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$\mu-$VALUES FOR MATRICES CORRESPONDING TO SYMMETRIES IN CONTROL SYSTEMS

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Abstract. In this article we consider numerical approximation of structured singular values ($\mu-$values). The lower bounds for $\mu-$values are approximated by using ordinary differential equations based technique. The structured singular values provide a vital tool to investigate stability of feedback systems. We also compute the lower bounds of $\mu-$values for certain matrices that correspond to symmetries in control systems.

1. Introduction

The Structured Singular Values known as μ-values is a well-known mathematical tool in control, introduced in 1981 by J. C. Doyle [13]. They can be used to discuss stability of linear systems subject to certain perturbations. Applications of structured singular values in engineering system are described in [14]. The exact computation of $\mu-$values is known to be an NP-hard problem see [2].

A considerable effort has been made to compute the lower and upper bound for structured singular values. The power method [10] provides a lower bound for $\mu-$values when we consider pure complex perturbations. It however fails to converge for pure real uncertainties for more details see [16]. The upper bound of μ-values provides critical information which guarantees the stability of feedback linear systems. The well-known Matlab function mussv available in MATLAB control toolbox approximates an upper bounds for structured singular values by means of diagonal balancing and Linear Matrix Inequality techniques [5]. The methodology proposed in [12] is based on a two level algorithm, inner and outer algorithm. In inner algorithm, we attempt

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to solve an optimization problem while outer algorithm allows us to compute an extremizer by varying a small parameter ϵ.

In [4], Danielson used symmetric groups to design model predictive controllers with reduced complexity for constrained linear control systems. In model predictive control, the control input is obtained by solving a constrained finite time optimal control problem. For a piecewise affine control law symmetries are state-space and input-space transformations that relate controller pieces. Using symmetry he could discard some of the pieces of a given controller. These discarded pieces can also be reconstructed using symmetry. Using symmetries of the control system he was able to reduce the complexity of the controller and save memory without sacrificing performance. It was also noted that the amount of reduction in complexity depends on the number of symmetries possessed by the system. For systems with large symmetry groups the techniques presented in [4] can significantly reduce the complexity of the piecewise affine control-law produced using explicit model predictive control.The goal of this article is to compute the $\mu-$values for matrices corresponding to a control system whose symmetry group is S_5. We present a comparison between lower bounds of μ-values approximated by mussv and the algorithm presented in [12].

2. Basic Framework

Let $\mathbb{C}^{n,n}$ ($\mathbb{R}^{n,n}$) denote the collection of $n \times n$ complex (real) matrices and let $M \in \mathbb{C}^{n,n}$. We denote a family of block diagonal matrices by

$$\Theta_{\mathbb{B}} = \{Diag(c_i I_i,\ \Gamma_j) : c_i \in \mathbb{C}(\mathbb{R}),\, \Gamma_j \in \mathbb{C}^{m_j,m_j}(\mathbb{R}^{m_j,m_j})\}.$$

In the above equation, I_i is an identity matrix having dimension i.

Definition 2.1. [9]. The structured singular values denoted by μ for a given matrix $M \in \mathbb{C}^{n,n}$ or $M \in \mathbb{R}^{n,n}$ and a set of block diagonal matrices $\Theta_{\mathbb{B}}$ is defined as

$$\mu_{\Theta_{\mathbb{B}}}(M) := \frac{1}{\min\{\|\Delta\|_2 : \Delta \in \Theta_{\mathbb{B}}, \det(I - M\Delta) = 0\}}. \tag{2.1}$$

In above definition 2.1, $det(\cdot)$ represent the determinant of a matrix $(I - M\Delta)$ while minimum is over an admissible perturbation Δ.

In this particular case we will denote the set of pure complex uncertainties by $\Theta_{\mathbb{B}}^{'}$. If $\Delta \in \Theta_{\mathbb{B}}^{'}$, there is a function $exp(i\Phi)\Delta \in \Theta_{\mathbb{B}}^{'}$ for any real number Φ and as a consequence we have $\Delta \in \Theta_{\mathbb{B}}^{'}$ such that the spectral radius of $M\Delta$ attains the exact value 1 iff there is $\Delta^* \in \Theta_{\mathbb{B}}^{'}$ such that $M\Delta^*$ has the eigenvalue 1. The perturbation Δ^* is constructed in such a way that it possesses a unit 2-norm and as result $\det(I - M\Delta^*) = 0$. The above construction allows us to write an alternate definition of μ-values when pure complex uncertainties are under consideration.

$$\Theta_{\mathbb{B}}^{'} = \frac{1}{\min\left\{\|\Delta\|_2 : \Delta \in \Theta_{\mathbb{B}}^{'},\ \rho(M\Delta) = 1\right\}}. \tag{2.2}$$

In Equu. (2.2), $\rho(\cdot)$ denotes the spectral radius of a matrix $M\Delta$.

