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ON APPROXIMATION SOLUTIONS OF THE CAUCHY-JENSEN AND THE ADDITIVE-QUADRATIC FUNCTIONAL EQUATION IN PARANORMED SPACES

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ABSTRACT. In this paper, we prove the generalized Hyers-Ulam-Rassias stability of the bi-Cauchy-Jensen functional equation and the bi-additive-quadratic functional equation in paranormed spaces. Moreover, we investigate the Hyers-Ulam-Rassias stability of the generalized Cauchy-Jensen equation in such spaces.

1. Introduction and Preliminaries

The stability problem of functional equations was initiated by Ulam in 1940 [17] arising from concerning the stability of group homomorphisms. These question form is the object of the stability theory. In 1941, Hyers [7] provided a first affirmative partial answer to Ulam's problem for the case of approximately additive mapping in Banach spaces. In 1978, Rassias [16] gave a generalization of Hyers's theorem for linear mapping by considering an unbounded Cauchy difference. A generalization of Rassias's result was developed by Găvruţa [6] in 1994 by replacing the unbounded Cauchy difference by a general control function. For more information on that subject and further references we refer to a survey paper [3] and to a recent monograph on Ulam stability [4].

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Let $\mathbb{R}$ and $\mathbb{N}$ be the set of real numbers and the set of natural numbers, respectively. Next, let X and Y be vector spaces and k be a positive integer, a function $f : X^k \to Y$ is called *k-additive functional equation* if and only if f satisfies the equation

$$f(x_1, x_2, \ldots, x_{i-1}, x_i + y, x_{i+1}, \ldots, x_k)$$
$$= f(x_1, x_1, \ldots, x_i) + f(x_1, x_2, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)$$

for all $i \in \mathbb{N}$, $1 \leq i \leq k$ and for every $x_1, x_2, \ldots, x_k, y \in X$, that is, f is additive in each of its variables $x_i \in X$ for all $i = 1, 2, \ldots, k$. Some fundamental properties on such mappings be mentioned in [10]. In particular, a 2-additive functional equation is called bi-additive functional equation.

A mapping $f : X \times X \to Y$ is called a *bi-additive-quadratic functional equation* (bi-AQE, shortly) if f satisfies the system equations

$$f(x + y, z) = f(x, z) + f(y, z),$$
$$f(x, y + z) + f(x, y - z) = 2f(x, y) + 2f(x, z) \tag{1.1}$$

for all $x, y, z \in X$. When $X = Y = \mathbb{R}$, the solution of (1.1) is given by the function $f(x, y) = cxy^2$ where $x, y, c \in \mathbb{R}$. For mapping $f : X \times X \to Y$ satisfies

$$f(x + y, z + w) + f(x + y, z - w) = 2f(x, z) + 2f(x, w) + 2f(y, z) + 2f(y, w) \tag{1.2}$$

for all $x, y, z, w \in X$. In 2005, Park, Bae and Chung [13] proved that the mapping $f : X \times X \to Y$ satisfies (1.1) if and only if it satisfies (1.2) and provided the general solution of (1.1) which is given by $f(x, y) = M(x, y, y)$ and $M(x, y, z) = M(x, z, y)$ for all $x, y, z \in X$ where $M : X \times X \times X \to Y$ is a multi-additive mapping.

A mapping $f : X \times X \to Y$ is called a *bi-Cauchy-Jensen functional equation* (bi-CJE, shortly) if f satisfies the system equations

$$f(x + y, z) = f(x, z) + f(y, z)$$
$$2f\left(x, \frac{y + z}{2}\right) = f(x, y) + f(x, z) \tag{1.3}$$

for all $x, y, z \in X$. In particular, For $X = Y = \mathbb{R}$, The solution of (1.3) is given by the function $f(x, y) = axy + bx$ where $x, y, a, b \in \mathbb{R}$. For mapping $f : X \times X \to Y$ satisfies

$$2f\left(x + y, \frac{z + w}{2}\right) = f(x, z) + f(x, w) + f(y, z) + f(y, w) \tag{1.4}$$

for all $x, y, z, w \in X$. In 2006, Park and Bae [12] showed that the mapping $f : X \times X \to Y$ satisfies (1.3) if and only if it satisfies (1.4) and gave the general solution of (1.4) which is given by $f(x, y) = B(x, y) + A(x)$ for all $x, y \in X$ where $B : X \times X \to Y$ is a bi-additive mapping and $A : X \to Y$ is an additive mapping.

Next, we recall the concepts of paranormed space and some basic facts on the Fréchet spaces.

Definition 1.1 ([18])**.** *Let X be a vector space. A paranorm on X is a function $P : X \to \mathbb{R}$ such that for all $x, y \in X$, the following conditions hold :*

(i) $P(0) = 0$;

(ii) $P(-x) = P(x)$;

(iii) $P(x + y) \leq P(x) + P(y)$ (*triangle inequality*);

(iv) *If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subseteq X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$.* (*continuity of scalar multication*)

The pair (X, P) is called a *paranormed space* if P is a paranorm on X. Note that

$$P(nx) \leq nP(x)$$

for all $n \in \mathbb{N}$ and all $x \in X$. The paranorm P on X is called *total* if, in addition, P satisfies

(v) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

In 2015, Bae and Park [2] proved the Hyers-Ulam stability of the functional equation (1.2) and (1.4) in paranormed spaces in the sense of Rassias [16]. We refer to some works of stability of the functional equation (1.2) and (1.4) and various functional equations in paranormed spaces with [1, 8, 9, 11, 13–15]. In the first section of main results, we investigate stability of the functional equation (1.2) and (1.4) in paranormed spaces in the sense of Găvruţa [6].

In 2009, Gao et al. [5] introduced the generalized Cauchy-Jensen functional equation and gave some useful properties. Let G be an n-divisible abelian group where $n \in \mathbb{N}$ and X be a normed space with norm $\|\cdot\|_X$. For a mapping $f : G \to X$ is called a *generalized Cauchy-Jensen functional equation* (GCJE, shortly) if it satisfies the equation

$$\alpha f\left(\frac{x+y}{\alpha} + z\right) = f(x) + f(y) + \alpha f(z) \tag{1.5}$$

for all $x, y, z \in X$ and for any fixed positive integer $\alpha \geq 2$. In particular, when $\alpha = 2$, it is called a *Cauchy-Jensen functional equation* (CJE, shortly).

Proposition 1.1 ([5])**.** *Let G be an n-divisible abelian group for some positive integer n and X be a normed space with norm $\|\cdot\|_X$. Then a mapping $f : G \to X$ is additive if and only if it satisfies*

$$\|f(x) + f(y) + nf(z)\|_X \leq \left\| nf(\frac{x+y}{n} + z) \right\|_X$$

for all $x, y, z \in G$.

The following corollary is an immediate consequence of Proposition 1.1.

Corollary 1.1 ([5]). *For a mapping $f : G \to X$, the following statements are equivalent.*

(a) f *is additive.*

(b) $f(x) + f(y) + nf(z) = nf(\frac{x+y}{n} + z)$, *for all* $x, y, z \in G$.

(c) $\|f(x) + f(y) + nf(z)\|_X \leq \|nf(\frac{x+y}{n} + z)\|_X$, *for all* $x, y, z \in G$.

Clearly, a vector space is n-divisible abelian group, so Corollary 1.1 is right when G is a vector space. In the second section of main results, we proved the stability of the functional equation (1.5) in paranormed spaces in the sense of Rassias [16].

Throughtout this paper, assume that (X, P) is a Fréchet space and that $(E, \|\cdot\|)$ is a Banach space.

