kappa_g^1 f_w \mu_4 + \kappa_g^2 f_w \mu_4 + \kappa_n f_y \mu_4 + \tau_g^2 f_y \mu_4 + \kappa_g^1 f_z \mu_5 - \kappa_g^2 f_z \mu_5, \\ \mu_8 &= \kappa_n f_z \mu_2 + \tau_g^2 f_z \mu_2 - \kappa_n f_x \mu_4 - \tau_g^2 f_x \mu_4, \\ \mu_9 &= \kappa_g^1 f_w \mu_2 - \kappa_g^2 f_w \mu_2 - \kappa_n f_y \mu_2 - \tau_g^2 f_y \mu_2 + \kappa_n f_x \mu_3 + \tau_g^2 f_x \mu_3 - \kappa_g^1 f_x \mu_5 + \kappa_g^2 f_x \mu_5, \\ \mu_{10} &= -\kappa_g^1 f_z \mu_2 + \kappa_g^2 f_z \mu_2 + \kappa_g^1 f_x \mu_4 - \kappa_g^2 f_x \mu_4, \\ \mu_{11} &= \mu_1 \mu_6 \|\nabla f\|. \end{aligned}$$

In the light of the above calculations, the curvature functions  $\bar{\kappa}_n, \bar{\kappa}_g^1, \bar{\kappa}_g^2, \bar{\tau}_g^1$  and  $\bar{\tau}_g^2$  of  $\beta$  are computed as in Eqs. (3.6). □

**Corollary 3.3.** *If  $r$  is an asymptotic curve. Then, the following equations hold:*

$$\begin{aligned} \bar{\mathbf{T}} &= \frac{(\kappa_g^1 - \kappa_g^2) \mathbf{P} + \tau_g^2 \mathbf{U}}{\mu_1}, \\ \bar{\mathbf{N}} &= \frac{f_x \mathbf{T} + f_y \mathbf{P} + f_z \mathbf{U} + f_w \mathbf{N}}{\|\nabla f\|}, \\ \bar{\mathbf{P}} &= \frac{\mu_{13} \mathbf{T} + \mu_{14} \mathbf{P} + \mu_{15} \mathbf{U} + \mu_{16} \mathbf{N}}{\mu_{17}}, \\ \bar{\mathbf{U}} &= \frac{\mu_{18} \mathbf{T} + \mu_{19} \mathbf{P} + \mu_{20} \mathbf{U} + \mu_{21} \mathbf{N}}{\mu_{22}}, \end{aligned}$$

and the curvature functions are obtained as follows:

$$\begin{aligned} \bar{\kappa}_n &= \frac{(\kappa_g^1 - \kappa_g^2)\tau_g^1 f_w + \kappa_g^1(-\kappa_g^1 + \kappa_g^2)f_x - \tau_g^1\tau_g^2 f_y - (-\kappa_g^1\kappa_g^2 + (\kappa_g^2)^2 + (\tau_g^2)^2)f_z}{\mu_{12}\|\nabla f\|}, \\ \bar{\kappa}_g^1 &= \frac{\kappa_g^1(-\kappa_g^1 + \kappa_g^2)\mu_{13} + (\kappa_g^1 - \kappa_g^2)\kappa_g^2\mu_{15} - \tau_g^2(\tau_g^1\mu_{14} + \tau_g^2\mu_{15}) + (\kappa_g^1 - \kappa_g^2)\tau_g^1\mu_{16}}{\mu_{12}\mu_{17}}, \\ \bar{\kappa}_g^2 &= \frac{1}{\mu_{17}^2\mu_{22}} \left( \begin{aligned} &\mu_{17}(\kappa_g^1\mu_{13}\mu_{19} - \kappa_g^2\mu_{15}\mu_{19} - \tau_g^1\mu_{16}\mu_{19} - \tau_g^2\mu_{16}\mu_{20} + \\ &\tau_g^2\mu_{15}\mu_{21} + \mu_{14}(-\kappa_g^1\mu_{18} + \kappa_g^2\mu_{20} + \tau_g^1\mu_{21}) + \mu_{18}\mu'_{13} + \mu_{19}\mu'_{14} + \\ &\mu_{20}\mu'_{15} + \mu_{21}\mu'_{16}) - (\mu_{13}\mu_{18} + \mu_{14}\mu_{19} + \mu_{15}\mu_{20} + \mu_{16}\mu_{21})\mu'_{17} \end{aligned} \right), \\ \bar{\tau}_g^1 &= \frac{1}{\mu_{17}^2\|\nabla f\|} \left( \begin{aligned} &-f_x(\kappa_g^1\mu_{14}\mu_{17} - \mu_{17}\mu'_{13} + \mu_{13}\mu'_{17}) + f_y(\mu_{17}(\kappa_g^1\mu_{13} - \\ &\kappa_g^2\mu_{15} - \tau_g^1\mu_{16} + \mu'_{14}) - \mu_{14}\mu'_{17}) + f_z(\mu_{17}(\kappa_g^2\mu_{14} - \tau_g^2\mu_{16} + \\ &\mu'_{15}) - \mu_{15}\mu'_{17}) + f_w(\mu_{17}(\tau_g^1\mu_{14} + \tau_g^2\mu_{15} + \mu'_{16}) - \mu_{16}\mu'_{17}) \end{aligned} \right), \\ \bar{\tau}_g^2 &= \frac{1}{\mu_{22}^2\|\nabla f\|} \left( \begin{aligned} &-f_x(\kappa_g^1\mu_{19}\mu_{22} - \mu_{22}\mu'_{18} + \mu_{18}\mu'_{22}) + f_y(\mu_{22}(\kappa_g^1\mu_{18} - \kappa_g^2\mu_{20} - \\ &\tau_g^1\mu_{21} + \mu'_{19}) - \mu_{19}\mu'_{22}) + f_z(\mu_{22}(\kappa_g^2\mu_{19} - \tau_g^2\mu_{21} + \mu'_{20}) - \\ &\mu_{20}\mu'_{22}) + f_w(\mu_{22}(\tau_g^1\mu_{19} + \tau_g^2\mu_{20} + \mu'_{21}) - \mu_{21}\mu'_{22}) \end{aligned} \right), \end{aligned}$$

where

$$\begin{aligned} \mu_{12} &= \sqrt{(\kappa_g^1 - \kappa_g^2)^2 + (\tau_g^2)^2}, \\ \mu_{13} &= \sqrt{2((\kappa_g^1)^2 - \kappa_g^1\kappa_g^2)\|\nabla f\| + 2\sigma_2 f_x}, \\ \mu_{14} &= \sqrt{2\tau_g^1\tau_g^2\|\nabla f\| + 2\sigma_2 f_y}, \\ \mu_{15} &= \sqrt{2(\kappa_g^2(-\kappa_g^1 + \kappa_g^2) + (\tau_g^2)^2)\|\nabla f\| + 2\sigma_2 f_z}, \\ \mu_{16} &= \sqrt{2(-\kappa_g^1 + \kappa_g^2)\tau_g^1\|\nabla f\| + 2\sigma_2 f_w}, \\ \mu_{17}^2 &= (\mu_{13})^2 + (\mu_{14})^2 + (\mu_{15})^2 + (\mu_{16})^2, \\ \mu_{18} &= -(\kappa_g^1\mu_{15}f_w - \kappa_g^2\mu_{15}f_w - \tau_g^2\mu_{15}f_y + \tau_g^2\mu_{14}f_z - \kappa_g^1\mu_{16}f_z + \kappa_g^2\mu_{16}f_z), \\ \mu_{19} &= -(\tau_g^2\mu_{15}f_x - \tau_g^2\mu_{13}f_z), \\ \mu_{20} &= -(-\kappa_g^1\mu_{13}f_w + \kappa_g^2\mu_{13}f_w - \tau_g^2\mu_{14}f_x + \kappa_g^1\mu_{16}f_x - \kappa_g^2\mu_{16}f_x + \tau_g^2\mu_{13}f_y), \\ \mu_{21} &= -(-\kappa_g^1\mu_{15}f_x + \kappa_g^2\mu_{15}f_x + \kappa_g^1\mu_{13}f_z - \kappa_g^2\mu_{13}f_z), \\ \mu_{22} &= \mu_{12} \mu_{17} \|\nabla f\|. \end{aligned}$$