2.1. **The μ-value based on a structured spectral value set.** For a given n-dimensional complex matrix $M \in \mathbb{C}^{n\times n}$ and a perturbation level ϵ the structured spectral value set is the collection of all eigenvalues of matrix $(\epsilon M\Delta)$ defined as

$$\Lambda_{\epsilon_0}^{\Theta_{\mathbb{B}}}(M) = \{\lambda \in \Lambda(\epsilon M\Delta) :\ \Delta \in \Theta_{\mathbb{B}}\}, \tag{2.3}$$

where $\Lambda(\cdot)$ denotes the spectrum of a matrix and $\|\Delta\|_2 = 1$. For mixed real and complex uncertainties, we let

$$\Sigma_{\epsilon}^{\Theta_{\mathbb{B}}}(M) = \{\eta = 1 - \lambda_1 : \lambda_1 \in \Lambda_{\epsilon}^{\Theta_{\mathbb{B}}}(M)\}. \tag{2.4}$$

The formulation in Equ. (2.4) allows us to write down structured singular values defined in Equ. (2.2) as follows:

$$\mu_{\Theta_{\mathbb{B}}}(M) = \frac{1}{\arg\ \min\{0 \in \Sigma_{\epsilon}^{\Theta_{\mathbb{B}}}(M)\}}. \tag{2.5}$$

While on the other hand when pure complex uncertainties are under consideration then Equ. (2.3) allows us to alternatively express μ-value as

$$\mu_{\Theta_{\mathbb{B}}^{'}}(M) = \frac{1}{\arg\ \min\{\max|\lambda_1| = 1\}}. \tag{2.6}$$

2.2. **Mathematical Problem.** We consider the following optimization problem,

$$\xi(\epsilon_0) = \arg\min|\eta|, \tag{2.7}$$

where $\eta \in \Sigma_{\epsilon_0}^{\Theta_{\mathbb{B}}}(M)$ for some fixed parameter $\epsilon_1 > 0$. It is clear from the above discussion that the μ-value $\mu_{\Theta_{\mathbb{B}}}(M)$ is the reciprocal of the minimum value of ϵ_1 for which $\eta(\epsilon_1) = 0$. Therefore we suggests a two-level algorithm that is inner and outer algorithm. For inner algorithm, we solve Equ. (2.7) by constructing and then solving a gradient system of ordinary differential equations. While for the case of outer algorithm, we make use of an iterative method to first vary the perturbation level ϵ_1. This gives the knowledge of the computation of derivative of a local extremizer say $\Delta(\epsilon_1)$ with respect to some fixed parameter ϵ_1.

We addressed the case of a purely complex uncertainties when $\Theta_{\mathbb{B}}^{*}$ by taking the inner algorithm in order to compute a local optimum for

$$\lambda(\epsilon_1) = \arg\max|\lambda_1|. \tag{2.8}$$

In Equ. (2.8), $\lambda_1 \in \Lambda_{\epsilon_1}^{\Theta_{\mathbb{B}}^{*}}(M)$ which yields a lower bound for the μ-value in case of pure complex perturbations that is $\mu_{\Delta_{\mathbb{B}}^{*}}(M)$.

3. Purely Complex uncertainties

In this section, we give a solution of the maximization problem (2.8) for $M \in \mathbb{C}^{n,n}$ while considering the set of pure complex uncertainties given below

$$\Theta_{\mathbb{B}}^{*} = \{diag(\alpha_1 I_1, ..., \alpha_n I_n; \Delta_1, ..., \Delta_F) : \alpha_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j, m_j}\}, \tag{3.1}$$

The following lemma describes the behavior of the eigenvalues of a matrix valued function.

Lemma 3.1. For a family of matrices $\Upsilon : \mathbb{R} \to \mathbb{C}^{n,n}$ suppose that $\lambda_1(t)$ is an eigenvalue of a matrix valued function $\Upsilon(t)$ which converges to a simple eigenvalue $\lambda^{'}$ of $\Upsilon_0 = \Upsilon(0)$ as $t \to 0$. Then $\lambda_1(t)$ is analytic near $t = 0$ with

$$\frac{d\lambda_1}{dt} = \frac{w_0^* \Upsilon_1 v_0}{w_0^* v_0},$$

where $\Upsilon_1 = \dot{\Upsilon}(0)$ and v_0, w_0 are right and left eigenvectors of Υ_0 associated to $\lambda^{'}$, that is, $(\Upsilon_0 - \lambda^{'} I)v_0 = 0$ and $w_0^*(\Upsilon_0 - \lambda^{'} I) = 0$.

To deal with the optimization problem (2.8) we need to compute an uncertainty Δ_{local} in such a way that $\rho(\epsilon_1 A \Delta_{local})$ has the maximum growth along $\Delta \in \Theta_{\mathbb{B}}^{*}$ with $\|\Delta\|_2 \leq 1$. In the following we call λ_1 the greatest eigenvalue if $|\lambda_1|$ equals to the spectral radius.

Definition 3.2. A matrix valued function $\Delta \in \Theta_{\mathbb{B}}^{*}$ such that $\|\Delta\|$ possesses a unit 2-norm and $(\epsilon_1 M \Delta)$ has greatest eigenvalue which maximizes the modulus of $\Lambda_{\epsilon_1}^{\Theta_{\mathbb{B}}^{*}}(M)$ is called a local extremizer. In the following theorem 3.3, we give the characterization of local extremizers towards a gradient system of ordinary differential equations.

