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SOME PROPERTIES OF ANALYTIC FUNCTIONS ASSOCIATED WITH CONIC TYPE REGIONS

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Abstract. The main purpose of this investigation is to define new subclasses of analytic functions with respect to symmetrical points. These functions map the open unit disk onto certain conic regions in the right half plane. We consider various corollaries and consequences of our main results. We also point out relevant connections to some of the earlier known developments.

1. Introduction and Definitions

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbb{E} = \{z : |z| < 1\}$. Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk $\mathbb{E}$ and satisfying the following conditions

$$f(0) = f'(0) - 1 = 0.$$

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Therefore, for $f \in \mathcal{A}$, one has

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\forall z \in \mathbb{E}). \tag{1.1}$$

Also let $\mathcal{S}$ be the subclass of $\mathcal{A}$ which consists of univalent functions in $\mathbb{E}$.

Moreover the class of starlike functions in $\mathbb{E}$ will be denoted by $\mathcal{S}^*$, which consists of normalized functions $f \in \mathcal{A}$ that satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (\forall\ z \in \mathbb{E}). \tag{1.2}$$

Similarly the class $\mathcal{C}$ of convex functions in $\mathbb{E}$ consists of normalized functions $f \in \mathcal{A}$ that satisfy the following inequality:

$$\Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0 \qquad (\forall\ z \in \mathbb{E}). \tag{1.3}$$

For two functions f and g, analytic in $\mathbb{E}$, we say that f is subordinate to g, denoted by

$$f(z) \prec g(z) \quad \text{or} \quad f \prec g,$$

if there exists a Schwarz function w which is analytic in $\mathbb{E}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| \leq |z|$$

such that

$$f(z) = g(w(z)).$$

Furthermore if the function g is univalent in $\mathbb{E}$, then one can find that

$$f(z) \prec g(z) \iff 0 = g(0) \quad \text{and} \quad f(\mathbb{E}) \subseteq g(\mathbb{E}).$$

We next denote by $\mathcal{P}$ the class of analytic functions p, which are normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.4}$$

such that

$$\Re(p(z)) > 0.$$

Definition 1.1. A function $f \in \mathcal{A}$ is said to belongs to the class $\mathcal{S}_s^*$, if and only if

$$\frac{zf'(z)}{f(z) - f(-z)} \prec \frac{1+z}{1-z} \qquad (\forall\ z \in \mathbb{E}).$$

The class $\mathcal{S}_s^*$, of starlike functions with respect to symmetrical points, was introduced by Sakaguchi in 1959, (see [23]).

Remark 1.1. For function $f \in \mathcal{A}$ the idea of Alexander's theorem [7] was used by Das and Singh [6] for defining the class $\mathcal{C}_s$ of convex functions with respect to symmetrical points, in the following way:

$$f(z) \in \mathcal{C}_s \iff zf'(z) \in \mathcal{S}_s^*.$$

Definition 1.2. A given function h with $h(0) = 1$ is said to belong to the class $\mathcal{P}[A, B]$ if and only if

$$h(z) \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1.$$

The analytic functions class $\mathcal{P}[A, B]$ was introduced by Janowski [9], who showed that $h(z) \in \mathcal{P}[A, B]$ if and only if there exist a function $p \in \mathcal{P}$ such that

$$h(z) = \frac{(A+1)\, p(z) - (A-1)}{(B+1)\, p(z) - (B-1)}, \quad -1 \leq B < A \leq 1.$$

Geometrically a function h belongs to $\mathcal{P}[A, B]$ if and only if it maps the open unit disk $\mathbb{E}$ onto the disk defined by the domain

$$\Omega[A, B] = \left\{ t \in \mathbb{C} : \left| t - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \right\}.$$

Historically speaking, the conic domain Ω_k, $k \geq 0$, was first introduced by Kanas and Wiśniowska (see [11] and [12]) as