2. The stability of the bi-Cauchy-Jensen functional equation and bi-additive-quadratic functional equation in paranormed spaces

The following result is the generalized Hyers-Ulam-Rassias stability of the functional equation (1.4).

Theorem 2.1. *Let $\varphi : E \times E \times E \times E \to [0, \infty)$ be a function and $f : E \times E \to X$ be a mapping satisfying $f(x, 0) = 0$ for all $x \in E$ such that*

$$\begin{aligned} &P\left(2f\left(x + y, \frac{z + w}{2}\right) - f(x, z) - f(x, w) - f(y, z) - f(y, w)\right) \\ \leq &\varphi(x, y, z, w) \end{aligned} \tag{2.1}$$

for all $x, y, z, w \in E$. Then there exists a unique mapping $F : E \times E \to X$ satisfying (1.4) *such that*

$$P\left(2f(x, y) - F(x, y)\right) \leq \tilde{\varphi}(x, x, y, y) \tag{2.2}$$

for all $x, y \in E$ where

$$\begin{aligned} &\tilde{\varphi}(x, y, z, w) \\ :=&\sum_{j=0}^{\infty} 6^j \left[6\varphi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{3^{j+1}}, -\frac{w}{3^{j+1}}\right) + 4\varphi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, -\frac{z}{3^{j+1}}, \frac{w}{3^{j+1}}\right)\right. \\ &+ 2\varphi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{3^{j+1}}, \frac{w}{3^{j+1}}\right) + 2\varphi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, -\frac{z}{3^{j+1}}, \frac{w}{3^{j}}\right) \\ &\left.+\varphi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{3^{j}}, \frac{w}{3^{j}}\right)\right] < \infty \end{aligned} \tag{2.3}$$

for all $x, y, z, w \in E$ and the mapping $F : E \times E \to X$ is given by

$$F(x, y) = \lim_{j\to\infty} 2\cdot 6^j f\left(\frac{x}{2^j}, \frac{y}{3^j}\right)$$

for all $x, y \in E$.

Proof. Letting $y = x$ in (2.1), we obtain that

$$P\left(2f\left(2x, \frac{z+w}{2}\right) - 2f(x,z) - 2f(x,w)\right) \leq \varphi(x,x,z,w) \tag{2.4}$$

for all $x, z, w \in E$. Letting $w = -z$ in (2.4), we get that

$$P\left(2f(x,z) + 2f(x,-z)\right) \leq \varphi(x,x,z,-z) \tag{2.5}$$

for all $x, z \in E$. Subtituting z by $-z$ and w by $-z$ in (2.4), we get

$$P(2f(2x,-z) - 4f(x,-z)) \leq \varphi(x,x,-z,-z) \tag{2.6}$$

for all $x, z \in E$. It follows from (2.5) and (2.6) that

$$\begin{aligned} &P(4f(x,z) + 2f(2x,-z)) \\ &= P(4f(x,z) + 4f(x,-z) - 4f(x,-z) + 2f(2x,-z)) \\ &\leq P(4f(x,z) + 4f(x,-z)) + P(2f(2x,-z) - 4f(x,-z)) \\ &\leq 2P(2f(x,z) + 2f(x,-z)) + P(2f(2x,-z) - 4f(x,-z)) \\ &\leq 2\varphi(x,x,z,-z) + \varphi(x,x,-z,-z) \end{aligned} \tag{2.7}$$

for all $x, z \in E$. Letting $w = -3z$ in (2.4), we have

$$P(2f(2x,-z) - 2f(x,-3z) - 2f(x,z)) \leq \varphi(x,x,z,-3z)$$

for all $x, z \in E$. By (ii) of definition 1.1, we have

$$P(2f(x,-3z) + 2f(x,z) - 2f(2x,-z)) \leq \varphi(x,x,z,-3z) \tag{2.8}$$

for all $x, z \in E$. By (2.7) and (2.8), we have

$$\begin{aligned} &P(6f(x,z) + 2f(x,-3z)) \\ =&P(4f(x,z) + 2f(2x,-z) + 2f(x,-3z) + 2f(x,z) - 2f(2x,-z)) \\ \leq&P(4f(x,z) + 2f(2x,-z)) + P(2f(x,-3z) + 2f(x,z) - 2f(2x,-z)) \\ \leq&2\varphi(x,x,z,-z) + \varphi(x,x,-z,-z) + \varphi(x,x,z,-3z) \end{aligned} \tag{2.9}$$

for all $x, z \in E$. Putting $z = 3z$ in (2.6), we obtain that

$$P(2f(2x,-3z) - 4f(x,-3z)) \leq \varphi(x,x,-3z,-3z) \tag{2.10}$$

for all $x, z \in E$. It follows from (2.9) and (2.10)

$$\begin{aligned} & P(12f(x,z)+2f(2x,-3z)) && (2.11)\\ =& P(12f(x,z)+4f(x,-3z)-4f(x,-3z)+2f(2x,-3z))\\ \le& P(12f(x,z)+4f(x,-3z))+P(2f(2x,-3z)-4f(x,-3z))\\ \le& 2P(6f(x,z)+2f(x,-3z))+P(2f(2x,-3z)-4f(x,-3z))\\ \le& 4\varphi(x,x,z,-z)+2\varphi(x,x,-z,-z)+2\varphi(x,x,z,-3z)\\ & +\varphi(x,x,-3z,-3z) \end{aligned}$$

for all $x, z \in E$. Replacing z by $-z$ in the above inequality, we get that

$$\begin{aligned} & P(12f(x,-z)+2f(2x,3z)) && (2.12)\\ \le& 4\varphi(x,x,-z,z)+2\varphi(x,x,z,z)+2\varphi(x,x,-z,3z)+\varphi(x,x,3z,3z) \end{aligned}$$

for all $x, z \in E$. By (2.5) and the above inequality, we have

$$\begin{aligned} & P(12f(x,z)-2f(2x,3z)) && (2.13)\\ =& P(12f(x,z)+12f(x,-z)-12f(x,-z)-2f(2x,3z))\\ \le& P(12f(x,z)+12f(x,-z))+P(-12f(x,-z)-2f(2x,3z))\\ =& P(12f(x,z)+12f(x,-z))+P(12f(x,-z)+2f(2x,3z))\\ \le& 6P(2f(x,z)+2f(x,-z))+P(12f(x,-z)+2f(2x,3z))\\ \le& 6\varphi(x,x,z,-z)+4\varphi(x,x,-z,z)+2\varphi(x,x,z,z)\\ & +2\varphi(x,x,-z,3z)+\varphi(x,x,3z,3z) \end{aligned}$$

for all $x, z \in E$. Replacing x by $\frac{x}{2^{j+1}}$ and z by $\frac{z}{3^{j+1}}$ in (2.13), we obtain that

$$\begin{aligned} & P\left(12f\left(\frac{x}{2^{j+1}},\frac{z}{3^{j+1}}\right)-2f\left(\frac{x}{2^j},\frac{z}{3^j}\right)\right) && (2.14)\\ \le& 6\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{z}{3^{j+1}},-\frac{z}{3^{j+1}}\right)+4\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},-\frac{z}{3^{j+1}},\frac{z}{3^{j+1}}\right)\\ & +2\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{z}{3^{j+1}},\frac{z}{3^{j+1}}\right)+2\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},-\frac{z}{3^{j+1}},\frac{z}{3^j}\right)\\ & +\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{z}{3^j},\frac{z}{3^j}\right) \end{aligned}$$

for all $x, z \in E$. By (2.14), for any integers l, m such that $0 \leq l < m$, we get that