Theorem 3.3 [12]. Let

$$\Delta_{local} = Diag(\alpha_1 I_1, ..., \alpha_n I_n; \Delta_1, ..., \Delta_F).$$

In above equation the Δ_{local} possesses a unit 2-norm and is a local extremizer of $\Lambda_{\epsilon_1}^{\Theta_{\mathbb{B}}^{*}}(A)$. Further suppose that the matrix $(\epsilon_1 M \Delta_{local})$ possesses a simple greatest eigenvalue that is $\lambda_1 = |\lambda_1| e^{i\theta}$, with the right and left eigenvectors v and w which are scaled so that $s = e^{i\theta} w^* v > 0$. partitioning of u and v yields

$$v = (v_1^{\mathrm{T}},\ \ldots,\ v_n^{\mathrm{T}},\ v_{n+1}^{\mathrm{T}},\ \ldots\ , v_{n+F}^{\mathrm{T}})^{\mathrm{T}};$$

$$u = A^* w = (u_1^{\mathrm{T}},\ \ldots\ , u_n^{\mathrm{T}},\ u_{n+1}^{\mathrm{T}},\ \ldots\ , u_{n+F}^{\mathrm{T}})^{\mathrm{T}}, \tag{3.2}$$

additionally we assume that

$$u_k^* v_k \neq 0 \quad \forall\ k = 1, \ldots, n, \tag{3.3}$$

$$\|u_{n+h}\|_2 \cdot \|v_{n+h}\|_2 \neq 0 \quad \forall\ h = 1, \ldots, F. \tag{3.4}$$

Then

$$|s_k| = 1 \quad \forall\ k = 1, \ldots, n \quad \text{and} \quad \|\Delta_h\|_2 = 1 \quad \forall h = 1, \ldots, F.$$

3.1. **System of ODEs to compute extremal points of $\Lambda_{\epsilon}^{\Delta_{\mathbb{B}}^*}(M)$.** We now compute a local maximizer for $|\lambda_1|$ where $\lambda_1 \in \Lambda_{\epsilon_1}^{\Theta_{\mathbb{B}}*}(M)$. For this we first construct a matrix valued function $\Delta(t)$ in such a way that the greatest eigenvalue $\lambda_{(t)}$ of the matrix $(\epsilon_1 M\Delta(t))$ has the maximum growth. We then derive and solve a gradient system of ordinary differential equations which satisfies the initial choice $\Delta(0)$.

3.2. **The local optimization problem.** Let $\lambda_1 = |\lambda_1|e^{i\theta}$ be a simple eigenvalue. Further suppose that v, w are normalized so that

$$\|w\| = \|v\| = 1, \qquad w^*v = |w^*v|e^{-i\theta}. \tag{3.5}$$

By making use of Lemma 3.1, we have

$$\frac{d}{dt}|\lambda_1|^2 = 2|\lambda_1| Re\Big(\frac{u^*\dot{\Delta}v}{e^{i\theta}w^*v}\Big) = \frac{2|\lambda_1|}{|w^*v|}Re(u^*\dot{\Delta}v), \tag{3.6}$$

where $u = M^*w$. For $\Delta \in \Theta_{\mathbb{B}}$ we aim to compute the direction $\dot{\Delta} = \tau$ that maximizes the local growth of the modulus of λ_1. We get

$$\tau = diag(\omega_1 I_{r_1}, \ldots, \omega_s I_{r_N}, \Omega_1, \ldots, \Omega_F) \tag{3.7}$$

as a solution of the optimization problem

$$\tau_* = \arg\max\{Re(u^*\tau x)\}$$

$$\text{subject to } \; Re(\overline{\delta}_i\omega_i) = 0, \;\; i = 1 : N,$$

$$\text{and } \quad Re\langle \Delta_j, \Omega_j\rangle = 0, \;\; j = 1 : F. \tag{3.8}$$

In the following Lemma 3.2.1, we give the solution τ_* of the optimization problem as discussed in the (3.8).

Lemma 3.2.1 [12].

$$\tau_* = Diag(\omega_1 I_{r_1}, \ldots, \omega_N I_{r_N}, \Omega_1, \ldots, \Omega_F), \tag{3.9}$$

with

$$\omega_i \;=\; \nu_i\left(v_i^* u_i - Re\left(v_i^* u_i \overline{s}_i\right) s_i\right), \quad i = 1, \ldots, N \tag{3.10}$$

$$\Omega_j \;=\; \zeta_j\left(u_{N+j}v_{N+j}^* - Re\langle \Delta_j, u_{N+j}v_{N+j}^*\rangle \Delta_j\right), \quad j = 1, \ldots, F. \tag{3.11}$$

The coefficient $\nu_i > 0$ is the reciprocal of the absolute value of the expression appearing in the right-hand side in Equ. (3.10) when it's different from zero and $\nu_i = 1$ else. While the coefficient $\zeta_j > 0$ is the reciprocal of the Frobenius norm of the matrix appearing in the right hand side of Equ. (3.11) if it's different from zero and $\zeta_j = 1$ else.