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

Moreover for fixed k this domain represents the right half plane ($k = 0$), a parabola ($k = 1$), the right branch of hyperbola ($0 < k < 1$) and an ellipse ($k > 1$), see also [17], [18] and recently [21]. Indeed the extremal functions for these conic regions are

$$p_k(z) = \begin{cases} \frac{1}{1-k^2} \cosh\left\{ \left(\frac{2}{\pi} \arccos k\right) \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right\} - \frac{k^2}{1-k^2} & (0 \leq k < 1) \\[2ex] 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 & (k = 1) \\[2ex] \frac{1}{k^2-1} \sin\left(\frac{\pi}{2K(\kappa)} \int_0^{\frac{u(z)}{\sqrt{\kappa}}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}} \right) + \frac{k^2}{k^2-1} & (k > 1), \end{cases} \tag{1.5}$$

where

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa} z} \qquad (\forall\ z \in \mathbb{E})$$

and $\kappa \in (0, 1)$ is chosen such that $k = \cosh(\pi K'(\kappa)/(4K(\kappa)))$. Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and $K'(\kappa) = K(\sqrt{1 - \kappa^2})$ i.e. $K'(t)$ is the complementary integral of $K(t)$, [1], [2]. Assume that

$$p_k(z) = 1 + T_1(k)z + T_2(k)z^2 + \ldots \qquad (\forall\ z \in \mathbb{E}).$$

Then it was shown in [13] that for (1.5) one can have

$$T_1 := T_1(k) = \begin{cases} \frac{2A^2}{1-k^2} & (0 \leq k < 1) \\ \\ \frac{8}{\pi^2} & (k = 1) \\ \\ \frac{\pi^2}{4Ķ^2(t)^2(1+t)\sqrt{t}} & (k > 1), \end{cases} \tag{1.6}$$

$$T_2 := T_2(k) = D(k)T_1(k),$$

where

$$D(k) = \begin{cases} \frac{A^2+2}{3} & (0 \leq k < 1) \\ \\ \frac{8}{\pi^2} & (k = 1) \\ \\ \frac{(4K(\kappa))^2\left(t^2+6t+1\right)-\pi^2}{24K(\kappa)^2(1+t)\sqrt{t}} & (k > 1) \end{cases} \tag{1.7}$$

with $A = \frac{2}{\pi} \arccos k$.

Noor et al. [16] combine the concepts of Janowski functions and the conic regions and define the following:

Definition 1.3. A function $h \in \mathcal{H}$ is said to be in the class k-$\mathcal{P}[A, B]$, if and only if

$$h(z) \prec \frac{(A+1)p_k(z) - (A-1)}{(B+1))p_k(z) - (B-1)} \qquad k \geq 0, \tag{1.8}$$

where $p_k(z)$ is defined by (1.5) and $-1 \leq B < A \leq 1$.

Geometrically, each function $h \in k$-$\mathcal{P}[A, B]$ takes all values in the domain $\Omega_k[A, B], -1 \leq B < A \leq 1, k \geq 0$ which is defined as

$$\Omega_k[A, B] = \left\{ w : \Re\left(\frac{(B-1)w - (A-1)}{(B+1))w - (A+1)}\right) > k\left|\frac{(B-1)w - (A-1)}{(B+1))w - (A+1)} - 1\right|\right\},$$

or equivalently $\Omega_k[A, B]$ is a set of numbers $w = u + iv$ such that

$$\begin{aligned} &\left[\left(B^2-1\right)\left(u^2+v^2\right) - 2\left(AB-1\right)u + \left(A^2-1\right)\right]^2 \\ &> k\left[\left(-2(B+1)\right)\left(u^2+v^2\right) + 2\left(A+B+2\right)u - 2(A+1)\right)^2 + 4(A-B)^2v^2\right]. \end{aligned}$$

This domain represents the conic type regions for detail (see [16]). One can observe that

$$0\text{-}\mathcal{P}[A, B] = \mathcal{P}[A, B],$$

introduced by Janowski (see [9]) and

$$k\text{-}\mathcal{P}[1, -1] = \mathcal{P}(p_k),$$

introduced by Kanas and Wiśniowska (see [11]).