**Corollary 3.4.** *If  $r$  is a line of curvature curve. Then, the following equations hold:*

$$\begin{aligned} \bar{\mathbf{T}} &= \frac{(\kappa_g^1 - \kappa_g^2)\mathbf{P} + \kappa_n\mathbf{N}}{\mu_{23}}, \\ \bar{\mathbf{N}} &= \frac{f_x\mathbf{T} + f_y\mathbf{P} + f_z\mathbf{U} + f_w\mathbf{N}}{\|\nabla f\|}, \\ \bar{\mathbf{P}} &= \frac{\mu_{24}\mathbf{T} + \mu_{25}\mathbf{P} + \mu_{26}\mathbf{U} + \mu_{27}\mathbf{N}}{\mu_{28}}, \\ \bar{\mathbf{U}} &= \frac{\mu_{29}\mathbf{T} + \mu_{30}\mathbf{P} + \mu_{31}\mathbf{U} + \mu_{32}\mathbf{N}}{\mu_{33}}, \end{aligned}$$

and the curvature functions are obtained as follows:

$$\begin{aligned} \bar{\kappa}_n &= \frac{-((\kappa_g^1)^2 - \kappa_g^1\kappa_g^2 + \kappa_n^2)f_x + (\kappa_g^1 - \kappa_g^2)\kappa_g^2f_z}{\mu_{23}\|\nabla f\|}, \\ \bar{\kappa}_g^1 &= \frac{-((\kappa_g^1)^2 - \kappa_g^1\kappa_g^2 + \kappa_n^2)\mu_{24} + (\kappa_g^1 - \kappa_g^2)\kappa_g^2\mu_{26}}{\mu_{23}\mu_{28}}, \\ \bar{\kappa}_g^2 &= \frac{1}{\mu_{17}^2\mu_{22}} \left( \begin{aligned} &\mu_{28}(-\kappa_n\mu_{27}\mu_{29} + \mu_{25}(-\kappa_g^1\mu_{29} + \kappa_g^2\mu_{31}) + \mu_{29}\mu'_{24} + \\ &\mu_{30}(\kappa_g^1\mu_{24} - \kappa_g^2\mu_{26} + \mu'_{25}) + \mu_{31}\mu'_{26} + \mu_{32}(\kappa_n\mu_{24} + \\ &\mu'_{27})) - (\mu_{24}\mu_{29} + \mu_{25}\mu_{30} + \mu_{26}\mu_{31} + \mu_{27}\mu_{32})\mu'_{28} \end{aligned} \right), \\ \bar{\tau}_g^1 &= \frac{1}{\mu_{28}^2\|\nabla f\|} \left( \begin{aligned} &-f_x(\mu_{28}(\kappa_g^1\mu_{25} + \kappa_n\mu_{27} - \mu'_{24}) + \mu_{24}\mu'_{28}) + \\ &f_y(\mu_{28}(\kappa_g^1\mu_{24} - \kappa_g^2\mu_{26} + \mu'_{25}) - \mu_{25}\mu'_{28}) + \\ &f_z(\mu_{28}(\kappa_g^2\mu_{25} + \mu'_{26}) - \mu_{26}\mu'_{28}) + \\ &f_w(\mu_{28}(\kappa_n\mu_{24} + \mu'_{27}) - \mu_{27}\mu'_{28}) \end{aligned} \right), \\ \bar{\tau}_g^2 &= \frac{1}{\mu_{33}^2\|\nabla f\|} \left( \begin{aligned} &-f_x(\mu_{33}(\kappa_g^1\mu_{30} + \kappa_n\mu_{32} - \mu'_{29}) + \mu_{29}\mu'_{33}) + \\ &f_y(\mu_{33}(\kappa_g^1\mu_{29} - \kappa_g^2\mu_{31} + \mu'_{30}) - \mu_{30}\mu'_{33}) + \\ &f_z(\mu_{33}(\kappa_g^2\mu_{30} + \mu'_{31}) - \mu_{31}\mu'_{33}) + \\ &f_w(\mu_{33}(\kappa_n\mu_{29} + \mu'_{32}) - \mu_{32}\mu'_{33}) \end{aligned} \right), \end{aligned}$$

where

$$\begin{aligned} \mu_{23} &= \sqrt{(\kappa_g^1 - \kappa_g^2)^2 + \kappa_n^2}, \quad \mu_{24} = \sqrt{2}((\kappa_g^1)^2 - \kappa_g^1\kappa_g^2 + \kappa_n^2)\|\nabla f\| + 2\sigma_2f_x, \\ \mu_{25} &= 2\sigma_2f_y, \quad \mu_{26} = \sqrt{2}\kappa_g^2(-\kappa_g^1 + \kappa_g^2)\|\nabla f\| + 2\sigma_2f_z, \\ \mu_{27} &= 2\sigma_2f_w, \quad \mu_{28}^2 = (\mu_{24})^2 + (\mu_{25})^2 + (\mu_{26})^2 + (\mu_{27})^2, \\ \mu_{29} &= -(\kappa_g^1\mu_{26}f_w - \kappa_g^2\mu_{26}f_w - \kappa_n\mu_{26}f_y + \kappa_n\mu_{25}f_z - \kappa_g^1\mu_{27}f_z + \kappa_g^2\mu_{27}f_z), \\ \mu_{30} &= -(\kappa_n\mu_{26}f_x - \kappa_n\mu_{25}f_z), \\ \mu_{31} &= -(-\kappa_g^1\mu_{24}f_w + \kappa_g^2\mu_{24}f_w - \kappa_n\mu_{25}f_x + \kappa_g^1\mu_{27}f_x - \kappa_g^2\mu_{27}f_x + \kappa_n\mu_{24}f_y), \\ \mu_{32} &= -(-\kappa_g^1\mu_{26}f_x + \kappa_g^2\mu_{26}f_x + \kappa_g^1\mu_{24}f_z - \kappa_g^2\mu_{24}f_z), \\ \mu_{33} &= \mu_{24} \mu_{28} \|\nabla f\|. \end{aligned}$$

### 3.3. PU-Smarandache curves.

**Definition 3.4.** Let  $M$  be an oriented hypersurface in  $\mathbb{E}^4$  and the Frenet curve  $\gamma = \gamma(s)$  lying fully on  $M$  with Darboux frame  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  and non-zero curvatures  $\kappa_n, \kappa_g^1, \kappa_g^2, \tau_g^1$  and  $\tau_g^2$ . Then the PU-Smarandache curve of  $\gamma$  is defined as

$$\gamma(u) = \frac{1}{\sqrt{2}}(\mathbf{P} + \mathbf{U}). \tag{3.8}$$

**Theorem 3.3.** Let  $r = r(u)$  be a Frenet curve lying on a hypersurface  $M$  in  $\mathbb{E}^4$  with Darboux frame  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  and non-zero curvatures  $\kappa_n, \kappa_g^1, \kappa_g^2, \tau_g^1$  and  $\tau_g^2$ . Then the curvature functions of the PU-Smarandache curve of  $r$  satisfy the following equations:

$$\begin{aligned} \bar{\kappa}_n &= \frac{-1}{\nu_1 \|\nabla f\|} \begin{pmatrix} (\kappa_g^1 \kappa_n + \kappa_g^2 (\tau_g^1 - \tau_g^2)) f_w + (-\kappa_g^1 \kappa_g^2 + \\ \kappa_n (\tau_g^1 + \tau_g^2)) f_x + ((\kappa_g^1)^2 + (\kappa_g^2)^2 + \tau_g^1 (\tau_g^1 + \\ \tau_g^2)) f_y + ((\kappa_g^2)^2 + \tau_g^2 (\tau_g^1 + \tau_g^2)) f_z \end{pmatrix}, \\ \bar{\kappa}_g^1 &= \frac{1}{\nu_1 \nu_6} \begin{pmatrix} (\kappa_g^1 \kappa_n + \kappa_g^2 (\tau_g^1 - \tau_g^2)) \nu_5 + (-\kappa_g^1 \kappa_g^2 + \\ \kappa_n (\tau_g^1 + \tau_g^2)) \nu_2 + ((\kappa_g^1)^2 + (\kappa_g^2)^2 + \tau_g^1 (\tau_g^1 + \\ \tau_g^2)) \nu_3 + ((\kappa_g^2)^2 + \tau_g^2 (\tau_g^1 + \tau_g^2)) \nu_4 \end{pmatrix}, \\ \bar{\kappa}_g^2 &= \frac{1}{\nu_6^2 \nu_{11}} \begin{pmatrix} -(\nu_2 \nu_7 + \nu_3 \nu_8 + \nu_4 \nu_9 + \nu_5 \nu_{10}) \nu'_6 + \nu_6 ((\kappa_g^1 \nu_2 - \kappa_g^2 \nu_4) \nu_8 \\ -\nu_5 (\kappa_n \nu_7 + \tau_g^1 \nu_8 + \tau_g^2 \nu_9) + \nu_3 (-\kappa_g^1 \nu_7 + \kappa_g^2 \nu_9 + \tau_g^1 \nu_{10}) + \\ \nu_7 \nu'_2 + \nu_8 \nu'_3 + \nu_9 \nu'_4 + \nu_{10} (\kappa_n \nu_2 + \tau_g^2 \nu_4 + \nu'_5)) \end{pmatrix}, \\ \bar{\tau}_g^1 &= \frac{1}{\nu_6^2 \|\nabla f\|} \begin{pmatrix} -(f_x \nu_2 + f_y \nu_3 + f_z \nu_4 + f_w \nu_5) \nu'_6 + \nu_6 (-\kappa_g^1 f_x \nu_3 + \kappa_g^2 f_z \nu_3 \\ -\kappa_n f_x \nu_5 - \tau_g^2 f_z \nu_5 + f_x \nu'_2 + f_y (\kappa_g^1 \nu_2 - \kappa_g^2 \nu_4 - \tau_g^1 \nu_5 \\ + \nu'_3) + f_z \nu'_4 + f_w (\kappa_n \nu_2 + \tau_g^1 \nu_3 + \tau_g^2 \nu_4 + \nu'_5)) \end{pmatrix}, \\ \bar{\tau}_g^2 &= \frac{1}{\nu_{11}^2 \|\nabla f\|} \begin{pmatrix} -(f_x \nu_7 + f_y \nu_8 + f_z \nu_9 + f_w \nu_{10}) \nu'_{11} + \nu_{11} (-\kappa_g^1 f_x \nu_8 + \kappa_g^2 f_z \nu_8 \\ -\kappa_n f_x \nu_{10} - \tau_g^2 f_z \nu_{10} + f_x \nu'_7 + f_y (\kappa_g^1 \nu_7 - \kappa_g^2 \nu_9 - \tau_g^1 \nu_{10} \\ + \nu'_8) + f_z \nu'_9 + f_w (\kappa_n \nu_7 + \tau_g^1 \nu_8 + \tau_g^2 \nu_9 + \nu'_{10})) \end{pmatrix}. \tag{3.9} \end{aligned}$$