We write down the result as obtained in the previous Lemma 1.2 as:

$$\tau_* = G_1 P_{\Theta_{\mathbb{B}}^*}(uv^*) - D_2\Delta. \tag{3.12}$$

In above equation $P_{\Theta^*_{\mathbb{B}}}(\cdot)$ is the orthogonal projection while $G_1, D_2 \in \Theta^*_{\mathbb{B}}$ are diagonal matrices while the matrix D_1 is positive.

3.3. **The gradient system of ordinary differential equations.** The result in the previous Lemma 3.2.1 allows us to consider the following differential equation on $\Theta^*_{\mathbb{B}}$:

$$\dot{\Delta} = G_1 P_{\Theta^*_{\mathbb{B}}}(uv^*) - D_2\Delta. \tag{3.13}$$

In the above equation $v(t)$ is an eigenvector having the unit 2-norm ans is associated to a simple eigenvalue $\lambda_{(t)}$ of the matrix $(\epsilon_1 M\Delta(t))$ for some fixed parameter $\epsilon_1 > 0$. The differential equation (3.13) is a gradient system of ordinary differential equations because it's the right-hand side is the projected gradient of $\tau \mapsto Re(u^*\tau v)$.

3.4. **Choice of initial value matrix and ϵ_0.** In order to compute ϵ_0 we choose the initial value matrix

$$\Delta_0 = D\, P_{\Delta_{\mathbb{B}}}(wv^*), \tag{3.14}$$

where D is the positive diagonal matrix such that $\Delta_0 \in \Theta_{\mathbb{B}}$. As a natural choice for the initialization of the perturbation level, we take ϵ_0 as

$$\epsilon_0 = \frac{1}{\widehat{\mu}_{\Theta_{\mathbb{B}}}(M)}. \tag{3.15}$$

where $\widehat{\mu}_{\Theta_{\mathbb{B}}}(M)$ is the upper bound of μ-value approximated by mussv.

4. Numerical Experimentation

In this section, we present the main contribution which is the numerical approximation of both lower and upper bounds of μ-values. These results are computed by well-known Matlab routine mussv and the algorithm [12].

Example 1. In table 1, we give comparison of numerical approximation to both lower and upper bounds of structured singular values approximated by the well-known MATLAB routine mussv and algorithm [12] for the matrix A_4. The matrix A_4 is gievn as below. In first column of table 1, we present the size of the matrix A_4. While in the next column, we present the family of block diagonal matrices which is denoted by $\Theta_{\mathbb{B}}$. In the third, fourth and fifth columns, we present both upper and lower bounds of SSV, that is, μ_u^{mussv}, μ_l^{mussv} approximated by MATLAB routine mussv and the lower bound μ_l^{New} approximated by algorithm [12] respectively.

$$A_4 = \begin{bmatrix} -0.5+1.4434i & -0.5774i & 0.5-0.2887i & -0.5+0.2887 \\ -0.5+0.8660i & 0 & 0 & 0 \\ -0.5+0.8660i & 0 & -0.5-0.8660i & -1 \\ -0.5-0.2887i & -0.5774i & 0.5774i & -0.8660i \end{bmatrix},$$

n	$\Theta_{\mathbb{B}}$	μ_u^{mussv}	μ_l^{mussv}	μ_l^{New}
04	$\{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{4,4}\}$	2.5031	2.5030	2.5030
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1, \delta_4 I_1) : \delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{R}\}$	0.6354	0.0000	0.6297
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1, \delta_4 I_1) : \delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{C}\}$	2.3780	2.3748	2.3748
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \Delta_2) : \delta_1, \delta_2 \in \mathbb{R}, \Delta_1 \in \mathbb{C}^{2,2}\}$	1.8114	1.7568	1.7552
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \Delta_2) : \delta_1, \delta_2 \in \mathbb{C}, \Delta_1 \in \mathbb{C}^{2,2}\}$	2.3832	2.3813	2.3811
04	$\{diag(\Delta_1, \Delta_2) : \Delta_1, \Delta_2 \in \mathbb{C}^{2,2}\}$	2.4047	2.4027	2.4029
04	$\{diag(\delta_1 I_1, \Delta_2) : \delta_1 \in \mathbb{R}, \Delta_2 \in \mathbb{C}^{3,3}\}$	1.8415	1.8415	1.8414

Table 1. Computation of bounds of μ-values

n	$\Theta_{\mathbb{B}}$	μ_u^{mussv}	μ_l^{mussv}	μ_l^{New}
04	$\{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{4,4}\}$	2.8765	2.8765	2.8763
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1, \delta_4 I_1) : \delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{R}\}$	1.5023	0.0000	1.0000
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1, \delta_4 I_1) : \delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{C}\}$	2.7375	2.7326	2.7326
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \Delta_2) : \delta_1, \delta_2 \in \mathbb{R}, \Delta_1 \in \mathbb{C}^{2,2}\}$	1.8732	1.8716	0.5373
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \Delta_2) : \delta_1, \delta_2 \in \mathbb{C}, \Delta_1 \in \mathbb{C}^{2,2}\}$	2.7375	2.7334	2.7336
04	$\{diag(\Delta_1, \Delta_2) : \Delta_1, \Delta_2 \in \mathbb{C}^{2,2}\}$	2.7375	2.7373	2.7372