In the recent years, several interesting subclasses of analytic functions have been introduced and investigated, see for example [3], [4], [5], [19] and [22]. Motivated and inspired by the recent research going on and from the above mentioned work, we now introduce some new subclasses of analytic functions as following:

Definition 1.4. A function $f \in \mathcal{S}$ is said to be in the class $k\text{-}\mathcal{US}_s\left[A, B\right]$, $k \geq 0$, $-1 \leq B < A \leq 1$, if and only if

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{(A+1)p_k(z) - (A-1)}{(B+1))p_k(z) - (B-1)} \qquad (\forall\ z \in \mathbb{E})\,. \tag{1.9}$$

Remark 1.2. First of all, it is easily seen that

$$0\text{-}\mathcal{US}_s\left[1, -1\right] = S_s^*,$$

the class of starlike functions with respect to symmetric points introduced and studied by Sakaguchi (see [23]). Secondly, we have

$$0\text{-}\mathcal{US}_s\left[A, B\right] = S_s^*\left[A, B\right],$$

the class of Janowski starlike functions with respect to symmetric points introduced by Goel and Mehrok in 1982 (see [8]). Thirdly, we have

$$k\text{-}\mathcal{US}_s\left[1, -1\right] = k\text{-}ST_s,$$

introduced and studied by Noor (see [20]).

Definition 1.5. A function $f \in \mathcal{S}$ is said to be in the class $k\text{-}\mathcal{UC}_s\left[A, B\right]$, $k \geq 0, -1 \leq B < A \leq 1$, if and only if

$$\frac{2\left(zf'(z)\right)'}{\left(f(z) - f(-z)\right)'} \prec \frac{(A+1)p_k(z) - (A-1)}{(B+1))p_k(z) - (B-1)} \qquad (\forall\ z \in \mathbb{E})\,.$$

Remark 1.3. From Definiton 1.5 it is readily observe that

$$0\text{-}\mathcal{UC}_s\left[1, -1\right] = \mathcal{C}_s,$$

the class of convex functions with respect to symmetric points introduced and studied by Das and Singh, (see [6]). Secondly we have

$$0\text{-}\mathcal{UC}_s\left[A, B\right] = \mathcal{C}_s^*\left[A, B\right],$$

the class of Janowski convex functions with respect to symmetric points introduced by Janteng and Halim in 2008 (see [10]) . And finally

$$k\text{-}\mathcal{UC}_s\left[1, -1\right] = k\text{-}\mathcal{UCV}_s,$$

introduced and studied by Noor (see [20]).

2. A Set of Lemmas

Each of the following lemmas will be needed in our present investigation.

Lemma 2.1. [14] *If a function $w \in \mathcal{H}$ is of the form*

$$w(z) = c_1 z + c_2 z^2 + \dots \quad and \quad |w(z)| \leq |z| \qquad (\forall\ z \in \mathbb{E}), \tag{2.1}$$

then for every complex number s, we have

$$\left|c_2 - sc_1^2\right| \leq 1 + (|s| - 1)\left|c_1^2\right|.$$

Lemma 2.2. *Let $k \in [0, \infty)$ be a fixed and*

$$q_k(z) = \frac{(A+1)p_k(z) - (A-1)}{(B+1))p_k(z) - (B-1)}.$$

Then

$$q_k(z) = 1 + H_1(k)z + H_2(k)z^2 + \dots \qquad (\forall\ z \in \mathbb{E}) \tag{2.2}$$

and

$$H_1 := H_1(k) = \frac{A-B}{2} T_1(k) \tag{2.3}$$

$$H_2 := H_2(k) = \frac{(A-B)T_1(k)}{4} \{2D(k) - (B+1)T_1(k)\} \tag{2.4}$$

where $T_1(k)$ and $D(k)$ are defined by (1.6) and (1.7).

Proof. We have

$$(A+1)p_k(z) - (A-1) = \{(B+1)p_k(z) - (B-1)\}\left\{1 + H_1 z + H_2 z^2 + \dots\right\}.$$

Therefore, we obtain

$$\begin{aligned} &2 + (A+1)\left\{T_1 z + T_2 z^2 + \dots\right\} \\ &= \left[2 + (B+1)\left\{T_1 z + T_2 z^2 + \dots\right\}\right]\left[1 + H_1 z + H_2 z^2 + \dots\right] \end{aligned} \tag{2.5}$$