$$\begin{aligned}
&P\left(2\cdot 6^m f\left(\frac{x}{2^m},\frac{z}{3^m}\right)-2\cdot 6^l f\left(\frac{x}{2^l},\frac{z}{3^l}\right)\right) \\
=&P\left(2\cdot 6^m f\left(\frac{x}{2^m},\frac{z}{3^m}\right)-2\cdot 6^{m-1} f\left(\frac{x}{2^{m-1}},\frac{z}{3^{m-1}}\right)+2\cdot 6^{m-1} f\left(\frac{x}{2^{m-1}},\frac{z}{3^{m-1}}\right)\right. \\
&-2\cdot 6^{m-2} f\left(\frac{x}{2^{m-2}},\frac{z}{3^{m-2}}\right)+2\cdot 6^{m-2} f\left(\frac{x}{2^{m-2}},\frac{z}{3^{m-2}}\right) \\
&\left.+\cdots+2\cdot 6^{l+1} f\left(\frac{x}{2^{l+1}},\frac{z}{3^{l+1}}\right)-2\cdot 6^l f\left(\frac{x}{2^l},\frac{z}{3^l}\right)\right) \\
\leq&P\left(2\cdot 6^m f\left(\frac{x}{2^m},\frac{z}{3^m}\right)-2\cdot 6^{m-1} f\left(\frac{x}{2^{m-1}},\frac{z}{3^{m-1}}\right)\right) \\
&+P\left(2\cdot 6^{m-1} f\left(\frac{x}{2^{m-1}},\frac{z}{3^{m-1}}\right)-2\cdot 6^{m-2} f\left(\frac{x}{2^{m-2}},\frac{z}{3^{m-2}}\right)\right) \\
&+\cdots+P\left(2\cdot 6^{l+1} f\left(\frac{x}{2^{l+1}},\frac{z}{3^{l+1}}\right)-2\cdot 6^l f\left(\frac{x}{2^l},\frac{z}{3^l}\right)\right) \\
\leq&6^{m-1}P\left(12 f\left(\frac{x}{2^m},\frac{z}{3^m}\right)-2 f\left(\frac{x}{2^{m-1}},\frac{z}{3^{m-1}}\right)\right) \\
&+6^{m-2}P\left(12 f\left(\frac{x}{2^{m-1}},\frac{z}{3^{m-1}}\right)-2 f\left(\frac{x}{2^{m-2}},\frac{z}{3^{m-2}}\right)\right) \\
&+\cdots+6^{l}P\left(12 f\left(\frac{x}{2^{l+1}},\frac{z}{3^{l+1}}\right)-2 f\left(\frac{x}{2^{l}},\frac{z}{3^{l}}\right)\right) \\
=&\sum_{j=l}^{m-1} 6^j P\left(12 f\left(\frac{x}{2^{j+1}},\frac{z}{3^{j+1}}\right)-2 f\left(\frac{x}{2^{j}},\frac{z}{3^{j}}\right)\right) \\
\leq&\sum_{j=l}^{\infty} 6^j \left[6\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{z}{3^{j+1}},-\frac{z}{3^{j+1}}\right)+4\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},-\frac{z}{3^{j+1}},\frac{z}{3^{j+1}}\right)\right. \\
&+2\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{z}{3^{j+1}},\frac{z}{3^{j+1}}\right)+2\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},-\frac{z}{3^{j+1}},\frac{z}{3^{j}}\right) \\
&\left.+\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{z}{3^{j}},\frac{z}{3^{j}}\right)\right]
\end{aligned} \tag{2.15}$$

for all $x, z \in E$. It follows from (2.3) that

$$\begin{aligned}
&\lim_{l\to\infty}\sum_{j=l}^{\infty} 6^j \left[6\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{z}{3^{j+1}},-\frac{z}{3^{j+1}}\right)+4\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},-\frac{z}{3^{j+1}},\frac{z}{3^{j+1}}\right)\right. \\
&+2\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{z}{3^{j+1}},\frac{z}{3^{j+1}}\right)+2\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},-\frac{z}{3^{j+1}},\frac{z}{3^{j}}\right) \\
&\left.+\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{z}{3^{j}},\frac{z}{3^{j}}\right)\right]=0
\end{aligned}$$

for all $x, z \in E$. This implies that the sequence $\{2\cdot 6^j f\left(\frac{x}{2^j},\frac{z}{3^j}\right)\}_{j=0}^{\infty}$ is a Cauchy sequence in X for all $x, z \in E$. Since X is complete paranormed space, the sequence $\{2\cdot 6^j f\left(\frac{x}{2^j},\frac{z}{3^j}\right)\}_{j=0}^{\infty}$ converges for all $x, z \in E$. Define $F : E \times E \to X$ by

$$F(x,z)=\lim_{j\to\infty} 2\cdot 6^j f\left(\frac{x}{2^j},\frac{z}{3^j}\right) \tag{2.16}$$

for all $x, z \in E$. By (2.3), we get that

$$\begin{aligned}
&\sum_{j=1}^{\infty}\frac{1}{6}\cdot 2\cdot 6^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{3^j},\frac{w}{3^j}\right)\\
=&\sum_{j=0}^{\infty}\frac{1}{6}\cdot 2\cdot 6^{j+1}\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{3^{j+1}},\frac{w}{3^{j+1}}\right)\\
=&\sum_{j=0}^{\infty}2\cdot 6^j\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{3^{j+1}},\frac{w}{3^{j+1}}\right)\\
\leq&\sum_{j=0}^{\infty}6\cdot 6^j\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{3^{j+1}},-\frac{w}{3^{j+1}}\right)+4\cdot\sum_{j=0}^{\infty}6^j\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},-\frac{z}{3^{j+1}},\frac{w}{3^{j+1}}\right)\\
&+2\cdot\sum_{j=0}^{\infty}6^j\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{3^{j+1}},\frac{w}{3^{j+1}}\right)+2\cdot\sum_{j=0}^{\infty}6^j\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},-\frac{z}{3^{j+1}},\frac{w}{3^j}\right)\\
&+\sum_{j=0}^{\infty}6^j\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{3^j},\frac{w}{3^j}\right)\\
\leq&\sum_{j=0}^{\infty}6^j\Bigg[6\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{3^{j+1}},-\frac{w}{3^{j+1}}\right)+4\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},-\frac{z}{3^{j+1}},\frac{w}{3^{j+1}}\right)\\
&+2\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{3^{j+1}},\frac{w}{3^{j+1}}\right)+2\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},-\frac{z}{3^{j+1}},\frac{w}{3^j}\right)\\
&+\varphi\left(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{3^j},\frac{w}{3^j}\right)\Bigg]\\
=&\tilde{\varphi}(x,y,z,w)<\infty
\end{aligned}$$

for all $x, y, z, w \in E$. This implies that

$$\lim_{j\to\infty}\frac{1}{6}\cdot 2\cdot 6^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{3^j},\frac{w}{3^j}\right)=0$$

for all $x, y, z, w \in E$, which implies

$$\lim_{j\to\infty}2\cdot 6^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{3^j},\frac{w}{3^j}\right)=0 \tag{2.17}$$

for all $x, y, z, w \in E$. It follows from (2.1), (2.16) and (2.17) that we have