Table 2. Computation of bounds of μ-values

In table 2, we give comparison of numerical approximation to both lower and upper bounds of structured singular values approximated by the well-known MATLAB routine mussv and algorithm [12] for the matrix B_4. The matrix B_4 is gievn as below. In first column of table 2, we present the size of the matrix B_4. While in the next column, we present the family of block diagonal matrices which is denoted by $\Theta_{\mathbb{B}}$. In the third, fourth and fifth columns we present both upper and lower bounds that is μ_u^{mussv}, μ_l^{mussv} approximated by MATLAB routine mussv and the lower bound μ_l^{New} approximated by algorithm [12] respectively.

$$B_4 = \begin{bmatrix} 0.5 + 0.8660i & -1 & -0.5 - 0.8660i & -1 \\ -0.5 + 0.8660i & 0 & -0.5 - 0.8660i & -1 \\ 0.5774i & -0.5 + 0.2887i & 0.5 - 0.2887i & -0.5 + 0.2887i \\ -0.5 + 0.2887i & -0.5774i & 0.5 - 0.2887i & 1 - 0.5774i \end{bmatrix}.$$

Example 2. Consider the following four dimensional matrix A_5.

$$A_5 = \begin{bmatrix} -0.5+0.8660i & 0.2500-0.4330i & 0.5 & 0.5 \\ 0 & -0.75-0.4330i & -0.5000 & 0+0.8660i \\ 0.5+0.8660i & 0 & 0 & 0 \\ -0.5-0.8660i & 0.25+0.4330i & 0.25+1.2990i & 0.25-0.4330i \end{bmatrix}.$$

Consider the perturbations set as

$$\Theta_{\mathbb{B}} = \{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1, \delta_4 I_1) : \delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{R}\}.$$

By applying MATLAB function mussv, we have obtained the perturbation $\widehat{\Delta}$ with

$$\widehat{\Delta} = 1.0e+050 \begin{bmatrix} 4.9136 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 4.9136 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 4.9136 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 4.9136 \end{bmatrix},$$

while $\|\widehat{\Delta}\|_2 = 4.9136e+050$. For this particular example, we have obtained an upper bound $\mu_{PD}^{upper} = 1.5797$ while the same lower bound as $\mu_{PD}^{lower} = 0.0000$. By making use of our algorithm [12], we have obtained the perturbation $\epsilon^* \Delta^*$ with

$$\Delta^* = \begin{bmatrix} -0.7049 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.7490 \end{bmatrix},$$

and $\epsilon^* = 0.7549$. while $\|\Delta^*\|_2 = 1$. The same lower bound can be obtained $\mu_{New}^{lower} = 1.3248$ as the one obtained by mussv.

In the table 3, we give comparison of numerical approximation to both lower and upper bounds of structured singular values approximated by the well-known MATLAB routine mussv and algorithm [12] for the matrix A_5. The matrix A_5 is gievn as below. In first column of table 3, we present the size of the matrix A_5. While in the next column, we present the family of block diagonal matrices which is denoted by $\Theta_{\mathbb{B}}$. In the third, fourth and fifth columns we present both upper and lower bounds that is μ_u^{mussv}, μ_l^{mussv} approximated by MATLAB routine mussv and the lower bound μ_l^{New} approximated by algorithm [12] respectively.

In the following table 4, we give comparison of numerical approximation to both lower and upper bounds of structured singular values approximated by the well-known MATLAB routine mussv and algorithm [12] for the matrix B_5. The matrix B_5 is gievn as below. In first column of table 4, we present the size of the matrix B_5. While in the next column, we present the family of block diagonal matrices which is denoted by $\Theta_{\mathbb{B}}$. In the third, fourth and fifth columns we present both upper and lower bounds that is

n	$\Theta_{\mathbb{B}}$	μ_u^{mussv}	μ_l^{mussv}	μ_l^{New}
04	$\{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{4,4}\}$	2.2701	2.2701	2.2699
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1, \delta_4 I_1) : \delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{C}\}$	1.8679	1.8679	1.8675
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \Delta_1) : \delta_1, \delta_2 \in \mathbb{C}, \Delta_1 \in \mathbb{C}^{2,2}\}$	2.0084	2.0084	2.0076
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \Delta_1) : \delta_1, \delta_2 \in \mathbb{R}, \Delta_1 \in \mathbb{C}^{2,2}\}$	2.1863	2.1863	2.1861
04	$\{diag(\delta_1 I_1, \Delta_1) : \delta_1 \in \mathbb{R}, \Delta_1 \in \mathbb{C}^{3,3}\}$	2.0242	2.0242	0.5000
04	$\{diag(\delta_1 I_1, \Delta_1) : \delta_1 \in \mathbb{C}, \Delta_1 \in \mathbb{C}^{3,3}\}$	2.1956	2.1932	2.1934