Comparing the coefficients at z gives

$$(A+1)T_1 = (B+1)T_1 + 2H_1,$$

so we obtain the first equality (2.3). Similarly, comparing the coefficients at z^2 gives

$$(A+1)T_2 = 2H_2 + (B+1)H_1 T_1 + (B+1)T_2,$$

so we have

$$(A-B)T_2 - (B+1)H_1 T_1 = 2H_2.$$

Applying (2.3) gives

$$\begin{aligned}H_2 &= \frac{(A-B)}{2}T_2 - \frac{(A-B)(B+1)}{4}T_1^2\\ &= \frac{(A-B)}{2}D(k)T_1 - \frac{(A-B)(B+1)}{4}T_1^2\\ &= \frac{(A-B)T_1}{4}\left\{2D(k)-(B+1)T_1\right\}.\end{aligned}$$

so, we obtain the second equality (2.4). This completes the proof. $\square$

3. Main Results

In this section, we will prove our main results. Throughout our discussion, we assume that

$$-1 \leq B < A \leq 1 \quad \text{and} \quad k \geq 0.$$

Theorem 3.1. *Let $f \in k\text{-}\mathcal{US}_s[A,B]$. Then the function*

$$\varphi(z) = \frac{1}{2}\left(f(z)-f(-z)\right), \tag{3.1}$$

belongs to $k\text{-}\mathcal{US}[A,B]$ in $\mathbb{E}$, where $k\text{-}\mathcal{US}[A,B]$ is the class of Janowski starlike functions $g(z) \in \mathcal{A}$ such that

$$\frac{zg'(z)}{g(z)} \in k\text{-}\mathcal{P}[A,B].$$

Proof. Taking logarithmic differentiation of (3.1), we have

$$\frac{z\varphi'(z)}{\varphi(z)} = \frac{z\left(f(z)\right)' + z\left(f(-z)\right)'}{\left(f(z)-f(-z)\right)}.$$

Then we find after some simplification that

$$\begin{aligned}\frac{z\varphi'(z)}{\varphi(z)} &= \frac{1}{2}\left[\frac{2z\left(f(z)\right)'}{\left(f(z)-f(-z)\right)} + \frac{2z\left(f(-z)\right)'}{\left(f(-z)-f(z)\right)}\right]\\ &= \frac{1}{2}\left[p_1(z)+p_2(z)\right], \quad p_1,p_2 \in k\text{-}P\left[A,B\right] \quad (\forall\ z \in \mathbb{E}).\end{aligned}$$

Moreover one can find that $k\text{-}\mathcal{P}[A,B]$ is a convex set (see [16]), it follows that

$$\frac{z\varphi'(z)}{\varphi(z)} \in k\text{-}\mathcal{P}[A,B]$$

and thus $\varphi(z) \in k\text{-}\mathcal{US}[A,B]$. $\square$

Remark 3.1. The above Theorem shows that the class $k\text{-}\mathcal{US}_s[A,B]$ is a subclass of the class of close-to-convex functions.

Theorem 3.2. *Let $0 \leq k < \infty$ be fixed. Assume that a function q_k defined in Lemma 2.2, has the form (2.2). If the function $h(z) = 1 + b_1 z + b_2 z^2 + \ldots$ is a member of the function class $k\text{-}\mathcal{P}[A,B]$, then for $-\infty < u < \infty$,*

$$\left|b_2 - u b_1^2\right| \leq \begin{cases} \frac{A-B}{2} T_1(k) \left\{ u \frac{(A-B)}{2} T_1(k) - \frac{1}{2} \left\{ [2D(k) - (B+1)] T_1(k) \right\} \right\} & (u > \alpha_1) \\ \\ \frac{A-B}{2} T_1(k), & (\alpha_1 \leq u \leq \alpha_2) \\ \\ \frac{A-B}{2} T_1(k) \left\{ \frac{1}{2} \left\{ [2D(k) - (B+1)] T_1(k)) \right\} - u \frac{(A-B)}{2} T_1(k) \right. & (u < \alpha_2), \end{cases} \tag{3.2}$$

where

$$\alpha_1 = \frac{[2 + 2D(k) - (B+1)] T_1(k)}{(A-B) T_1(k)},$$
$$\alpha_2 = \frac{[2D(k) - (B+1)] T_1(k) - 2}{(A-B) T_1(k)},$$

and T_1, $D(k)$ are defined by (1.6) and (1.7).