$$\begin{aligned}
&P\left(2F\left(x+y,\frac{z+w}{2}\right)-F(x,z)-F(x,w)-F(y,z)-F(y,w)\right)\\
\leq&P\left(2\lim_{j\to\infty}2\cdot 6^j f\left(\frac{x+y}{2^j},\frac{\frac{z+w}{2}}{3^j}\right)-\lim_{j\to\infty}2\cdot 6^j f\left(\frac{x}{2^j},\frac{z}{3^j}\right)\right.\\
&\left.-\lim_{j\to\infty}2\cdot 6^j f\left(\frac{x}{2^j},\frac{w}{3^j}\right)-\lim_{j\to\infty}2\cdot 6^j f\left(\frac{y}{2^j},\frac{z}{3^j}\right)-\lim_{j\to\infty}2\cdot 6^j f\left(\frac{y}{2^j},\frac{w}{3^j}\right)\right)\\
\leq&\lim_{j\to\infty}2\cdot 6^j P\left(2f\left(\frac{x}{2^j}+\frac{y}{2^j},\frac{\frac{z}{3^j}+\frac{w}{3^j}}{2}\right)-f\left(\frac{x}{2^j},\frac{z}{3^j}\right)-f\left(\frac{x}{2^j},\frac{w}{3^j}\right)\right.\\
&\left.-f\left(\frac{y}{2^j},\frac{z}{3^j}\right)-f\left(\frac{y}{2^j},\frac{w}{3^j}\right)\right)\\
\leq&\lim_{j\to\infty}2\cdot 6^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{3^j},\frac{w}{3^j}\right)=0
\end{aligned}$$

for all $x, y, z, w \in E$. Since X is total, we have

$$2F\left(x+y, \frac{z+w}{2}\right) = F(x,z) + F(x,w) + F(y,z) + F(y,w)$$

for all $x, y, z, w \in E$. Setting $l = 0$ and taking $m \to \infty$ in (2.15), this implies that the inequality (2.2). Next, we will show that F is unique. Let $G : E \times E \to X$ be another mapping satisfying (1.4) and (2.2). By [12], there exists bi-additive mapping $B, B' : E \times E \to X$ and additive mapping $A, A' : E \to X$ such that $F(x, y) = B(x, y) + A(x)$ and $G(x, y) = B'(x, y) + A'(x)$ for all $x, y \in E$. Since B is bi-additive mapping, A is additive mapping and $f(x, 0) = 0$ for all $x \in E$, we have

$$\begin{aligned}
F(x,y) - 6F\left(\frac{x}{2}, \frac{x}{3}\right) &= [B(x,y) + A(x)] - 6\left[B\left(\frac{x}{2}, \frac{y}{3}\right) + A\left(\frac{x}{2}\right)\right] \\
&= B(x,y) + A(x) - 6B\left(\frac{x}{2}, \frac{y}{3}\right) - 6A\left(\frac{x}{2}\right) \\
&= B(x,y) + A(x) - B(x,y) - 3A(x) \\
&= -2A(x) \\
&= -2B(x,0) - 2A(x) \\
&= -2F(x,0) \\
&= -2 \lim_{j\to\infty} 2 \cdot 6^j f\left(\frac{x}{2^j}, 0\right) = 0
\end{aligned}$$

for all $x, y \in E$, that is,

$$F(x,y) = 6F\left(\frac{x}{2}, \frac{y}{3}\right) \tag{2.18}$$

for all $x, y \in E$. Replacing x by $\frac{x}{2}$ and y by $\frac{y}{3}$ in (2.18), we have

$$F\left(\frac{x}{2}, \frac{y}{3}\right) = 6F\left(\frac{x}{2^2}, \frac{y}{3^2}\right)$$

for all $x, y \in E$. Continuing this process, we have $F(x, y) = 6^n F\left(\frac{x}{2^n}, \frac{y}{3^n}\right)$ for all $x, y \in E$ and for all $n \in \mathbb{N}$. Similarly, we get that $G(x, y) = 6^n G\left(\frac{x}{2^n}, \frac{y}{3^n}\right)$ for all $x, y \in E$ and for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
&P(F(x,y)-G(x,y)) \qquad (2.19)\\
=&P\left(6^nF\left(\frac{x}{2^n},\frac{y}{3^n}\right)-6^nG\left(\frac{x}{2^n},\frac{y}{3^n}\right)\right)\\
=&P\left(6^nF\left(\frac{x}{2^n},\frac{y}{3^n}\right)-2\cdot 6^nf\left(\frac{x}{2^n},\frac{y}{3^n}\right)+2\cdot 6^nf\left(\frac{x}{2^n},\frac{y}{3^n}\right)-6^nG\left(\frac{x}{2^n},\frac{y}{3^n}\right)\right)\\
\le&P\left(6^nF\left(\frac{x}{2^n},\frac{y}{3^n}\right)-2\cdot 6^nf\left(\frac{x}{2^n},\frac{y}{3^n}\right)\right)+P\left(2\cdot 6^nf\left(\frac{x}{2^n},\frac{y}{3^n}\right)-6^nG\left(\frac{x}{2^n},\frac{y}{3^n}\right)\right)\\
\le&6^nP\left(F\left(\frac{x}{2^n},\frac{y}{3^n}\right)-2f\left(\frac{x}{2^n},\frac{y}{3^n}\right)\right)+6^nP\left(2f\left(\frac{x}{2^n},\frac{y}{3^n}\right)-G\left(\frac{x}{2^n},\frac{y}{3^n}\right)\right)\\
\le&2\cdot 6^n\tilde{\varphi}\left(\frac{x}{2^n},\frac{x}{2^n},\frac{y}{3^n},\frac{y}{3^n}\right)\\
=&2\cdot 6^n\sum_{j=0}^{\infty}6^j\left[6\varphi\left(\frac{\frac{x}{2^n}}{2^{j+1}},\frac{\frac{x}{2^n}}{2^{j+1}},\frac{\frac{y}{3^n}}{3^{j+1}},-\frac{\frac{y}{3^n}}{3^{j+1}}\right)+4\varphi\left(\frac{\frac{x}{2^n}}{2^{j+1}},\frac{\frac{x}{2^n}}{2^{j+1}},-\frac{\frac{y}{3^n}}{3^{j+1}},\frac{\frac{y}{3^n}}{3^{j+1}}\right)\right.\\
&+2\varphi\left(\frac{\frac{x}{2^n}}{2^{j+1}},\frac{\frac{x}{2^n}}{2^{j+1}},\frac{\frac{y}{3^n}}{3^{j+1}},\frac{\frac{y}{3^n}}{3^{j+1}}\right)+2\varphi\left(\frac{\frac{x}{2^n}}{2^{j+1}},\frac{\frac{x}{2^n}}{2^{j+1}},-\frac{\frac{y}{3^n}}{3^{j+1}},\frac{\frac{y}{3^n}}{3^{j}}\right)\\
&\left.+\varphi\left(\frac{\frac{x}{2^n}}{2^{j+1}},\frac{\frac{x}{2^n}}{2^{j+1}},\frac{\frac{y}{3^n}}{3^{j}},\frac{\frac{y}{3^n}}{3^{j}}\right)\right]\\
=&2\sum_{j=0}^{\infty}6^{n+j}\left[6\varphi\left(\frac{x}{2^{n+j+1}},\frac{x}{2^{n+j+1}},\frac{y}{3^{n+j+1}},-\frac{y}{3^{n+j+1}}\right)\right.\\
&+4\varphi\left(\frac{x}{2^{n+j+1}},\frac{x}{2^{n+j+1}},-\frac{y}{3^{n+j+1}},\frac{y}{3^{n+j+1}}\right)+2\varphi\left(\frac{x}{2^{n+j+1}},\frac{x}{2^{n+j+1}},\frac{y}{3^{n+j+1}},\frac{y}{3^{n+j+1}}\right)\\
&\left.+2\varphi\left(\frac{x}{2^{n+j+1}},\frac{x}{2^{n+j+1}},-\frac{y}{3^{n+j+1}},\frac{y}{3^{n+j}}\right)+\varphi\left(\frac{x}{2^{n+j+1}},\frac{x}{2^{n+j+1}},\frac{y}{3^{n+j}},\frac{y}{3^{n+j}}\right)\right]\\
=&2\sum_{i=n}^{\infty}6^i\left[6\varphi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{y}{3^{i+1}},-\frac{y}{3^{i+1}}\right)+4\varphi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},-\frac{y}{3^{i+1}},\frac{y}{3^{i+1}}\right)\right.\\
&+2\varphi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{y}{3^{i+1}},\frac{y}{3^{i+1}}\right)+2\varphi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},-\frac{y}{3^{i+1}},\frac{y}{3^{i}}\right)\\
&\left.+\varphi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{y}{3^{i}},\frac{y}{3^{i}}\right)\right]
\end{aligned}$$