*Proof.* Since  $\gamma = \gamma(s)$  is a PU-Smarandache curve reference to Frenet curve  $r$ . Then, by differentiating Eq. (3.8), we get

$$\begin{aligned} \gamma' &= \frac{-\kappa_g^1 \mathbf{T} + \kappa_g^2 \mathbf{P} + \kappa_g^2 \mathbf{U} + (\tau_g^1 + \tau_g^2) \mathbf{N}}{\sqrt{2}}, \\ \gamma'' &= \frac{1}{\sqrt{2}} (\kappa_g^1 \kappa_g^2 - \kappa_n (\tau_g^1 + \tau_g^2)) \mathbf{T} + (-(\kappa_g^1)^2 - (\kappa_g^2)^2 - \tau_g^1 (\tau_g^1 + \tau_g^2)) \mathbf{P} \\ &\quad + (-(\kappa_g^2)^2 - \tau_g^2 (\tau_g^1 + \tau_g^2)) \mathbf{U} + (-\kappa_g^1 \kappa_n - \kappa_g^2 \tau_g^1 + \kappa_g^2 \tau_g^2) \mathbf{N}. \tag{3.10} \end{aligned}$$

Using Eqs. (3.10), we have

$$\begin{aligned} \bar{\mathbf{T}} &= \frac{-\kappa_g^1 \mathbf{T} + \kappa_g^2 \mathbf{P} + \kappa_g^2 \mathbf{U} + (\tau_g^1 + \tau_g^2) \mathbf{N}}{\nu_1}, \\ \bar{\mathbf{N}} &= \frac{f_x \mathbf{T} + f_y \mathbf{P} + f_z \mathbf{U} + f_w \mathbf{N}}{\|\nabla f\|}, \end{aligned}$$

where

$$\nu_1 = \sqrt{2(\kappa_g^2)^2 + (\tau_g^1 + \tau_g^2)^2 + (\kappa_g^1)^2}.$$

On the other hand, we get

$$\bar{\mathbf{P}} = \frac{\nu_2 \mathbf{T} + \nu_3 \mathbf{P} + \nu_4 \mathbf{U} + \nu_5 \mathbf{N}}{\nu_6},$$

where

$$\begin{aligned} \nu_2 &= \sqrt{2}(-\kappa_g^1 \kappa_g^2 + \kappa_n(\tau_g^1 + \tau_g^2))\|\nabla f\| + 2\sigma_3 f_x, \\ \nu_3 &= \sqrt{2}((\kappa_g^1)^2 + (\kappa_g^2)^2 + \tau_g^1(\tau_g^1 + \tau_g^2))\|\nabla f\| + 2\sigma_3 f_y, \\ \nu_4 &= \sqrt{2}((\kappa_g^2)^2 + \tau_g^2(\tau_g^1 + \tau_g^2))\|\nabla f\| + 2\sigma_3 f_z, \\ \nu_5 &= \sqrt{2}(\kappa_g^1 \kappa_n + \kappa_g^2 \tau_g^1 - \kappa_g^2 \tau_g^2)\|\nabla f\| + 2\sigma_3 f_w, \\ \nu_6^2 &= \nu_2^2 + \nu_3^2 + \nu_4^2 + \nu_5^2, \\ \sigma_3 &= \langle \gamma'', \mathbf{N} \rangle = \frac{1}{2\|\nabla f\|} \begin{pmatrix} -(\kappa_g^1 \kappa_n + \kappa_g^2(\tau_g^1 - \tau_g^2))f_w + (\kappa_g^1 \kappa_g^2 - \\ \kappa_n(\tau_g^1 + \tau_g^2))f_x - ((\kappa_g^1)^2 + (\kappa_g^2)^2 + \tau_g^1(\tau_g^1 + \\ \tau_g^2))f_y - ((\kappa_g^2)^2 + \tau_g^2(\tau_g^1 + \tau_g^2))f_z \end{pmatrix}. \end{aligned}$$

Also, we get

$$\bar{\mathbf{U}} = \frac{\nu_7 \mathbf{T} + \nu_8 \mathbf{P} + \nu_9 \mathbf{U} + \nu_{10} \mathbf{N}}{\nu_{11}},$$

where

$$\begin{aligned} \nu_7 &= \kappa_g^2 f_w \nu_3 - \tau_g^1 f_z \nu_3 - \tau_g^2 f_z \nu_3 + \kappa_g^2 f_w \nu_4 + \tau_g^1 f_y \nu_4 + \tau_g^2 f_y \nu_4 - \kappa_g^2 (f_y + f_z) \nu_5, \\ \nu_8 &= -f_w (\kappa_g^2 \nu_2 + \kappa_g^1 \nu_4) + f_z ((\tau_g^1 + \tau_g^2) \nu_2 + \kappa_g^1 \nu_5) + f_x (-(\tau_g^1 + \tau_g^2) \nu_4 + \kappa_g^2 \nu_5), \\ \nu_9 &= f_w (-\kappa_g^2 \nu_2 + \kappa_g^1 \nu_3) - f_y ((\tau_g^1 + \tau_g^2) \nu_2 + \kappa_g^1 \nu_5) + f_x ((\tau_g^1 + \tau_g^2) \nu_3 + \kappa_g^2 \nu_5), \\ \nu_{10} &= \kappa_g^2 f_y \nu_2 + \kappa_g^2 f_z \nu_2 - \kappa_g^2 f_x \nu_3 - \kappa_g^1 f_z \nu_3 - \kappa_g^2 f_x \nu_4 + \kappa_g^1 f_y \nu_4, \\ \nu_{11} &= \nu_1 \|\nabla f\| \nu_6. \end{aligned}$$

In the light of the above calculations, the curvature functions  $\bar{\kappa}_n, \bar{\kappa}_g^1, \bar{\kappa}_g^2, \bar{\tau}_g^1$  and  $\bar{\tau}_g^2$  of  $\gamma$  are computed as in Eqs. (3.9). □