Table 3. Computation of bounds of μ-values

n	$\Delta_{\mathbb{B}}$	μ_u^{mussv}	μ_l^{mussv}	μ_l^{New}
04	$\{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{4,4}\}$	2.9458	2.9458	2.9456
04	$\{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1, \delta_4 I_1) : \delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{R}\}$	1.5155	0.0000	1.0000
04	$\{diag(\delta_1 I_2, \delta_2 I_2) : \delta_1, \delta_2 \in \mathbb{R}\}$	1.2164	0.0000	1.0894
04	$\{diag(\delta_1 I_1, \Delta_2) : \delta_1 \in \mathbb{R}, \Delta_2 \in \mathbb{R}^{3,3}\}$	1.1641	0.0000	1.0000
04	$\{diag(\Delta_1) : \Delta_1 \in \mathbb{R}^{4,4}\}$	1.2165	0.0000	0.7275

Table 4. Computation of bounds of μ-values

μ_u^{mussv}, μ_l^{mussv} approximated by MATLAB routine mussv and the lower bound μ_l^{New} approximated by algorithm [12] respectively.

$$B_5 = \begin{bmatrix} -1 & 0.75 - 0.4330i & 0.25 + 1.2990i & -0.75 - 0.4330i \\ 0 & -0.5 & 0.25 + 1.2990i & -0.25 - 1.2990i \\ 0 & -0.8660i & -0.2500 - 1.2990i & -0.75 + 0.4330i \\ 0 & 0.75 + 0.4330i & -0.75 - 0.4330i & -0.2500 + 0.4330i \end{bmatrix}.$$

Example 3. Consider the following four dimensional matrix A_6.

$$A_6 = \begin{bmatrix} 0.3819 & 0.3819 & 1.0000 & 0 & 0 & 0 \\ 1.2360 & -0.3819 & 0.1680 & 0 & 0 & 0 \\ -0.6180 & 1.0000 & -0.6180 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3819 & -1.2360 & 0.3819 \\ 0 & 0 & 0 & 1.2360 & -0.3819 & 0.6180 \\ 0 & 0 & 0 & -1.0000 & 0.6180 & 0.6180 \end{bmatrix},$$

Consider the perturbation set as

n	$\Delta_{\mathbb{B}}$	μ_u^{mussv}	μ_l^{mussv}	μ_l^{New}
06	$\{diag(\delta_1 I_6) : \delta_1 \in \mathbb{C}\}$	1.3061	1.3037	1.3037
06	$\{diag(\delta_1 I_6) : \delta_1 \in \mathbb{R}\}\}$	0.9560	0.0000	0.9560
06	$\{diag(\delta_i I_i) : \delta_i \in \mathbb{R},\ \forall i = 1:6\}$	1.9330	1.9330	1.9330
06	$\{diag(\delta_i I_i) : \delta_i \in \mathbb{C},\ \forall i = 1:6\}$	1.1875	0.0000	1.1875
06	$\{diag(\delta_1 I_3, \delta_2 I_3) : \delta_1, \delta_2 \in \mathbb{R}\}$	0.9560	0.0000	0.9560

Table 5. Computation of bounds of μ-values

$$\Theta_{\mathbb{B}} = \{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{6,6}\}.$$

By applying the MATLAB function mussv, we have obtained the perturbation $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2136 & 0.2414 & -0.2030 \\ 0 & 0 & 0 & -0.1765 & -0.1994 & 0.1678 \\ 0 & 0 & 0 & 0.0386 & 0.0436 & -0.0367 \end{bmatrix},$$

while $\|\widehat{\Delta}\|_2 = 0.4989$. For this particular example, we have obtained an upper bound $\mu_{PD}^{upper} = 2.0043$ while the same lower bound as $\mu_{PD}^{lower} = 2.0043$. By making use of our algorithm [12], we have obtained the perturbation $\epsilon^* \Delta^*$ with

$$\Delta^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4282 & 0.4837 & -0.4069 \\ 0 & 0 & 0 & -0.3538 & -0.3997 & 0.3363 \\ 0 & 0 & 0 & 0.0773 & 0.0874 & -0.0735 \end{bmatrix},$$

and $\epsilon^* = 0.4989$. while $\|\Delta^*\|_2 = 1$. The same lower bound can be obtained $\mu_{New}^{lower} = 2.0043$ as the one obtained by mussv.

In table 5, we give comparison of numerical approximation to both lower and upper bounds of structured singular values approximated by the well-known MATLAB routine mussv and algorithm [12] for the matrix A_6. The matrix A_6 is gievn as below. In first column of table 5, we present the size of the matrix A_6. While in the next column, we present the family of block diagonal matrices which is denoted by $\Theta_{\mathbb{B}}$. In the third, fourth and fifth columns we present both upper and lower bounds that is μ_u^{mussv}, μ_l^{mussv} approximated by MATLAB routine mussv and the lower bound μ_l^{New} approximated by algorithm [12] respectively.