for all $x,y\in E$. By (2.3), we obtain that

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=n}^{\infty}6^i\left[6\varphi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{y}{3^{i+1}},-\frac{y}{3^{i+1}}\right)+4\varphi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},-\frac{y}{3^{i+1}},\frac{y}{3^{i+1}}\right)\right. \qquad (2.20)\\
&+2\varphi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{y}{3^{i+1}},\frac{y}{3^{i+1}}\right)+2\varphi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},-\frac{y}{3^{i+1}},\frac{y}{3^{i}}\right)\\
&\left.+\varphi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{y}{3^{i}},\frac{y}{3^{i}}\right)\right]=0
\end{aligned}$$

for all $x,y\in E$. From (2.20), taking limit $n\to\infty$ in (2.19), we obtain that

$$\lim_{n\to\infty}P(F(x,y)-G(x,y))=0$$

for all $x,y\in E$. Since paranorm P on X is total, we have $F(x,y)-G(x,y)=0$ for all $x,y\in E$. Hence F is a unique mapping satisfying (1.4) and (2.2). □

Remark 2.1. *Let r, θ be positive real numbers with $r > \log_2 6$. If we set $\varphi(x,y,z,w) = \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r)$ for all $x,y,z,w \in E$, then Theorem 2.1 recovers Theorem 2.1 in [2].*

The following result is the generalized Hyers-Ulam-Rassias stability of the functional equation (1.2).

Theorem 2.2. *Let $\varphi : E \times E \times E \times E \to [0,\infty)$ be a function and $f : E \times E \to X$ be a mapping satisfying $f(x,0) = 0$ for all $x \in E$ such that*

$$P\left(f(x+y,z+w) + f(x+y,z-w) - 2f(x,z) - 2f(x,w) - 2f(y,z) - 2f(y,w)\right) \le \varphi(x,y,z,w) \tag{2.21}$$

for all $x,y,z,w \in E$. Then there exists a unique mapping $F : E \times E \to X$ satisfying (1.2) such that

$$P\left(f(x,y) - F(x,y)\right) \le \tilde{\varphi}(x,x,y,y) \tag{2.22}$$

for all $x,y \in E$ where

$$\tilde{\varphi}(x,y,z,w) := \sum_{j=0}^{\infty} 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{w}{2^{j+1}}\right) < \infty \tag{2.23}$$

for all $x,y,z,w \in E$ where the mapping $F : E \times E \to X$ is given by

$$F(x,y) = \lim_{j\to\infty} 8^j f\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$$

for all $x,y \in E$.

Proof. Letting $y = x$ and $w = z$ in (2.21), we obtain that

$$P(f(2x,2z) - 8f(x,z)) \le \varphi(x,x,z,z)$$

for all $x,z \in E$. Replacing x by $\frac{x}{2^{j+1}}$ and z by $\frac{z}{2^{j+1}}$ in the above inequality, we get that

$$P\left(f\left(\frac{x}{2^j}, \frac{z}{2^j}\right) - 8f\left(\frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}\right)\right) \le \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{z}{2^{j+1}}\right) \tag{2.24}$$

for all nonnegative integer j and for all $x,z \in E$. It follows from (2.24) that we have

$$\begin{aligned} &P\left(8^j f\left(\frac{x}{2^j}, \frac{z}{2^j}\right) - 8^{j+1} f\left(\frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}\right)\right) \\ \le& 8^j P\left(f\left(\frac{x}{2^j}, \frac{z}{2^j}\right) - 8f\left(\frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}\right)\right) \\ \le& 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{z}{2^{j+1}}\right) \end{aligned} \tag{2.25}$$

for all nonnegative integer j and for all $x, z \in E$. By (2.25), for any integers l and m such that $0 \leq l < m$, we have

$$\begin{aligned}
& P\left(8^l f\left(\frac{x}{2^l}, \frac{y}{2^l}\right) - 8^m f\left(\frac{x}{2^m}, \frac{y}{2^m}\right)\right) \qquad (2.26)\\
=& P\left(8^l f\left(\frac{x}{2^l}, \frac{y}{2^l}\right) - 8^{l+1} f\left(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}\right) + 8^{l+1} f\left(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}\right)\right.\\
& \left. + \cdots + 8^{m-1} f\left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right) - 8^m f\left(\frac{x}{2^m}, \frac{y}{2^m}\right)\right)\\
\leq & P\left(8^l f\left(\frac{x}{2^l}, \frac{y}{2^l}\right) - 8^{l+1} f\left(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}\right)\right)\\
& + P\left(8^{l+1} f\left(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}\right) - 8^{l+2} f\left(\frac{x}{2^{l+2}}, \frac{y}{2^{l+2}}\right)\right)\\
& + \cdots + P\left(8^{m-1} f\left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right) - 8^m f\left(\frac{x}{2^m}, \frac{y}{2^m}\right)\right)\\
=& \sum_{j=l}^{m-1} 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{z}{2^{j+1}}\right)\\
\leq & \sum_{j=l}^{\infty} 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{z}{2^{j+1}}\right)
\end{aligned}$$

for all $x, z \in E$. It follows from (2.23) that we obtain that

$$\lim_{l\to\infty} \sum_{j=l}^{\infty} 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{z}{2^{j+1}}\right) = 0$$

for all $x, z \in E$. This implies that $\{8^j f\left(\frac{x}{2^j}, \frac{z}{2^j}\right)\}$ is Cauchy seqeunce in X for all $x, z \in E$. By completeness of X, the sequence $\{8^j f\left(\frac{x}{2^j}, \frac{z}{2^j}\right)\}$ is convergent sequence for all $x, y \in E$. Define $F : E \times E \to X$ by