**Corollary 3.5.** *If  $r$  is an asymptotic curve. Then, the following equations hold:*

$$\begin{aligned} \bar{\mathbf{T}} &= \frac{-\kappa_g^1 \mathbf{T} - \kappa_g^2 \mathbf{P} + \kappa_g^2 \mathbf{U} + (\tau_g^1 + \tau_g^2) \mathbf{N}}{\nu_{12}}, \\ \bar{\mathbf{N}} &= \frac{f_x \mathbf{T} + f_y \mathbf{P} + f_z \mathbf{U} + f_w \mathbf{N}}{\|\nabla f\|}, \\ \bar{\mathbf{P}} &= \frac{\nu_{13} \mathbf{T} + \nu_{14} \mathbf{P} + \nu_{15} \mathbf{U} + \nu_{16} \mathbf{N}}{\nu_{17}}, \\ \bar{\mathbf{U}} &= \frac{\nu_{18} \mathbf{T} + \nu_{19} \mathbf{P} + \nu_{20} \mathbf{U} + \nu_{21} \mathbf{N}}{\nu_{22}}, \end{aligned}$$

and the curvature functions are obtained as follows:

$$\begin{aligned} \bar{\kappa}_n &= \frac{1}{\nu_{17} \|\nabla f\|} \left( \begin{array}{l} \kappa_g^2 (-\tau_g^1 + \tau_g^2) f_w + \kappa_g^1 \kappa_g^2 f_x - ((\kappa_g^1)^2 + (\kappa_g^2)^2 + \\ \tau_g^1 (\tau_g^1 + \tau_g^2)) f_y - ((\kappa_g^2)^2 + \tau_g^2 (\tau_g^1 + \tau_g^2)) f_z \end{array} \right), \\ \bar{\kappa}_g^1 &= \frac{1}{\nu_{12} \nu_{17}} \left( \begin{array}{l} \kappa_g^1 \kappa_g^2 \nu_{13} - ((\kappa_g^1)^2 + (\kappa_g^2)^2 + \tau_g^1 (\tau_g^1 + \tau_g^2)) \nu_{14} - \\ ((\kappa_g^2)^2 + \tau_g^2 (\tau_g^1 + \tau_g^2)) \nu_{15} + \kappa_g^2 (-\tau_g^1 + \tau_g^2) \nu_{16} \end{array} \right), \\ \bar{\kappa}_g^2 &= \frac{1}{\nu_{17}^2 \nu_{22}} \left( \begin{array}{l} \nu_{17} (\kappa_g^1 \nu_{13} \nu_{19} - \kappa_g^2 \nu_{15} \nu_{19} - \tau_g^1 \nu_{16} \nu_{19} - \tau_g^2 \nu_{16} \nu_{20} + \\ \tau_g^2 \nu_{15} \nu_{21} + \nu_{14} (-\kappa_g^1 \nu_{18} + \kappa_g^2 \nu_{20} + \tau_g^1 \nu_{21}) + \nu_{18} \nu'_{13} + \nu_{19} \nu'_{14} + \\ \nu_{20} \nu'_{15} + \nu_{21} \nu'_{16}) - (\nu_{13} \nu_{18} + \nu_{14} \nu_{19} + \nu_{15} \nu_{20} + \nu_{16} \nu_{21}) \nu'_{17} \end{array} \right), \\ \bar{\tau}_g^1 &= \frac{1}{\nu_{17}^2 \|\nabla f\|} \left( \begin{array}{l} -f_x (\kappa_g^1 \nu_{14} \nu_{17} - \nu_{17} \nu'_{13} + \nu_{13} \nu'_{17}) + f_y (\nu_{17} (\kappa_g^1 \nu_{13} - \kappa_g^2 \nu_{15} - \\ \tau_g^1 \nu_{16} + \nu'_{14}) - \nu_{14} \nu'_{17}) + f_z (\nu_{17} (\kappa_g^2 \nu_{14} - \tau_g^2 \nu_{16} + \nu'_{15}) - \\ \nu_{15} \nu'_{17}) + f_w (\nu_{17} (\tau_g^1 \nu_{14} + \tau_g^2 \nu_{15} + \nu'_{16}) - \nu_{16} \nu'_{17}) \end{array} \right), \\ \bar{\tau}_g^2 &= \frac{1}{\nu_{22}^2 \|\nabla f\|} \left( \begin{array}{l} -f_x (\kappa_g^1 \nu_{19} \nu_{22} - \nu_{22} \nu'_{18} + \nu_{18} \nu'_{22}) + f_y (\nu_{22} (\kappa_g^1 \nu_{18} - \\ \kappa_g^2 \nu_{20} - \tau_g^1 \nu_{21} + \nu'_{19}) - \nu_{19} \nu'_{22}) + f_z (\nu_{22} (\kappa_g^2 \nu_{19} - \tau_g^2 \nu_{21} + \\ \nu'_{20}) - \nu_{20} \nu'_{22}) + f_w (\nu_{22} (\tau_g^1 \nu_{19} + \tau_g^2 \nu_{20} + \nu'_{21}) - \nu_{21} \nu'_{22}) \end{array} \right), \end{aligned}$$

where

$$\begin{aligned} \nu_{12} &= \sqrt{2(\kappa_g^2)^2 + (\tau_g^1 + \tau_g^2)^2 + (\kappa_g^1)^2}, \quad \nu_{13} = \sqrt{2}(-\kappa_g^1 \kappa_g^2) \|\nabla f\| + 2\sigma_3 f_x, \\ \nu_{14} &= \sqrt{2}((\kappa_g^1)^2 + (\kappa_g^2)^2 + \tau_g^1 (\tau_g^1 + \tau_g^2)) \|\nabla f\| + 2\sigma_3 f_y, \\ \nu_{15} &= \sqrt{2}((\kappa_g^2)^2 + \tau_g^2 (\tau_g^1 + \tau_g^2)) \|\nabla f\| + 2\sigma_3 f_z, \\ \nu_{16} &= \sqrt{2}(\kappa_g^2 \tau_g^1 - \kappa_g^2 \tau_g^2) \|\nabla f\| + 2\sigma_3 f_w, \quad \nu_{17}^2 = \nu_{12}^2 + \nu_{13}^2 + \nu_{14}^2 + \nu_{15}^2, \\ \nu_{18} &= -\kappa_g^2 \nu_{16} f_y + \nu_{15} (\kappa_g^2 f_w + (\tau_g^1 + \tau_g^2) f_y) - \kappa_g^2 \nu_{16} f_z + \nu_{14} (\kappa_g^2 f_w - (\tau_g^1 + \tau_g^2) f_z), \\ \nu_{19} &= -\kappa_g^2 \nu_{13} f_w + \kappa_g^2 \nu_{16} f_x - \nu_{15} (\kappa_g^1 f_w + (\tau_g^1 + \tau_g^2) f_x) + (\tau_g^1 + \tau_g^2) \nu_{13} f_z + \kappa_g^1 \nu_{16} f_z, \\ \nu_{20} &= -\kappa_g^2 \nu_{13} f_w + \kappa_g^2 \nu_{16} f_x + \nu_{14} (\kappa_g^1 f_w + (\tau_g^1 + \tau_g^2) f_x) - \tau_g^1 \nu_{13} f_y - \tau_g^2 \nu_{13} f_y - \kappa_g^1 \nu_{16} f_y, \\ \nu_{21} &= \kappa_g^2 \nu_{13} f_y + \nu_{15} (-\kappa_g^2 f_x + \kappa_g^1 f_y) + \kappa_g^2 \nu_{13} f_z - \nu_{14} (\kappa_g^2 f_x + \kappa_g^1 f_z), \\ \nu_{22} &= \nu_{12} \nu_{17} \|\nabla f\|. \end{aligned}$$



**Corollary 3.6.** *If  $r$  is a line of curvature. Then, the following equations hold:*

$$\begin{aligned} \bar{\mathbf{T}} &= \frac{\kappa_g^1 \mathbf{T} + \kappa_g^2 \mathbf{P} - \kappa_g^2 \mathbf{U}}{\nu_{12}}, \\ \bar{\mathbf{N}} &= \frac{f_x \mathbf{T} + f_y \mathbf{P} + f_z \mathbf{U} + f_w \mathbf{N}}{\|\nabla f\|}, \\ \bar{\mathbf{P}} &= \frac{\nu_{13} \mathbf{T} + \nu_{14} \mathbf{P} + \nu_{15} \mathbf{U} + \nu_{16} \mathbf{N}}{\nu_{17}}, \\ \bar{\mathbf{U}} &= \frac{\nu_{18} \mathbf{T} + \nu_{19} \mathbf{P} + \nu_{20} \mathbf{U} + \nu_{21} \mathbf{N}}{\nu_{22}}, \end{aligned}$$

and the curvature functions are obtained as follows:

$$\begin{aligned} \bar{\kappa}_n &= \frac{-1}{\nu_{23} \|\nabla f\|} \left( \kappa_g^1 \kappa_n f_w - \kappa_g^1 \kappa_g^2 f_x + ((\kappa_g^1)^2 + (\kappa_g^2)^2) f_y + (\kappa_g^2)^2 f_z \right), \\ \bar{\kappa}_g^1 &= \frac{-1}{\nu_{23} \nu_{28}} \left( -\kappa_g^1 \kappa_g^2 \nu_{24} + ((\kappa_g^1)^2 + (\kappa_g^2)^2) \nu_{25} + (\kappa_g^2)^2 \nu_{26} + \kappa_g^1 \kappa_n \nu_{27} \right), \\ \bar{\kappa}_g^2 &= \frac{1}{\nu_{28}^2 \nu_{33}} \left( \begin{aligned} &\nu_{28} (-\kappa_n \nu_{27} \nu_{29} + \nu_{25} (-\kappa_g^1 \nu_{29} + \kappa_g^2 \nu_{31}) + \nu_{29} \nu'_{24} + \\ &\nu_{30} (\kappa_g^1 \nu_{24} - \kappa_g^2 \nu_{26} + \nu'_{25}) + \nu_{31} \nu'_{26} + \nu_{32} (\kappa_n \nu_{24} + \nu'_{27})) - \\ &(\nu_{24} \nu_{29} + \nu_{25} \nu_{30} + \nu_{26} \nu_{31} + \nu_{27} \nu_{32}) \nu'_{28} \end{aligned} \right), \\ \bar{\tau}_g^1 &= \frac{1}{\nu_{28}^2 \|\nabla f\|} \left( \begin{aligned} &-f_x (\nu_{28} (\kappa_g^1 \nu_{25} + \kappa_n \nu_{27} - \nu'_{24}) + \nu_{24} \nu'_{28}) + f_y (\nu_{28} (\kappa_g^1 \nu_{24} - \\ &\kappa_g^2 \nu_{26} + \nu'_{25}) - \nu_{25} \nu'_{28}) + f_z (\nu_{28} (\kappa_g^2 \nu_{25} + \nu'_{26}) - \nu_{26} \nu'_{28}) + \\ &f_w (\nu_{28} (\kappa_n \nu_{24} + \nu'_{27}) - \nu_{27} \nu'_{28}) \end{aligned} \right), \\ \bar{\tau}_g^2 &= \frac{1}{\nu_{33}^2 \|\nabla f\|} \left( \begin{aligned} &-f_x (\nu_{33} (\kappa_g^1 \nu_{30} + \kappa_n \nu_{32} - \nu'_{29}) + \nu_{29} \nu_{33}) + f_y (\nu_{33} (\kappa_g^1 \nu_{29} - \\ &\kappa_g^2 \nu_{31} + \nu'_{30}) - \nu_{30} \nu'_{33}) + f_z (\nu_{33} (\kappa_g^2 \nu_{30} + \nu'_{31}) - \nu_{31} \nu'_{33}) + \\ &f_w (\nu_{33} (\kappa_n \nu_{29} + \nu'_{32}) - \nu_{32} \nu'_{33}) \end{aligned} \right), \end{aligned}$$

where

$$\begin{aligned} \nu_{23} &= \sqrt{2(\kappa_g^2)^2 + (\kappa_g^1)^2}, \quad \nu_{24} = -\sqrt{2}\kappa_g^1 \kappa_g^2 \|\nabla f\| + 2\sigma_3 f_x, \\ \nu_{25} &= \sqrt{2}((\kappa_g^1)^2 + (\kappa_g^2)^2) \|\nabla f\| + 2\sigma_3 f_y, \quad \nu_{26} = \sqrt{2}(\kappa_g^2)^2 \|\nabla f\| + 2\sigma_3 f_z, \\ \nu_{27} &= \sqrt{2}\kappa_g^1 \kappa_n \|\nabla f\| + 2\sigma_3 f_w, \quad \nu_{28}^2 = \nu_{12}^2 + \nu_{13}^2 + \nu_{14}^2 + \nu_{15}^2, \\ \nu_{29} &= \kappa_g^2 \nu_{25} f_w + \kappa_g^2 \nu_{26} f_w - \kappa_g^2 \nu_{27} f_y - \kappa_g^2 \nu_{27} f_z, \\ \nu_{19} &= -\kappa_g^2 \nu_{24} f_w - \kappa_g^1 \nu_{26} f_w + \kappa_g^2 \nu_{27} f_x + \kappa_g^1 \nu_{27} f_z, \\ \nu_{20} &= -\kappa_g^2 \nu_{24} f_w + \kappa_g^1 \nu_{25} f_w + \kappa_g^2 \nu_{27} f_x - \kappa_g^1 \nu_{27} f_y, \\ \nu_{21} &= \kappa_g^2 \nu_{24} f_y + \nu_{26} (-\kappa_g^2 f_x + \kappa_g^1 f_y) + \kappa_g^2 \nu_{24} f_z - \nu_{25} (\kappa_g^2 f_x + \kappa_g^1 f_z), \\ \nu_{22} &= \nu_{23} \nu_{28} \|\nabla f\|. \end{aligned}$$

3.4. PN-Smarandache curves.

**Definition 3.5.** Let  $M$  be an oriented hypersurface in  $\mathbb{E}^4$  and the Frenet curve  $\delta = \delta(s)$  lying fully on  $M$  with Darboux frame  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  and non-zero curvatures  $\kappa_n, \kappa_g^1, \kappa_g^2, \tau_g^1$  and  $\tau_g^2$ . Then the PN-Smarandache curve of  $\delta$  is defined as

$$\delta(u) = \frac{1}{\sqrt{2}}(\mathbf{P} + \mathbf{N}). \tag{3.11}$$

**Theorem 3.4.** Let  $r = r(u)$  be a Frenet curve lying on a hypersurface  $M$  in  $\mathbb{E}^4$  with Darboux frame  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  and non-zero curvatures  $\kappa_n, \kappa_g^1, \kappa_g^2, \tau_g^1$  and  $\tau_g^2$ . Then the curvature functions of the PN-Smarandache curve of  $r$  satisfy the following equations:

$$\begin{aligned} \bar{\kappa}_n &= \frac{-1}{\xi_1 \|\nabla f\|} \left( \begin{aligned} &(\kappa_n(\kappa_g^1 + \kappa_n) + (\tau_g^1)^2 + \tau_g^2(-\kappa_g^2 + \tau_g^2))f_w + (-\kappa_g^1 + \kappa_n)\tau_g^1 f_x \\ &+ ((\kappa_g^2)^2 + \kappa_g^1(\kappa_g^1 + \kappa_n) + (\tau_g^1)^2 - \kappa_g^2\tau_g^2)f_y + \tau_g^1(\kappa_g^2 + \tau_g^2)f_z \end{aligned} \right), \\ \bar{\kappa}_g^1 &= \frac{-1}{\xi_1 \xi_6} \left( \begin{aligned} &(\kappa_n(\kappa_g^1 + \kappa_n) + (\tau_g^1)^2 + \tau_g^2(-\kappa_g^2 + \tau_g^2))\xi_5 + (-\kappa_g^1 + \kappa_n)\tau_g^1 \xi_2 + \\ &((\kappa_g^2)^2 + \kappa_g^1(\kappa_g^1 + \kappa_n) + (\tau_g^1)^2 - \kappa_g^2\tau_g^2)\xi_3 + \tau_g^1(\kappa_g^2 + \tau_g^2)\xi_4 \end{aligned} \right), \\ \bar{\kappa}_g^2 &= \frac{1}{\xi_6^2 \xi_{11}} \left( \begin{aligned} &-(\xi_2 \xi_7 + \xi_3 \xi_8 + \xi_4 \xi_9 + \xi_5 \xi_{10})\xi'_6 + \xi_6((\kappa_g^1 \xi_2 - \kappa_g^2 \xi_4)\xi_8 \\ &-\xi_5(\kappa_n \xi_7 + \tau_g^1 \xi_8 + \tau_g^2 \xi_9) + \xi_3(-\kappa_g^1 \xi_7 + \kappa_g^2 \xi_9 + \tau_g^1 \xi_{10}) + \\ &\xi_7 \xi'_2 + \xi_8 \xi'_3 + \xi_9 \xi'_4 + \xi_{10}(\kappa_n \xi_2 + \tau_g^2 \xi_4 + \xi'_5)) \end{aligned} \right), \\ \bar{\tau}_g^1 &= \frac{1}{\xi_6^2 \|\nabla f\|} \left( \begin{aligned} &-(f_x \xi_2 + f_y \xi_3 + f_z \xi_4 + f_w \xi_5)\xi'_6 + \xi_6(-\kappa_g^1 f_x \xi_3 + \kappa_g^2 f_z \xi_3 \\ &-\kappa_n f_x \xi_5 - \tau_g^2 f_z \xi_5 + f_x \xi'_2 + f_y(\kappa_g^1 \xi_2 - \kappa_g^2 \xi_4 - \tau_g^1 \xi_5 \\ &+ \xi'_3) + f_z \xi'_4 + f_w(\kappa_n \xi_2 + \tau_g^1 \xi_3 + \tau_g^2 \xi_4 + \xi'_5)) \end{aligned} \right), \\ \bar{\tau}_g^2 &= \frac{1}{\xi_{11}^2 \|\nabla f\|} \left( \begin{aligned} &-(f_x \xi_7 + f_y \xi_8 + f_z \xi_9 + f_w \xi_{10})\xi'_{11} + \xi_{11}(-\kappa_g^1 f_x \xi_8 + \\ &\kappa_g^2 f_z \xi_8 - \kappa_n f_x \xi_{10} - \tau_g^2 f_z \xi_{10} + f_x \xi'_7 + f_y(\kappa_g^1 \xi_7 - \\ &\kappa_g^2 \xi_9 - \tau_g^1 \xi_{10} + \xi'_8) + f_z \xi'_9 + f_w(\kappa_n \xi_7 + \tau_g^1 \xi_8 + \tau_g^2 \xi_9 + \xi'_{10})) \end{aligned} \right). \tag{3.12} \end{aligned}$$