Proof. If $f \in k\text{-}\mathcal{P}[A,B]$ then it follows that

$$h(z) \prec q_k(z) = 1 + \frac{A-B}{2} T_1(k) z + \frac{(A-B)\left[2D(k) - (B+1)\right] T_1(k)}{4} T_1(k) z^2 + \ldots \qquad (\forall z \in \mathbb{E}). \tag{3.3}$$

Now by the definition of subordination there exists a function w analytic in $\mathbb{E}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad \text{such that} \quad w(z) = c_1 z + c_2 z^2 + \cdots$$

and

$$h(z) = 1 + \frac{A-B}{2} T_1(k) w(z) + \frac{(A-B)\left[2D(k) - (B+1)\right] T_1(k)}{4} T_1(k) w^2(z) + \ldots. \tag{3.4}$$

Now from (2.1), (3.3) and (3.4), we have

$$b_1 = \frac{A-B}{2} T_1(k) c_1,$$
$$b_2 = \frac{A-B}{2} T_1(k) \left\{ c_2 + \frac{\left[2D(k) - (B+1)\right] T_1(k)}{2} c_1^2 \right\}.$$

Therefore, we obtain

$$b_2 - u b_1^2 = \frac{A-B}{2} T_1(k) \left\{ c_2 + \left\{ \frac{\left[2D(k) - (B+1)\right] T_1(k)}{2} - u \frac{A-B}{2} T_1(k) \right\} c_1^2 \right\}. \tag{3.5}$$

This gives

$$\left|b_2 - u b_1^2\right| = \frac{A-B}{2} T_1(k) \left| c_2 - c_1^2 + \left\{ 1 + \frac{\left[2D(k) - (B+1)\right] T_1(k)}{2} - u \frac{A-B}{2} T_1(k) \right\} c_1^2 \right|.$$

Suppose that $u > \alpha_1$, then using the estimate $\left|c_2 - c_1^2\right| \leq 1$ from Lemma 2.1 and the well known estimate $|c_1| \leq 1$ of the Schwarz Lemma, we obtain

$$\left|b_2 - ub_1^2\right| \leq \frac{A-B}{2}T_1(k)\left\{u\frac{(A-B)}{2}T_1(k) - \frac{(2D(k)-(B+1))T_1(k))}{2}\right\}.$$

This is the first inequality in (3.2).

On the other hand if $u < \alpha_2$, then (3.5) gives

$$\left|b_2 - ub_1^2\right| \leq \frac{A-B}{2}T_1(k)\left\{|c_2| + \left\{\frac{(2D(k)-(B+1))T_1(k))}{2} - u\frac{(A-B)}{2}T_1(k)\right\}|c_1|^2\right\}.$$

Applying the estimates $|c_2| \leq 1 - |c_1|^2$ of Lemma 2.1 and $|c_1| \leq 1$, we have

$$\begin{aligned}\left|b_2 - ub_1^2\right| &\leq \frac{A-B}{2}T_1(k)\left\{1 + \left\{\frac{(2D(k)-(B+1))T_1(k))}{2} - u\frac{(A-B)}{2}T_1(k) - 1\right\}|c_1|^2\right\}\\ &\leq \frac{A-B}{2}T_1(k)\left\{\frac{(2D(k)-(B+1))T_1(k))}{2} - u\frac{(A-B)}{2}T_1(k)\right\}.\end{aligned}$$

This is the last inequality in (3.2). Finally if $\alpha_1 < u < \alpha_2$, then

$$\left|\frac{(2D(k)-(B+1))T_1(k))}{2} - u\frac{(A-B)}{2}T_1(k)\right| \leq 1.$$

Therefore (3.5), yields

$$\left|b_2 - ub_1^2\right| \leq \frac{A-B}{2}T_1(k)\left\{|c_2| + |c_1|^2\right\} \leq \frac{A-B}{2}T_1(k)\left\{1 - |c_1|^2 + |c_1|^2\right\} = \frac{A-B}{2}T_1(k).$$

We get the middle inequality in (3.2). This completes the proof. □

Remark 3.2. In above Theorem if we set

$$A = 1 \quad \text{and} \quad B = -1$$

we have the result given in [15].