Consider the following four dimensional matrix B_6.

$$B_6 = \begin{bmatrix} 0 & 0 & 0 & -1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.0000 & 0 \\ 0 & 0 & 0 & 0.3819 & 0.3819 & 1.0000 \\ -1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.0000 & 0 & 0 & 0 & 0 \\ 0.3819 & 0.3819 & 1.0000 & 0 & 0 & 0 \end{bmatrix},$$

Consider the perturbation set as

$$\Theta_{\mathbb{B}} = \{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{6,6}\}.$$

By applying MATLAB function mussv, we have obtained the perturbation $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2136 & 0.2414 & -0.2030 \\ 0 & 0 & 0 & -0.1765 & -0.1994 & 0.1678 \\ 0 & 0 & 0 & 0.0386 & 0.0436 & -0.0367 \end{bmatrix},$$

and $\|\widehat{\Delta}\|_2 = 0.7658$. For this example, one can obtain the upper bound $\mu_{PD}^{upper} = 1.3059$ while the same lower bound as $\mu_{PD}^{lower} = 1.3059$. By making use of our algorithm [12], we have obtained the perturbation $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} 0 & 0 & 0 & -0.1848 & -0.1848 & 0.3413 \\ 0 & 0 & 0 & -0.1848 & -0.1848 & 0.3413 \\ 0 & 0 & 0 & -0.2002 & -0.2002 & 0.3696 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\epsilon^* = 0.7658$. while $\|\Delta^*\|_2 = 1$. The same lower bound can be obtained $\mu_{New}^{lower} = 1.3059$ as the one obtained by mussv.

In the following table 6, we give comparison of numerical approximation to both lower and upper bounds of structured singular values approximated by the well-known MATLAB routine mussv and algorithm [12] for the matrix B_6. The matrix B_6 is gievn as below. In first column of table 6, we present the size of the matrix B_6. While in the next column, we present the family of block diagonal matrices which is denoted by $\Theta_{\mathbb{B}}$. In the third, fourth and fifth columns we present both upper and lower bounds that is μ_u^{mussv}, μ_l^{mussv}

n	$\Theta_{\mathbb{B}}$	μ_u^{mussv}	μ_l^{mussv}	μ_l^{New}
06	$\{diag(\delta_1 I_6) : \delta_1 \in \mathbb{C}\}$	1.0016	1.0000	0.5000
06	$\{diag(\delta_1 I_6) : \delta_1 \in \mathbb{R}\}\}$	0.9560	0.0000	1.0000
06	$\{diag(\delta_i I_i) : \delta_i \in \mathbb{R},\ \forall i = 1 : 6\}$	1.0000	0.0000	1.0000
06	$\{diag(\delta_i I_i) : \delta_i \in \mathbb{C},\ \forall i = 1 : 6\}$	1.0041	1.0000	0.7453
06	$\{diag(\delta_1 I_3, \delta_2 I_3) : \delta_1, \delta_2 \in \mathbb{R}\}$	1.0813	0.0000	1.0000

Table 6. Computation of bounds of μ-values

approximated by MATLAB routine mussv and the lower bound μ_l^{New} approximated by algorithm [12] respectively.

Example 4. In the following example, we consider a five dimensional complex matrix A_7 given as,

$$A_7 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

Consider the perturbation set as

$$\Theta_{\mathbb{B}} = \{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{5,5}\}.$$

By applying Apply the MATLAB function mussv, we have obtained the perturbation $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0.1028 & -0.0978 & 0.0226 & -0.0000 & 0.0441 \\ 0.0802 & -0.0763 & 0.0176 & -0.0000 & 0.0344 \\ 0.2006 & -0.1909 & 0.0441 & -0.0000 & 0.0860 \\ 0.1644 & -0.1565 & 0.0361 & -0.0000 & 0.0705 \end{bmatrix},$$

and $\|\widehat{\Delta}\|_2 = 0.4244$. For this example, one can obtain the upper bound $\mu_{PD}^{upper} = 2.3563$ while the same lower bound as $\mu_{PD}^{lower} = 2.3563$. By making use of our algorithm [12], we have obtained the perturbation $\epsilon^* \Delta^*$

with

$$\Delta^* = \begin{bmatrix} 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.2421 & -0.2304 & 0.0532 & 0.0000 & 0.1038 \\ 0.1889 & -0.1798 & 0.0415 & 0.0000 & 0.0810 \\ 0.4726 & -0.4498 & 0.1038 & 0.0000 & 0.2026 \\ 0.3874 & -0.3687 & 0.0851 & 0.0000 & 0.1661 \end{bmatrix},$$

and $\epsilon^* = 0.4244$ while $\|\Delta^*\|_2 = 1$ The same lower bound can be obtained $\mu_{New}^{lower} = 2.3563$ as the one obtained by mussv.