*Proof.* Let  $\delta = \delta(s)$  be a PN-Smarandache curve reference to Frenet curve  $r$ . Then, by differentiating Eq. (3.11), we obtain

$$\begin{aligned} \delta' &= \frac{-\kappa_g^1 + \kappa_n}{\sqrt{2}}\mathbf{T} - \tau_g^1 \mathbf{P} + \frac{\kappa_g^2 - \tau_g^2}{\sqrt{2}}\mathbf{U} + \tau_g^1 \mathbf{N}, \\ \delta'' &= \frac{1}{\sqrt{2}}(\kappa_g^1 - \kappa_n)\tau_g^1 \mathbf{T} + (-(\kappa_g^2)^2 - \kappa_g^1(\kappa_g^1 + \kappa_n) - (\tau_g^1)^2 + \kappa_g^2\tau_g^2)\mathbf{P} \\ &\quad - \tau_g^1(\kappa_g^2 + \tau_g^2)\mathbf{U} + (-\kappa_n(\kappa_g^1 + \kappa_n) - (\tau_g^1)^2 + \kappa_g^2\tau_g^2 - (\tau_g^2)^2)\mathbf{N}. \tag{3.13} \end{aligned}$$

Therefore, from Eqs. (3.13), we get

$$\begin{aligned} \bar{\mathbf{T}} &= \frac{-\kappa_g^1 + \kappa_n}{\xi_1}\mathbf{T} - \tau_g^1 \mathbf{P} + \frac{\kappa_g^2 - \tau_g^2}{\xi_1}\mathbf{U} + \tau_g^1 \mathbf{N}, \\ \bar{\mathbf{N}} &= \frac{f_x \mathbf{T} + f_y \mathbf{P} + f_z \mathbf{U} + f_w \mathbf{N}}{\|\nabla f\|}, \end{aligned}$$

where

$$\xi_1 = \sqrt{2(\tau_g^1)^2 + (\kappa_g^1 + \kappa_n)^2 + (\kappa_g^2 - \tau_g^2)^2}.$$

On the other hand, we obtain

$$\bar{P} = \frac{\xi_2 \mathbf{T} + \xi_3 \mathbf{P} + \xi_4 \mathbf{U} + \xi_5 \mathbf{N}}{\xi_6},$$

where

$$\begin{aligned} \xi_2 &= \sqrt{2}(-\kappa_g^1 + \kappa_n)\tau_g^1 \|\nabla f\| + 2\sigma_4 f_x, \\ \xi_3 &= \sqrt{2}((\kappa_g^2)^2 + \kappa_g^1(\kappa_g^1 + \kappa_n) + (\tau_g^1)^2 - \kappa_g^2\tau_g^2) \|\nabla f\| + 2\sigma_4 f_y, \\ \xi_4 &= \sqrt{2}\tau_g^1(\kappa_g^2 + \tau_g^2) \|\nabla f\| + 2\sigma_4 f_z, \\ \xi_5 &= \sqrt{2}(\kappa_n(\kappa_g^1 + \kappa_n) + (\tau_g^1)^2 - \kappa_g^2\tau_g^2 + (\tau_g^2)^2) \|\nabla f\| + 2\sigma_4 f_w, \\ \xi_6^2 &= \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2, \\ \sigma_4 &= \langle \delta'', \mathbf{N} \rangle = \frac{1}{2\|\nabla f\|} \begin{pmatrix} -(\kappa_n(\kappa_g^1 + \kappa_n) + (\tau_g^1)^2 + \tau_g^2(-\kappa_g^2 + \tau_g^2))f_w + \\ (\kappa_g^1 - \kappa_n)\tau_g^1 f_x - ((\kappa_g^2)^2 + \kappa_g^1(\kappa_g^1 + \kappa_n) + \\ (\tau_g^1)^2 - \kappa_g^2\tau_g^2)f_y - \tau_g^1(\kappa_g^2 + \tau_g^2)f_z \end{pmatrix}. \end{aligned}$$

Also, we have

$$\bar{U} = \frac{\xi_7 \mathbf{T} + \xi_8 \mathbf{P} + \xi_9 \mathbf{U} + \xi_{10} \mathbf{N}}{\xi_{11}},$$

where,

$$\begin{aligned} \xi_7 &= \kappa_g^2 f_w \xi_3 - \tau_g^2 f_w \xi_3 - \tau_g^1 f_z \xi_3 + \tau_g^1 f_w \xi_4 + \tau_g^1 f_y \xi_4 + (-\kappa_g^2 - \tau_g^2) f_y - \tau_g^1 f_z \xi_5, \\ \xi_8 &= f_w ((-\kappa_g^2 + \tau_g^2) \xi_2 - (\kappa_g^1 + \kappa_n) \xi_4) + f_z (\tau_g^1 \xi_2 + (\kappa_g^1 + \kappa_n) \xi_5) - f_x (\tau_g^1 \xi_4 + (-\kappa_g^2 + \tau_g^2) \xi_5), \\ \xi_9 &= f_w (-\tau_g^1 \xi_2 + (\kappa_g^1 + \kappa_n) \xi_3) + \tau_g^1 f_x (\xi_3 + \xi_5) - f_y (\tau_g^1 \xi_2 + (\kappa_g^1 + \kappa_n) \xi_5), \\ \xi_{10} &= (\kappa_g^2 f_y \xi_2 - \tau_g^2 f_y \xi_2 + \tau_g^1 f_z \xi_2 - \kappa_g^2 f_x \xi_3 + \tau_g^2 f_x \xi_3 - \kappa_g^1 f_z \xi_3 - \kappa_n f_z \xi_3 - \\ &\quad \tau_g^1 f_x \xi_4 + \kappa_g^1 f_y \xi_4 + \kappa_n f_y \xi_4), \\ \xi_{11} &= \xi_1 \|\nabla f\| \xi_6. \end{aligned}$$

In the light of the above calculations, the curvature functions  $\bar{\kappa}_n, \bar{\kappa}_g^1, \bar{\kappa}_g^2, \bar{\tau}_g^1$  and  $\bar{\tau}_g^2$  of  $\delta$  are computed as in Eqs. (3.12). □

**Corollary 3.7.** *If  $r$  be asymptotic curve. Then, the following equations hold:*

$$\begin{aligned} \bar{\mathbf{T}} &= \frac{-\kappa_g^1 \mathbf{T} - \tau_g^1 \mathbf{P} + (\kappa_g^2 - \tau_g^2) \mathbf{U} + \tau_g^1 \mathbf{N}}{\xi_{12}}, \\ \bar{\mathbf{N}} &= \frac{f_x \mathbf{T} + f_y \mathbf{P} + f_z \mathbf{U} + f_w \mathbf{N}}{\|\nabla f\|}, \\ \bar{\mathbf{P}} &= \frac{\xi_{13} \mathbf{T} + \xi_{14} \mathbf{P} + \xi_{15} \mathbf{U} + \xi_{16} \mathbf{N}}{\xi_{17}}, \\ \bar{\mathbf{U}} &= \frac{\xi_{18} \mathbf{T} + \xi_{19} \mathbf{P} + \xi_{20} \mathbf{U} + \xi_{21} \mathbf{N}}{\xi_{22}}. \end{aligned}$$

Thus, the curvature functions can be computed as follows:

$$\begin{aligned} \bar{\kappa}_n &= \frac{-1}{\|\nabla f\| \xi_{12}} \left( \frac{((\tau_g^1)^2 + \tau_g^2(-\kappa_g^2 + \tau_g^2))f_w - \kappa_g^1 \tau_g^1 f_x + ((\kappa_g^1)^2 + (\kappa_g^2)^2 + (\tau_g^1)^2 - \kappa_g^2 \tau_g^2) f_y + \tau_g^1 (\kappa_g^2 + \tau_g^2) f_z}{\xi_{12}} \right), \\ \bar{\kappa}_g^1 &= \frac{-1}{\xi_{12} \xi_{17}} \left( \frac{((\tau_g^1)^2 + \tau_g^2(-\kappa_g^2 + \tau_g^2)) \xi_{16} - \kappa_g^1 \tau_g^1 \xi_{13} + ((\kappa_g^1)^2 + (\kappa_g^2)^2 + (\tau_g^1)^2 - \kappa_g^2 \tau_g^2) \xi_{14} + \tau_g^1 (\kappa_g^2 + \tau_g^2) \xi_{15}}{\xi_{17}} \right), \\ \bar{\kappa}_g^2 &= \frac{1}{\xi_{17}^2 \xi_{22}} \left( \frac{\xi_{17}(\kappa_g^1 \xi_{13} \xi_{19} - \kappa_g^2 \xi_{15} \xi_{19} - \tau_g^1 \xi_{16} \xi_{19} - \tau_g^2 \xi_{16} \xi_{20} + \tau_g^2 \xi_{15} \xi_{21} + \xi_{14}(-\kappa_g^1 \xi_{18} + \kappa_g^2 \xi_{20} + \tau_g^1 \xi_{21}) + \xi_{18} \xi'_{13} + \xi_{19} \xi'_{14} + \xi_{20} \xi'_{15} + \xi_{21} \xi'_{16}) - (\xi_{13} \xi_{18} + \xi_{14} \xi_{19} + \xi_{15} \xi_{20} + \xi_{16} \xi_{21}) \xi'_{17}}{\xi_{17}^2 \xi_{22}} \right), \\ \bar{\tau}_g^1 &= \frac{1}{\xi_{17}^2 \|\nabla f\|} \left( \frac{-f_x(\kappa_g^1 \xi_{14} \xi_{17} - \xi_{17} \xi'_{13} + \xi_{13} \xi'_{17}) + f_y(\xi_{17}(\kappa_g^1 \xi_{13} - \kappa_g^2 \xi_{15} - \tau_g^1 \xi_{16} + \xi'_{14}) - \xi_{14} \xi'_{17}) + f_z(\xi_{17}(\kappa_g^2 \xi_{14} - \tau_g^2 \xi_{16} + \xi'_{15}) - \xi_{15} \xi'_{17}) + f_w(\xi_{17}(\tau_g^1 \xi_{14} + \tau_g^2 \xi_{15} + \xi'_{16}) - \xi_{16} \xi'_{17})}{\xi_{17}^2 \|\nabla f\|} \right), \\ \bar{\tau}_g^2 &= \frac{1}{\xi_{22}^2 \|\nabla f\|} \left( \frac{-f_x(\kappa_g^1 \xi_{19} \xi_{22} - \xi_{22} \xi'_{18} + \xi_{18} \xi'_{22}) + f_y(\xi_{22}(\kappa_g^1 \xi_{18} - \kappa_g^2 \xi_{20} - \tau_g^1 \xi_{21} + \xi'_{19}) - \xi_{19} \xi'_{22}) + f_z(\xi_{22}(\kappa_g^2 \xi_{19} - \tau_g^2 \xi_{21} + \xi'_{20}) - \xi_{20} \xi'_{22}) + f_w(\xi_{22}(\tau_g^1 \xi_{19} + \tau_g^2 \xi_{20} + \xi'_{21}) - \xi_{21} \xi'_{22})}{\xi_{22}^2 \|\nabla f\|} \right), \end{aligned}$$

where

$$\begin{aligned} \xi_{12} &= \sqrt{2(\tau_g^1)^2 + (\kappa_g^1)^2 + (\kappa_g^2 - \tau_g^2)^2}, \quad \xi_{13} = \sqrt{2}(-\kappa_g^1) \tau_g^1 \|\nabla f\| + 2\sigma_4 f_x, \\ \xi_{14} &= \sqrt{2}((\kappa_g^2)^2 + \kappa_g^1(\kappa_g^1) + (\tau_g^1)^2 - \kappa_g^2 \tau_g^2) \|\nabla f\| + 2\sigma_4 f_y, \\ \xi_{15} &= \sqrt{2} \tau_g^1 (\kappa_g^2 + \tau_g^2) \|\nabla f\| + 2\sigma_4 f_z, \\ \xi_{16} &= \sqrt{2}((\tau_g^1)^2 - \kappa_g^2 \tau_g^2 + (\tau_g^2)^2) \|\nabla f\| + 2\sigma_4 f_w, \quad \xi_{17}^2 = \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2, \\ \xi_{18} &= \kappa_g^2 f_w \xi_{14} - \tau_g^2 f_w \xi_{14} - \tau_g^1 f_z \xi_{14} + \tau_g^1 f_w \xi_{15} + \tau_g^1 f_y \xi_{15} + (-\kappa_g^2 - \tau_g^2) f_y - \tau_g^1 f_z \xi_{16}, \\ \xi_{19} &= f_w((-\kappa_g^2 + \tau_g^2) \xi_{14} - (\kappa_g^1) \xi_{15}) + f_z(\tau_g^1 \xi_{14} + (\kappa_g^1) \xi_{16}) - f_x(\tau_g^1 \xi_{15} + (-\kappa_g^2 + \tau_g^2) \xi_{16}), \\ \xi_{20} &= f_w(-\tau_g^1 \xi_{14} + (\kappa_g^1) \xi_{14}) + \tau_g^1 f_x(\xi_{14} + \xi_{16}) - f_y(\tau_g^1 \xi_{14} + (\kappa_g^1) \xi_{16}), \\ \xi_{21} &= \kappa_g^2 f_y \xi_{13} - \tau_g^2 f_y \xi_{13} + \tau_g^1 f_z \xi_{13} - \kappa_g^2 f_x \xi_{14} + \tau_g^2 f_x \xi_{14} - \kappa_g^1 f_z \xi_{14} - \tau_g^1 f_x \xi_{15} + \kappa_g^1 f_y \xi_{15}, \\ \xi_{22} &= \xi_{12} \|\nabla f\| \xi_{17}. \end{aligned}$$

**Corollary 3.8.** *If  $r$  be line of curvature. Then, the following equations hold:*

$$\begin{aligned} \bar{\mathbf{T}} &= \frac{-(\kappa_g^1 + \kappa_n)\mathbf{T} + k_g^2\mathbf{U}}{\xi_{23}}, \\ \bar{\mathbf{N}} &= \frac{f_x\mathbf{T} + f_y\mathbf{P} + f_z\mathbf{U} + f_w\mathbf{N}}{\|\nabla f\|}, \\ \bar{\mathbf{P}} &= \frac{\xi_{24}\mathbf{T} + \xi_{25}\mathbf{P} + \xi_{26}\mathbf{U} + \xi_{27}\mathbf{N}}{\xi_{28}}, \\ \bar{\mathbf{U}} &= \frac{\xi_{29}\mathbf{T} + \xi_{30}\mathbf{P} + \xi_{31}\mathbf{U} + \xi_{32}\mathbf{N}}{\xi_{33}}. \end{aligned}$$

Thus, the curvature functions can be computed as follows:

$$\begin{aligned} \bar{\kappa}_n &= -\frac{\kappa_n(\kappa_g^1 + \kappa_n)f_w + ((\kappa_g^2)^2 + \kappa_g^1(\kappa_g^1 + \kappa_n))f_y}{\|\nabla f\|\xi_{23}}, \\ \bar{\kappa}_g^1 &= -\frac{\kappa_n(\kappa_g^1 + \kappa_n)\xi_{25} + ((\kappa_g^2)^2 + \kappa_g^1(\kappa_g^1 + \kappa_n))\xi_{27}}{\xi_{23}\xi_{28}}, \\ \bar{\kappa}_g^2 &= \frac{1}{\xi_{28}^2 \xi_{33}} \left( \begin{aligned} &\xi_{28}(-\kappa_n\xi_{27}\xi_{29} + \xi_{25}(-\kappa_g^1\xi_{29} + \kappa_g^2\xi_{31}) + \xi_{29}\xi'_{24} + \\ &\xi_{30}(\kappa_g^1\xi_{24} - \kappa_g^2\xi_{26} + \xi'_{25}) + \xi_{31}\xi'_{26} + \xi_{32}(\kappa_n\xi_{24} + \\ &\xi'_{27})) - (\xi_{24}\xi_{29} + \xi_{25}\xi_{30} + \xi_{26}\xi_{31} + \xi_{27}\xi_{32})\xi'_{28} \end{aligned} \right), \\ \bar{\tau}_g^1 &= \frac{1}{\xi_{28}^2 \|\nabla f\|} \left( \begin{aligned} &-f_x(\xi_{28}(\kappa_g^1\xi_{25} + \kappa_n\xi_{27} - \xi'_{24}) + \xi_{24}\xi'_{28}) + f_y(\xi_{28}(\kappa_g^1\xi_{24} - \\ &\kappa_g^2\xi_{26} + \xi'_{25}) - \xi_{25}\xi'_{28}) + f_z(\xi_{28}(\kappa_g^2\xi_{25} + \xi'_{26}) - \xi_{26}\xi'_{28}) + \\ &f_w(\xi - 28(\kappa_n\xi_{24} + \xi'_{27}) - \xi_{27}\xi'_{28}) \end{aligned} \right), \\ \bar{\tau}_g^2 &= \frac{1}{\xi_{33}^2 \|\nabla f\|} \left( \begin{aligned} &-f_x(\xi_{33}(\kappa_g^1\xi_{30} + \kappa_n\xi_{32} - \xi'_{29}) + \xi_{29}\xi'_{33}) + f_y(\xi_{33}(\kappa_g^1\xi_{29} - \\ &\kappa_g^2\xi_{31} + \xi'_{30}) - \xi_{30}\xi'_{33}) + f_z(\xi_{33}(\kappa_g^2\xi_{30} + \xi'_{31}) - \xi_{31}\xi'_{33}) + \\ &f_w(\xi - 33(\kappa_n\xi_{29} + \xi'_{32}) - \xi_{32}\xi'_{33}) \end{aligned} \right), \end{aligned}$$