Theorem 3.3. *Let the function f given by (2.1) be in the class k-$\mathcal{US}_s[A,B]$. Then*

$$\left|\mu a_2^2 - a_3\right| \leq \frac{1}{2}\begin{cases}\frac{A-B}{2}T_1(k)\left\{\frac{\mu(A-B)}{4}T_1(k) - \frac{(2D(k)-(B+1))T_1(k))}{2}\right\} & (u > \delta_1)\\[2ex] \frac{A-B}{2}T_1(k) & (\delta_1 \leq u \leq \delta_2)\\[2ex] \frac{A-B}{2}T_1(k)\left\{\frac{(2D(k)-(B+1))T_1(k))}{2} - \frac{\mu(A-B)}{4}T_1(k)\right\} & (u < \delta_2),\end{cases}$$

where

$$\delta_1 = \frac{2\left(2 + 2D(k) - (B+1))T_1(k)\right)}{(A-B)T_1(k)},$$

$$\delta_1 = \frac{2\left(2D(k) - (B+1))T_1(k) - 2\right)}{(A-B)T_1(k)}.$$

and T_1, $D(k)$ are defined by (1.6) and (1.7).

Proof. By definition of the class k-$\mathcal{US}_s\,[A,B]$, there exists a function $h \in \mathcal{S}$, represented by $h(z) = 1 + b_1 z + b_2 z^2 + \ldots$ and subordinate to q_k, where q_k is given by (2.2), such that

$$\frac{2zf'(z)}{f(z)-f(-z)} = h(z) \qquad (\forall\ z \in \mathbb{E})\,.$$

Substituting the corresponding series expansions and by equating coefficients of z and z^2, we obtain

$$a_2 = \frac{1}{2}b_1, \qquad a_3 = \frac{1}{2}b_2.$$

Therefore

$$\left|\mu a_2^2 - a_3\right| \le \frac{1}{2}\left|\frac{\mu b_1^2}{2} - b_2\right|.$$

An application of Theorem 3.2, with $u = \frac{\mu}{2}$, we obtain the result asserted by Theorem 3.3. □

Theorem 3.4. *A function $f \in k$-$\mathcal{US}_s\,[A,B]$, if and only if*

$$\frac{1}{z}\left\{f(z) * \left[\frac{z - Mz^2}{(1-z)^2(1+z)}\right]\right\} \ne 0 \qquad (\forall\ z \in \mathbb{E})\,, \quad (0 < \theta < 2\pi)\,, \tag{3.6}$$

where

$$M = \frac{(A+B+2)p_k(e^{i\theta}) - (A+B-2)}{(A-B)p_k(e^{i\theta}) + (A-B)}. \tag{3.7}$$

Proof. If $f \in k$-$\mathcal{US}_s\,[A,B]$, then we have

$$F(z) := \frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{(A+1)p_k(z) - (A-1)}{(B+1)p_k(z) - (B-1)} \qquad (\forall\ z \in \mathbb{E})\,. \tag{3.8}$$

For $0 \le k \le 1$ the function $p_k(z)$ has a pole at $z = 1$ and the curve $p_k(e^{i\theta})$, $\theta \in (0, 2\pi)$, is the imaginary axis, a hyperbola or an ellipse. For $k > 1$ the function $p_k(z)$ is analytic on the unit disk. In each of the cases, if $f(z) \in k - \mathcal{US}_s\,[A,B]$, then $F(|z| < 1)$ lies on the right with respect this curve, or

$$\frac{2zf'(z)}{f(z)-f(-z)} \ne \frac{(A+1)p_k(e^{i\theta}) - (A-1)}{(B+1)p_k(e^{i\theta}) - (B-1)} \qquad (\forall\ z \in \mathbb{E} \ \text{ and } \ 0 < \theta < 2\pi)\,.$$

A simple computations gives

$$\frac{1}{z}\left\{\begin{array}{c} zf'(z)\left[(B+1)p_k(e^{i\theta}) - (B-1)\right] - \frac{1}{2}\left[f(z) - f(-z)\right] \times \\ \left[(A+1)p_k(e^{i\theta}) - (A-1) + (B+1)p_k(e^{i\theta}) - (B-1)\right] \end{array}\right\} \ne 0, \tag{3.9}$$

for $z \in \mathbb{E}$, $\theta \in (0, 2\pi)$. Using the convolution properties

$$f(z) * \frac{z}{(1-z)^2} = zf'(z) \quad \text{and} \quad f(z) * \frac{z}{1-z^2} = \frac{1}{2}\left[f(z) - f(-z)\right] \qquad (\forall\ z \in \mathbb{E})\,,$$

we have that

$$\frac{1}{z}\left\{\begin{array}{l} f(z) * \left[\frac{z\left[(B+1)p_k(e^{i\theta}) - (B-1)\right]}{(1-z)^2}\right. \\ \qquad \left. - \frac{z\left[(A+1)p_k(e^{i\theta}) - (A-1) + (B+1)p_k(e^{i\theta}) - (B-1)\right]}{1-z^2}\right] \end{array}\right\} \ne 0.$$