$$F(x, z) = \lim_{j\to\infty} 8^j f\left(\frac{x}{2^j}, \frac{z}{2^j}\right)$$

for all $x, z \in E$. By (2.21), we obtain that

$$\begin{aligned}
& P(F(x+y, z+w) + F(x+y, z-w) - 2F(x, z) - 2F(x, w) - 2F(y, z) - 2F(y, w))\\
=& P\left(\lim_{j\to\infty} 8^j f\left(\frac{x+y}{2^j}, \frac{z+w}{2^j}\right) + \lim_{j\to\infty} 8^j f\left(\frac{x+y}{2^j}, \frac{z-w}{2^j}\right) - 2\cdot \lim_{j\to\infty} 8^j f\left(\frac{x}{2^j}, \frac{z}{2^j}\right)\right.\\
& \left. -2\cdot \lim_{j\to\infty} 8^j f\left(\frac{x}{2^j}, \frac{w}{2^j}\right) - 2\cdot \lim_{j\to\infty} 8^j f\left(\frac{y}{2^j}, \frac{z}{2^j}\right) - 2\cdot \lim_{j\to\infty} 8^j f\left(\frac{y}{2^j}, \frac{w}{2^j}\right)\right)\\
=& \lim_{j\to\infty} P\left(8^j f\left(\frac{x+y}{2^j}, \frac{z+w}{2^j}\right) + 8^j f\left(\frac{x+y}{2^j}, \frac{z-w}{2^j}\right) - 2\cdot 8^j f\left(\frac{x}{2^j}, \frac{z}{2^j}\right)\right.\\
& \left. -2\cdot 8^j f\left(\frac{x}{2^j}, \frac{w}{2^j}\right) - 2\cdot 8^j f\left(\frac{y}{2^j}, \frac{z}{2^j}\right) - 2\cdot 8^j f\left(\frac{y}{2^j}, \frac{w}{2^j}\right)\right)\\
\leq & \lim_{j\to\infty} 8^j P\left(f\left(\frac{x+y}{2^j}, \frac{z+w}{2^j}\right) + f\left(\frac{x+y}{2^j}, \frac{z-w}{2^j}\right) - 2f\left(\frac{x}{2^j}, \frac{z}{2^j}\right) - 2f\left(\frac{x}{2^j}, \frac{w}{2^j}\right)\right.\\
& \left. -2f\left(\frac{y}{2^j}, \frac{z}{2^j}\right) - 2f\left(\frac{y}{2^j}, \frac{w}{2^j}\right)\right)\\
\leq & \lim_{j\to\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{w}{2^j}\right) = 0
\end{aligned}$$

for all $x, y, z, w \in E$. Since X is total, we have

$$F(x+y, z+w) + F(x+y, z-w) = 2F(x,z) + 2F(x,w) + 2F(y,z) + 2F(y,w)$$

for all $x, y, z, w \in E$. Setting $l = 0$ and letting $m \to \infty$ in (2.26), the inequality (2.26) holds. Next, we will show that F is unique.

Let $G : E \times E \to X$ be a another mapping satisfying (1.2) and (2.22). It follows from Theorem 3 in [13] that there exists multi-additive mapping $M, M' : E \times E \times E \to X$ such that $F(x,y) = M(x,y,y)$, $G(x,y) = M'(x,y,y)$, $M(x,y,z) = M(x,z,y)$ and $M'(x,y,z) = M'(x,z,y)$ for all $x, y, z \in E$. For any $n \in \mathbb{N}$, we get that

$$\begin{aligned}
P(F(x,y) - G(x,y)) =& P\left(M(x,y,y) - M'(x,y,y)\right) \\
=& P\left(M\left(\frac{2^n x}{2^n}, \frac{2^n y}{2^n}, \frac{2^n y}{2^n}\right) - M'\left(\frac{2^n x}{2^n}, \frac{2^n y}{2^n}, \frac{2^n y}{2^n}\right)\right) \\
=& P\left(8^n\left[M\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\right)\right] - 8^n\left[M'\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\right)\right]\right) \\
=& P\left(8^n\left[M\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\right) - M'\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\right)\right]\right) \\
\leq& 8^n P\left(M\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\right) - M'\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\right)\right) \\
\leq& 8^n P\left(F\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - G\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) \\
\leq& 8^n\left[P\left(F\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) + P\left(f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right.\right. \\
&\left.\left. - G\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right)\right] \\
\leq& 2 \cdot 8^n \tilde{\varphi}\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\right) \\
=& 2 \cdot 8^n \sum_{j=0}^{\infty} 8^j \varphi\left(\frac{\frac{x}{2^n}}{2^{j+1}}, \frac{\frac{x}{2^n}}{2^{j+1}}, \frac{\frac{y}{2^n}}{2^{j+1}}, \frac{\frac{y}{2^n}}{2^{j+1}}\right) \\
=& 2 \cdot \sum_{j=0}^{\infty} 8^{n+j} \varphi\left(\frac{x}{2^{n+j+1}}, \frac{x}{2^{n+j+1}}, \frac{y}{2^{n+j+1}}, \frac{y}{2^{n+j+1}}\right) \\
=& 2 \cdot \sum_{i=n}^{\infty} 8^i \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}, \frac{y}{2^{i+1}}\right)
\end{aligned}$$

for all $x, y \in E$. By (2.23), we get that

$$\lim_{n \to \infty} \sum_{i=n}^{\infty} 8^i \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}, \frac{y}{2^{i+1}}\right) = 0$$

for all $x, y \in E$. Hence

$$\lim_{n \to \infty} P(F(x,y) - G(x,y)) = 0$$

for all $x, y \in E$. Since paranorm P on X is total, we have $F(x,y) - G(x,y) = 0$ for all $x, y \in E$. Hence F is a unique mapping satisfying (1.2) and (2.22). $\square$

Remark 2.2. *Let r, θ be positive real numbers with $r > 3$. If we set $\varphi(x, y, z, w) = \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r)$ for all $x, y, z, w \in E$, then Theorem 2.2 recovers Theorem 3.1 in [2].*

3. Stability of the generalized Cauchy-Jensen functional equation in paranormed space

The following result is the Hyers-Ulam-Rassias stability of the functional equation (1.5).

Theorem 3.1. *Let r be a positive real number with $r > 1$, and let $f : E \to X$ be a mapping satisfying*

$$P\left(\alpha f\left(\frac{x+y}{\alpha} + z\right) - f(x) - f(y) - \alpha f(z)\right) \leq \theta\left(\|x\|^r + \|y\|^r + \|z\|^r\right) \tag{3.1}$$

for all $x, y, z \in E$. Then there exists a unique mapping $F : E \to X$ satisfying (1.5) such that

$$P\left(f(x) - F(x)\right) \leq \left(\frac{3\alpha^r + 1}{\alpha^r - \alpha}\right)\theta\|x\|^r \tag{3.2}$$

for all $x \in E$ where the mapping $F : E \to X$ is given by

$$F(x) = \lim_{n\to\infty} \alpha^n f\left(\frac{x}{\alpha^n}\right)$$

for all $x \in E$.

Proof. Putting $x = y = z = 0$ in (3.1), we have $P(f(0)) \leq 0$. Since X is total, we obtain that $f(0) = 0$. Letting $x = -\frac{x}{\alpha}$, $y = \frac{x}{\alpha}$ and $z = 0$ in (3.1), we obtain that

$$\begin{aligned}
&P\left(f\left(-\frac{x}{\alpha}\right) + f\left(\frac{x}{\alpha}\right)\right) \\
=&P\left(-f\left(-\frac{x}{\alpha}\right) - f\left(\frac{x}{\alpha}\right)\right) \\
=&P\left(\alpha f\left(\frac{-\frac{x}{\alpha} + \frac{x}{\alpha}}{\alpha} + 0\right) - f\left(-\frac{x}{\alpha}\right) - f\left(\frac{x}{\alpha}\right) - \alpha f(0)\right) \\
\leq&\theta\left(\left\|-\frac{x}{\alpha}\right\|^r + \left\|\frac{x}{\alpha}\right\|^r + \|0\|^r\right) \\
=&\frac{2\theta}{\alpha^r}\|x\|^r
\end{aligned} \tag{3.3}$$

for all $x \in E$. Replacing x by αx in the inequality (3.3), we get that

$$P\left(f(-x) + f(x)\right) \leq 2\theta\|x\|^r \tag{3.4}$$

for all $x \in E$. Replacing $x = -x$, $y = 0$, and $z = \frac{x}{\alpha}$ in (3.1), we have