where

$$\begin{aligned} \xi_{23} &= \sqrt{(\kappa_g^1 + \kappa_n)^2 - (\kappa_g^2)^2}, \quad \xi_{24} = 2\sigma_4 f_x, \\ \xi_{25} &= \sqrt{2((\kappa_g^2)^2 + \kappa_g^1(\kappa_g^1 + \kappa_n))\|\nabla f\|} + 2\sigma_4 f_y, \quad \xi_{26} = 2\sigma_4 f_z, \\ \xi_{27} &= \sqrt{2}\kappa_n(\kappa_g^1 + \kappa_n)\|\nabla f\| + 2\sigma_4 f_w, \quad \xi_{28}^2 = \xi_{24}^2 + \xi_{25}^2 + \xi_{26}^2 + \xi_{27}^2, \\ \xi_{29} &= (kg2\xi_{25}f_w - \kappa_g^2\xi_{27}f_y), \\ \xi_{30} &= -\kappa_g^2\xi_{24}f_w - \kappa_g^1\xi_{26}f_w - \kappa_n\xi_{26}[u]f_w + \kappa_g^2\xi_{27}f_x + (\kappa_g^1 + \kappa_n)\xi_{27}f_z, \\ \xi_{31} &= (\kappa_g^1 + \kappa_n)\xi_{25}f_w - \kappa_g^1\xi_{27}f_y - \kappa_n\xi_{27}f_y, \\ \xi_{32} &= \kappa_g^2\xi_{24}f_y + \kappa_g^1\xi_{26}f_y + \kappa_n\xi_{26}f_y - \xi_{25}(\kappa_g^2f_x + (\kappa_g^1 + \kappa_n)f_z), \\ \xi_{33} &= \xi_{23}\|\nabla f\|\xi_{28}. \end{aligned}$$

4. EXAMPLES

**Example 4.1.** Consider the curve  $r(u)$  given by

$$r(u) = (\cos(u), \sin(u), \cos(2u), \sin(2u)).$$

By using the definition 2.6, we can calculate the Darboux frame  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  as follows:

$$\begin{aligned} \mathbf{T} &= \left( -\frac{\sin(u)}{5}, \frac{\cos(u)}{5}, -\frac{2}{5} \sin(2u), \frac{2}{5} \cos(2u) \right), \\ \mathbf{N} &= \left( \frac{\cos(u)}{\sqrt{2}}, \frac{\sin(u)}{\sqrt{2}}, \frac{\cos(2u)}{\sqrt{2}}, \frac{\sin(2u)}{\sqrt{2}} \right), \\ \mathbf{P} &= \left( \frac{\cos(u)}{\sqrt{2}}, \frac{\sin(u)}{\sqrt{2}}, -\frac{\cos(2u)}{\sqrt{2}}, -\frac{\sin(2u)}{\sqrt{2}} \right), \\ \mathbf{U} &= \left( \frac{2 \sin(u)}{\sqrt{5}}, -\frac{2 \cos(u)}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \sin(2u), \frac{1}{\sqrt{5}} \cos(2u) \right). \end{aligned}$$

The curvature functions of the curve  $r$  can be computed as follows:

$$\kappa_n = -\sqrt{\frac{5}{2}}, \kappa_g^1 = \frac{3}{\sqrt{10}}, \kappa_g^2 = -2\sqrt{\frac{2}{5}}, \tau_g^1 = \tau_g^2 = 0.$$

Therefore, we can obtain **TU**-Smarndache curve as

$$\phi = \frac{1}{\sqrt{2}} (\mathbf{T} + \mathbf{U}) = \left( \frac{\sin(u)}{\sqrt{10}}, \frac{-\cos(u)}{\sqrt{10}}, \frac{-3 \sin(2u)}{\sqrt{10}}, \frac{3 \cos(2u)}{\sqrt{10}} \right).$$

By using the definition 2.6, we can obtain

$$\begin{aligned} \bar{\mathbf{T}} &= \frac{1}{\sqrt{37}} (\cos(u), \sin(u), -6 \cos(2u), -6 \sin(2u)), \\ \bar{\mathbf{N}} &= \frac{1}{2\sqrt{5}} (\sin(u), \cos(u), 3 \sin(2u), 3 \cos(2u)), \\ \bar{\mathbf{P}} &= \frac{1}{\sqrt{16930}} (17 \sin(u), -17 \cos(u), 129 \sin(2u), -129 \cos(2u)), \\ \bar{\mathbf{U}} &= \frac{1}{\sqrt{8465}} (-54 \cos(u), -54 \sin(u), -9 \cos(2u), -9 \sin(2u)). \end{aligned}$$

The curvature functions of the Smarandache curve  $\phi$  can be computed as follows:

$$\bar{\kappa}_n = \frac{-1}{2} \sqrt{\frac{37}{5}}, \bar{\kappa}_g^1 = \frac{1531}{\sqrt{626410}}, \bar{\kappa}_g^2 = \frac{-324\sqrt{2}}{1693}, \bar{\tau}_g^1 = \bar{\tau}_g^2 = 0.$$

**Example 4.2.** Consider the unit speed curve given by

$$r(u) = \left( \frac{\cos(u)}{2}, \frac{\sin(u)}{2}, \frac{u}{2}, \frac{u}{\sqrt{2}} \right).$$

By using the definition 2.6 we can calculate the Darboux frame  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}, \mathbf{N}\}$  as follow:

$$\begin{aligned} \mathbf{T} &= \left( \frac{-\sin(u)}{2}, \frac{\cos(u)}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right), \\ \mathbf{N} &= \left( \frac{\cos(u)}{2}, \frac{\sin(u)}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right), \end{aligned}$$

$$\mathbf{P} = \left( \frac{-\sqrt{3}}{2} \cos(u), \frac{-\sqrt{3}}{2} \sin(u), \frac{1}{2\sqrt{3}}, \frac{-1}{\sqrt{6}} \right),$$

$$\mathbf{U} = \left( -\sqrt{\frac{2}{3}} \sin(u), \sqrt{\frac{2}{3}} \cos(u), \frac{-1}{\sqrt{6}}, -\frac{1}{2\sqrt{3}} \right),$$

The curvature functions of the curve  $r$  can be computed as follows

$$\kappa_n = -\frac{1}{4}, \kappa_g^1 = \frac{\sqrt{3}}{4}, \kappa_g^2 = -\frac{1}{\sqrt{2}}, \tau_g^1 = 0, \tau_g^2 = -\frac{1}{\sqrt{6}}.$$

Therefore we can obtain **TN**-Smarandache curve as

$$\beta = \left( \frac{\cos(u) - \sin(u)}{2\sqrt{2}}, \frac{\cos(u) + \sin(u)}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right).$$

By using the definition 2.6, we can obtain

$$\bar{\mathbf{T}} = \left( -\frac{\cos(u) + \sin(u)}{2\sqrt{2}}, \frac{\cos(u) - \sin(u)}{2\sqrt{2}}, 0, 0 \right),$$

$$\bar{\mathbf{N}} = \left( \frac{\cos(u) - \sin(u)}{2\sqrt{2}}, \frac{\cos(u) + \sin(u)}{2\sqrt{2}}, \frac{1}{2}, \frac{1}{2} \right),$$

$$\bar{\mathbf{P}} = \left( \frac{-3(\cos(u) - \sin(u))}{\sqrt{22}}, \frac{3(\cos(u) + \sin(u))}{\sqrt{22}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right),$$

$$\bar{\mathbf{U}} = \left( 0, 0, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right).$$

The curvature functions of the Smarandache curve  $\beta$  can be computed as

$$\bar{\kappa}_n = \frac{-1}{2}, \bar{\kappa}_g^1 = \frac{3}{\sqrt{11}}, \bar{\kappa}_g^2 = 0, \bar{\tau}_g^1 = \bar{\tau}_g^2 = 0.$$

### 5. CONCLUSION

In the four dimensional Euclidean space  $\mathbb{E}^4$ , some special Smarandache curves lying on a hypersurface are investigated. Also, the differential geometric properties of these curves are obtained. Furthermore, some computational examples in support of our main results are given.

### ACKNOWLEDGMENT

This research was supported by Islamic University of Madinah. We would like to thank our colleagues from Deanship of Scientific Research who provided insight and expertise that greatly assisted the research.

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