Hence it follows that

$$\frac{1}{z}\left\{f(z)*\frac{z-\frac{(B+1)p_k(e^{i\theta})-(B-1)+(A+1)p_k(e^{i\theta})-(A-1)}{(B+1)p_k(e^{i\theta})-(B-1)-(A+1)p_k(e^{i\theta})-(A-1)}z^2}{(1-z)^2(1+z)}\right\}\neq 0 \tag{3.10}$$

for $z\in\mathbb{E}$, $\theta\in(0,2\pi)$, which is the required conditions (3.6) and (3.7).

Conversely, suppose that the condition (3.6) holds. Therefore we have

$$\frac{2zf'(z)}{f(z)-f(-z)}\neq\frac{(A+1)p_k(e^{i\theta})-(A-1)}{(B+1)p_k(e^{i\theta})-(B-1)} \qquad (\forall\ z\in\mathbb{E}). \tag{3.11}$$

Suppose that

$$H(z)=\frac{(A+1)p_k(z)-(A-1)}{(B+1))p_k(z)-(B-1)} \qquad (\forall\ z\in\mathbb{E}).$$

Now from relation (3.11), it is clear that

$$H(\partial\mathbb{E})\cap F(\mathbb{E})=\emptyset.$$

Therefore the simply connected domain $F(\mathbb{E})$ is contained in a connected component of $\mathbb{C}\backslash H(\partial\mathbb{E})$. The univalence of the function H together with the fact $H(0)=h(0)=1$ shows that $F\prec H$ which shows that $f\in k\text{-}\mathcal{US}_s[A,B]$. □

In its special case when $k=0$, Theorem 3.4 yields the following known result.

Corollary 3.1. *For $\lambda=0,-1\leq B<A\leq 1$. A function $f\in k\text{-}\mathcal{US}_s^{\lambda}[A,B]$, if and only if*

$$\frac{1}{z}\left\{f(z)*\left[\frac{z+\frac{[(B+A+2)p_k(e^{i\theta})-(B+A-2)]}{(B-A)(p_k(e^{i\theta})-1)}z^2}{(1-z)^2(1+z)}\right]\right\}\neq 0 \qquad (\forall\ z\in\mathbb{E}\quad \textit{and}\quad 0\leq\theta<2\pi).$$

If, in Theorem 3.4, we set

$$-B=1=A \quad \text{and} \quad k=0,$$

we obtain the following result.

Corollary 3.2. *A function $f\in 0\text{-}\mathcal{US}_s[1,-1]$, if and only if*

$$\frac{1}{z}\left\{f(z)*\left[\frac{z\left(1-ze^{-i\theta}\right)}{(1-z)^2(1+z)}\right]\right\}\neq 0 \qquad (\forall\ z\in\mathbb{E}),\quad 0<\theta<2\pi.$$

Theorem 3.5. *If $f\in\mathcal{S}$, then $f\in k\text{-}\mathcal{UC}_s[A,B]$, if and only if*

$$\frac{1}{z}\left\{f(z)*\frac{1+2z^3+[M-3]z^2-3Mz^4}{z(1-z)^3(1+z)^2}\right\}\neq 0 \qquad (\forall\ z\in\mathbb{E}),\quad 0<\theta<2\pi,$$

where M is given by (3.7).

Proof. Let

$$g(z) = \frac{z + Mz^2}{(1-z)^2(1+z)},$$

then

$$zg'(z) = \frac{z + Mz^4 + (M+2)z^3 + (2M+1)z^2}{(1-z)^3(1+z)^2}.$$

Now using the Alexander type relation between $k\text{-}\mathcal{US}_s[A,B]$ and $k\text{-}\mathcal{UC}_s[A,B]$, the identity

$$zf'(z) * g(z) = f(z) * zg'(z),$$

and Theorem 3.4, we obtain the required result. □

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