$$\begin{aligned} &P\left(f(-x) + \alpha f\left(\frac{x}{\alpha}\right)\right) \\ =&P\left(-f(-x) - \alpha f\left(\frac{x}{\alpha}\right)\right) \\ =&P\left(\alpha f\left(\frac{-x+0}{\alpha} + \left(-\frac{x}{\alpha}\right)\right) - f(-x) - f(0) - \alpha f\left(\frac{x}{\alpha}\right)\right) \\ \leq&\theta\left(\|-x\|^r + \|0\|^r + \left\|\frac{x}{\alpha}\right\|^r\right) \\ =&\left(1 + \frac{1}{\alpha^r}\right)\theta\|x\|^r \end{aligned} \tag{3.5}$$

for all $x \in E$. It follows from (3.4) and (3.5) that we have

$$\begin{aligned} P\left(\alpha f\left(\frac{x}{\alpha}\right) - f(x)\right) &= P\left(\alpha f\left(\frac{x}{\alpha}\right) + f(-x) - f(-x) - f(x)\right) \\ &\leq P\left(\alpha f\left(\frac{x}{\alpha}\right) + f(-x)\right) + P\left(f(-x) + f(x)\right) \\ &\leq \left(1 + \frac{1}{\alpha^r}\right)\theta\|x\|^r + 2\theta\|x\|^r \\ &\leq \left(3 + \frac{1}{\alpha^r}\right)\theta\|x\|^r \end{aligned} \tag{3.6}$$

for all $x \in E$. For $i \in \mathbb{N}$, replacing $x = \frac{x}{\alpha^i}$ in (3.6), we get that

$$\begin{aligned} P\left(\alpha^{i+1} f\left(\frac{x}{\alpha^{i+1}}\right) - \alpha^i f\left(\frac{x}{\alpha^i}\right)\right) &\leq \alpha^i P\left(\alpha f\left(\frac{x}{\alpha^{i+1}}\right) - f\left(\frac{x}{\alpha^i}\right)\right) \\ &\leq \alpha^i\left(3 + \frac{1}{\alpha^r}\right)\theta\left\|\frac{x}{\alpha^i}\right\|^r \\ &= \left(\frac{1}{\alpha^{r-1}}\right)^i\left(3 + \frac{1}{\alpha^r}\right)\theta\|x\|^r \end{aligned} \tag{3.7}$$

for all $x \in E$. For given nonnegative integer l, m such that $l < m$, we have

$$\begin{aligned} &P\left(\alpha^m f\left(\frac{x}{\alpha^m}\right) - \alpha^l f\left(\frac{x}{\alpha^l}\right)\right) \\ =&P\left(\alpha^m f\left(\frac{x}{\alpha^m}\right) - \alpha^{m-1} f\left(\frac{x}{\alpha^{m-1}}\right) + \alpha^{m-1} f\left(\frac{x}{\alpha^{m-1}}\right)\right. \\ &\left. + \cdots + \alpha^{l+1} f\left(\frac{x}{\alpha^{l+1}}\right) - \alpha^l f\left(\frac{x}{\alpha^l}\right)\right) \\ \leq&\sum_{j=l}^{m-1} P\left(\alpha^{j+1} f\left(\frac{x}{\alpha^{j+1}}\right) - \alpha^j f\left(\frac{x}{\alpha^j}\right)\right) \\ \leq&\sum_{j=l}^{m-1}\left(\frac{1}{\alpha^{r-1}}\right)^i\left(3 + \frac{1}{\alpha^r}\right)\theta\|x\|^r \\ \leq&\left(3 + \frac{1}{\alpha^r}\right)\theta\|x\|^r\sum_{j=0}^{\infty}\left(\frac{1}{\alpha^{r-1}}\right)^i \end{aligned} \tag{3.8}$$

for all $x \in E$. Since $r > 1$, we have $\frac{1}{\alpha^{r-1}} < 1$. Since $\frac{1}{\alpha^{r-1}} < 1$, the sequence $\{\alpha^n f\left(\frac{x}{\alpha^n}\right)\}$ is Cauchy sequence for all $x \in E$. By completeness of X, the sequence $\{\alpha^n f\left(\frac{x}{\alpha^n}\right)\}$ converges. Define $F : E \to X$ by

$$F(x) = \lim_{n\to\infty} \alpha^n f\left(\frac{x}{\alpha^n}\right) \tag{3.9}$$

for all $x \in E$. Moreover, letting $l = 0$ and taking limit $m \to \infty$ in (3.8), we can obtain that inequality (3.2). It follows from (3.1) and (3.9) that

$$\begin{aligned}
&P\left(\alpha F\left(\frac{x+y}{\alpha}+z\right) - F(x) - F(y) - \alpha F(z)\right)\\
=&P\left(\alpha\cdot \lim_{n\to\infty} \alpha^n f\left(\frac{\frac{x+y}{\alpha}+z}{\alpha^n}\right) - \lim_{n\to\infty} \alpha^n f\left(\frac{x}{\alpha^n}\right) - \lim_{n\to\infty} \alpha^n f\left(\frac{y}{\alpha^n}\right)\right.\\
&\left.-\alpha \lim_{n\to\infty} \alpha^n f\left(\frac{z}{\alpha^n}\right)\right)\\
=&\lim_{n\to\infty} \alpha^n P\left(\alpha f\left(\frac{\frac{x}{\alpha^n}+\frac{y}{\alpha^n}}{\alpha}+\frac{z}{\alpha^n}\right) - f\left(\frac{x}{\alpha^n}\right) - f\left(\frac{y}{\alpha^n}\right) - \alpha f\left(\frac{z}{\alpha^n}\right)\right)\\
\leq&\lim_{n\to\infty} \alpha^n \theta\left(\left\|\frac{x}{\alpha^n}\right\|^r + \left\|\frac{x}{\alpha^n}\right\|^r + \left\|\frac{x}{\alpha^n}\right\|^r\right)\\
=&\theta\|x\|^r \lim_{n\to\infty}\left(\frac{1}{\alpha^{r-1}}\right)^n = 0
\end{aligned}$$

for all $x, y, z \in E$. Since X is total, we have

$$\alpha F\left(\frac{x+y}{\alpha}+z\right) = F(x) + F(y) + \alpha F(z)$$

for all $x, y, z \in E$. By Corollary 1.1, F is additive. Next, we will show that F is unique. Let G be another mapping satisfying (1.5) and (3.2). Then, we consider

$$\begin{aligned}
P(F(x) - G(x)) &= P\left(nF\left(\frac{x}{n}\right) - nf\left(\frac{x}{n}\right) + nf\left(\frac{x}{n}\right) - nG\left(\frac{x}{n}\right)\right)\\
&\leq n\left(P\left(F\left(\frac{x}{n}\right) - f\left(\frac{x}{n}\right)\right) + P\left(f\left(\frac{x}{n}\right) - G\left(\frac{x}{n}\right)\right)\right)\\
&\leq 2n\theta\left(\frac{3\alpha^r+1}{\alpha^r-\alpha}\right)\left\|\frac{x}{n}\right\|^r\\
&= \left(\frac{1}{n^{r-1}}\right)2\theta\left(\frac{3\alpha^r+1}{\alpha^r-\alpha}\right)\|x\|^r
\end{aligned} \tag{3.10}$$

for all $x \in E$. Since $r - 1 > 0$, taking limit $n \to \infty$ in (3.10), we have $P(F(x) - G(x)) = 0$ for all $x \in E$. Since X is total, we have $F(x) = G(x)$ for all $x \in E$, that is F is unique. □