In the following example, we consider a five dimensional complex matrix B_7 given as,

$$B_7 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

Consider the perturbation set as

$$\Delta_{\mathbb{B}} = \{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{5,5}\}.$$

By applying MATLAB function mussv, we have obtained the perturbation $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} 0.1057 & -0.2887 & 0.0000 & 0.1057 & 0.0000 \\ 0.0774 & -0.2113 & 0.0000 & 0.0774 & 0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.1057 & -0.2887 & 0.0000 & 0.1057 & 0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix},$$

and $\|\widehat{\Delta}\|_2 = 0.5176$. For this example, one can obtain the upper bound $\mu_{PD}^{upper} = 1.9319$ while the same lower bound as $\mu_{PD}^{lower} = 1.9319$. By making use of our algorithm [12], we have obtained the perturbation $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -0.5000 & 0.0000 & 0.5000 \\ 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & 0.5000 & -0.0000 & -0.5000 \end{bmatrix},$$

and $\epsilon^* = 1.0000$ while $\|\Delta^*\|_2 = 1.0000$ The same lower bound can be obtained $\mu_{New}^{lower} = 1.0000$ as the one obtained by mussv.

Example 5. In the following table 7, we give comparison of numerical approximation to both lower and upper bounds of structured singular values approximated by the well-known MATLAB routine mussv and algorithm [12] for the matrix A_8. The matrix A_8 is gievn as below. In first column of table 7, we present the size of the matrix A_8. While in the next column, we present the family of block diagonal matrices which is denoted by $\Theta_{\mathbb{B}}$. In the third, fourth and fifth columns we present both upper and lower bounds that is μ_u^{mussv}, μ_l^{mussv} approximated by MATLAB routine mussv and the lower bound μ_l^{New} approximated by algorithm [12] respectively.

$$A_8 = \begin{bmatrix} 0 & 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

In the following example, we consider a five dimensional complex matrix B_8 given as,

$$B_8 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix},$$

Consider the perturbation set as

$$\Theta_{\mathbb{B}} = \{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{5,5}\}.$$

By Applying the MATLAB function mussv, we have obtained the perturbation $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} -0.0288 & 0.0576 & 0.0288 & 0.1091 & 0.1091 \\ 0.0576 & -0.1151 & -0.0576 & -0.2182 & -0.2182 \\ 0.0288 & -0.0576 & -0.0288 & -0.1091 & -0.1091 \\ -0.0228 & 0.0455 & 0.0228 & 0.0863 & 0.0863 \\ -0.0228 & 0.0455 & 0.0228 & 0.0863 & 0.0863 \end{bmatrix},$$

n	$\Theta_{\mathbb{B}}$	μ_u^{mussv}	μ_l^{mussv}	μ_l^{New}
05	$\{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{5,5}\}$	2.4142	2.4142	2.4142
05	$\{diag(\delta_i I_i) : \delta_i \in \mathbb{R},\ \forall i = 1:5\}$	2.1616	2.1516	2.0609
05	$\{diag(\delta_i I_i) : \delta_i \in \mathbb{C},\ \forall i = 1:5\}$	2.1628	2.1537	1.9511
05	$\{diag(\delta_1 I_1, \delta_2 I_1, \Delta_2) : \delta_1, \delta_2 \in \mathbb{R}, \Delta_1 \in \mathbb{C}^{3,3}\}$	2.2182	2.2174	2.2176
05	$\{diag(\delta_1 I_1, \delta_2 I_1, \Delta_2) : \delta_1, \delta_2 \in \mathbb{C}, \Delta_1 \in \mathbb{C}^{3,3}\}$	2.7178	2.7176	0.5000
05	$\{diag(\Delta_1, \delta_1 I_1, \Delta_2) : \delta_1 \in \mathbb{C}, \Delta_1, \Delta_2 \in \mathbb{C}^{2,2}\}$	2.2592	2.2571	2.2589
05	$\{diag(\Delta_1, \delta_1 I_1, \Delta_2) : \delta_1 \in \mathbb{R}, \Delta_1, \Delta_2 \in \mathbb{C}^{2,2}\}$	2.2592	2.2592	2.2592

Table 7. Computation of bounds of μ-values

and $\|\widehat{\Delta}\|_2 = 0.4569$. For this example, one can obtain the upper bound $\mu_{PD}^{upper} = 2.1889$ while the same lower bound as $\mu_{PD}^{lower} = 2.1889$. By making use of our algorithm [12], one can obtain the perturbation $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} -0.0630 & 0.1260 & 0.0630 & 0.2388 & 0.2388 \\ 0.1260 & -0.2520 & -0.1260 & -0.4777 & -0.4777 \\ 0.0630 & -0.1260 & -0.0630 & -0.2388 & -0.2388 \\ -0.0498 & 0.0997 & 0.0498 & 0.1890 & 0.1890 \\ -0.0498 & 0.0997 & 0.0498 & 0.1890 & 0.1890 \end{bmatrix},$$

and $\epsilon^* = 0.4569$ while $\|\Delta^*\|_2 = 1.0000$. The same lower bound can be obtained $\mu_{New}^{lower} = 2.1889$ as the one obtained by mussv.

5. Conclusion

In this article we have considered the numerical approximation of μ-values for the matrix representations of finite symmetric groups S_n over the filed of complex numbers by using well-known MATLAB function mussv and algorithm [12]. The experimental results indicates the different behaviors of lower bounds of μ-values with once computed by mussv and our algorithm.

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