Theorem 3.2. *Let r be a positive real number with $r < 1$ and let $f : X \to E$ be a mapping satisfying*

$$\left\|\alpha f\left(\frac{x+y}{\alpha}+z\right) - f(x) - f(y) - \alpha f(z)\right\| \leq P(x)^r + P(y)^r + P(z)^r \tag{3.11}$$

for all $x, y, z \in X$. Then there exists a unique mapping $F : X \to E$ satisfying (1.5) such that

$$\|f(x) - F(x)\| \leq \frac{2+3\alpha^r}{\alpha-\alpha^r} P(x)^r \tag{3.12}$$

for all $x \in X$ where the mapping $F : X \to E$ is given by

$$F(x) = \lim_{n\to\infty} \frac{1}{\alpha^n} f(\alpha^n x)$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (3.11), we get that

$$\begin{aligned}\|2f(0)\| &= \left\|\alpha f\left(\frac{0+0}{\alpha}+0\right) - f(0) - f(0) - \alpha f(0)\right\| \\ &\le P(0)^r + P(0)^r + P(0)^r \\ &= 0\end{aligned}$$

So $f(0) = 0$. Subtituting $x = -\alpha x$, $y = 0$ and $z = x$ in (3.11), we obtain that

$$\begin{aligned}\|f(-\alpha x) + \alpha f(x)\| &= \left\|\alpha f\left(\frac{-\alpha x+0}{\alpha}+x\right) - f(-\alpha x) - f(0) - \alpha f(x)\right\| \\ &\le P(-\alpha x)^r + P(0)^r + P(x)^r \\ &\le (1+\alpha^r)P(x)^r\end{aligned}$$

for all $x \in X$. Letting $x = -\alpha x$, $y = \alpha x$ and $z = x$, we get that

$$\begin{aligned}\|f(-\alpha x) + f(\alpha x)\| &= \left\|\alpha f\left(\frac{-\alpha x+\alpha x}{\alpha}+x\right) - f(-\alpha x) - f(\alpha x) - \alpha f(x)\right\| \\ &\le P(-\alpha x)^r + P(\alpha x)^r + P(x)^r \\ &\le (1+2\alpha^r)P(x)^r\end{aligned}$$

for all $x \in X$. Then we have

$$\begin{aligned}\|f(\alpha x) - \alpha f(x)\| &= \|f(\alpha x) + f(-\alpha x) - f(-\alpha x) - \alpha f(x)\| \\ &= \|f(\alpha x) + f(-\alpha x)\| + \|f(-\alpha x) + \alpha f(x)\| \\ &\le (1+\alpha^r)P(x)^r + (1+2\alpha^r)P(x)^r \\ &= (2+3\alpha^r)P(x)^r\end{aligned}$$

and so

$$\left\|\frac{1}{\alpha} f(\alpha x) - f(x)\right\| \le \frac{2+3\alpha^r}{\alpha} P(x)^r \tag{3.13}$$

for all $x \in X$. Replacing $x = \alpha^i x$ and multiplying by $\frac{1}{\alpha^i}$ in (3.13), we have

$$\begin{aligned}\left\|\frac{1}{\alpha^{i+1}} f(\alpha^{i+1}x) - \frac{1}{\alpha^i} f(\alpha^i x)\right\| &\le \frac{1}{\alpha^i} \cdot \frac{2+3\alpha^r}{\alpha} P(\alpha^i x)^r \\ &\le \frac{2+3\alpha^r}{\alpha} P(x)^r \cdot \left(\frac{1}{\alpha^{1-r}}\right)^i\end{aligned} \tag{3.14}$$

for all $x \in X$. By (3.14), for any integers l, m such that $0 \le l < m$, we obtain that

$$\begin{aligned}
&\left\| \frac{1}{\alpha^m} f(\alpha^m x) - \frac{1}{\alpha^l} f(\alpha^l x) \right\| \qquad (3.15)\\
=&\left\| \frac{1}{\alpha^m} f(\alpha^m x) + \frac{1}{\alpha^{m-1}} f(\alpha^{m-1} x) - \frac{1}{\alpha^{m-1}} f(\alpha^{m-1} x) + \cdots + \frac{1}{\alpha^{l+1}} f(\alpha^{l+1} x) \right.\\
&\left. - \frac{1}{\alpha^l} f(\alpha^l x) \right\|\\
\le& \sum_{i=l}^{m-1} \frac{2+3\alpha^r}{\alpha} P(x)^r \cdot \left(\frac{1}{\alpha^{1-r}} \right)^i\\
\le& \frac{2+3\alpha^r}{\alpha} P(x)^r \cdot \sum_{i=0}^{\infty} \left(\frac{1}{\alpha^{1-r}} \right)^i
\end{aligned}$$

for all $x \in X$. Since $\frac{1}{\alpha^{1-r}} < 1$, we have $\sum_{i=0}^{\infty} \left(\frac{1}{\alpha^{1-r}} \right)^i < \infty$. It follows from (3.15) that the sequence $\{\frac{1}{\alpha^n} f(\alpha^n x)\}$ is Cauchy sequence for all $x \in X$. Since E is complete, the sequence $\{\frac{1}{\alpha^n} f(\alpha^n x)\}$ is convergent sequence. We define a mapping $F : X \to E$ by

$$F(x) = \lim_{n\to\infty} \frac{1}{\alpha^n} f(\alpha^n x) \qquad (3.16)$$

for all $x \in X$. Moreover, letting $l = 0$ and taking limit $m \to \infty$ in (3.15), we can obtain that inequality (3.12). It follows from (3.11) and (3.16) that we have

$$\begin{aligned}
&\left\| \alpha F\left(\frac{x+y}{\alpha} + z \right) - F(x) - F(y) - \alpha F(z) \right\|\\
=&\left\| \alpha \lim_{n\to\infty} \frac{1}{\alpha^n} f\left(\alpha^n \left(\frac{x+y}{\alpha} + z \right) \right) - \lim_{n\to\infty} \frac{1}{\alpha^n} f(\alpha^n y) - \lim_{n\to\infty} \frac{1}{\alpha^n} f(\alpha^n z) \right.\\
&\left. - \alpha \lim_{n\to\infty} \frac{1}{\alpha^n} f(\alpha^n z) \right\|\\
\le& \lim_{n\to\infty} \frac{1}{\alpha^n} \left\| \alpha f\left(\frac{\alpha^n x + \alpha^n y}{\alpha} + \alpha^n z \right) - f(\alpha^n y) - f(\alpha^n z) - \alpha f(\alpha^n z) \right\|\\
\le& \lim_{n\to\infty} \frac{1}{\alpha^n} \left\| \alpha f\left(\frac{\alpha^n x + \alpha^n y}{\alpha} + \alpha^n z \right) - f(\alpha^n y) - f(\alpha^n z) - \alpha f(\alpha^n z) \right\|\\
\le& \lim_{n\to\infty} \frac{1}{\alpha^n} \left(P(\alpha^n x)^r + P(\alpha^n y)^r + P(\alpha^n z)^r \right)\\
\le& \lim_{n\to\infty} \frac{\alpha^{nr}}{\alpha^n} \left(P(x)^r + P(y)^r + P(z)^r \right)\\
\le& \left(P(x)^r + P(y)^r + P(z)^r \right) \lim_{n\to\infty} \left(\frac{1}{\alpha^{1-r}} \right)^n = 0
\end{aligned}$$

for all $x, y, z \in X$. Hence

$$\alpha F\left(\frac{x+y}{\alpha} + z \right) = F(x) + F(y) + \alpha F(z)$$

for all $x, y, z \in X$, that is F is the generalized Cauchy-Jensen functional equation. By tha same reasoning as in the proof of Theorem 3.1, F is unique. □

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