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$\mathcal{Q}_K$-TYPE SPACES OF QUATERNION-VALUED FUNCTIONS

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Abstract. In this paper we develop the necessary tools to generalize the $\mathcal{Q}_K$-type function classes to the case of the monogenic functions defined in the unit ball of $\mathbb{R}^3$, some important basic properties of these functions are also considered. Further, we show some relations between $\mathcal{Q}_K(p,q)$ and α-Bloch spaces of quaternion-valued functions.

1. Introduction

1.1. **Analytic function spaces.** The so called $\mathcal{Q}_K$-type spaces of analytic functions on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit open complex disk, were introduced by Wulan and Zhou in [12]. For $K : [0,\infty) \to [0,\infty)$ is a non-decreasing and non-negative function, and $0 < p < \infty$, $-2 < q < \infty$, an analytic function f in $\mathbb{D}$ belongs to the $\mathcal{Q}_K(p,q)$ if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(1-|\varphi_a(z)|^2)dx\,dy < \infty.$$

Moreover, if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(1-|\varphi_a(z)|^2)dx\,dy = 0,$$

then $f \in \mathcal{Q}_{K,0}(p,q)$, where $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ is the automorphism of the unit disk $\mathbb{D}$ that changes 0 and a. The $\mathcal{Q}_K(p,q)$ class is Banach under the norm $\|f\| = \|f\|_{\mathcal{Q}_K(p,q)} + |f(0)|$, when $p \geq 1$. If $K(t) = t^s, 0 \leq s < \infty$, then $\mathcal{Q}_K(p,q) = F(p,q,s)$ see [3]. For more results of $\mathcal{Q}_K(p,q)$ classes see [3] and [7].

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1.2. **Quaternion function spaces.** We will work throughout this paper in the field $\mathbb{H}$ (the skew field of quaternion-valued functions), i.e. each element $a \in \mathbb{H}$ with basis $1, e_1, e_2, e_3$, can be given in the form

$$a := a_0 + a_1e_1 + a_2e_2 + a_3e_3, \quad a_k \in \mathbb{R}, k = 0, 1, 2, 3$$

The multiplication rules of these elements are given by

$$\begin{aligned} e_1^2 &= e_2^2 = e_3^2 = -1, \\ e_1e_2 &= -e_2e_1 = e_3, \\ e_2e_3 &= -e_3e_2 = e_1, \\ e_3e_1 &= -e_1e_3 = e_2. \end{aligned}$$

The conjugation element $\bar{a}$ of an element a is $\bar{a} = a_0 - a_1e_1 - a_2e_2 - a_3e_3$, with the property

$$a\bar{a} = \bar{a}a = |a|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

If $a \in \mathbb{H} \setminus \{0\}$, then $a^{-1} := \bar{a}/|a|^2$ and $|ab| = |a||b|$ for each $a, b \in \mathbb{H}$. Let $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ of the form $x = x_0 + x_1e_1 + x_2e_2$ be a quaternion point.

Given $\Omega \subset \mathbb{R}^3$ a domain and let $f : \Omega \longrightarrow \mathbb{H}$ the quaternion-valued functions defined in Ω. For $p \in \mathbb{N} \cup \{0\}$, thus the notation $C^p(\Omega; \mathbb{H})$ has the usual componentwise meaning. We consider D and $\overline{D}$ the generalization of a Cauchy-Riemann operator and it's conjugate, respectively, and they are defined on $C^1(\Omega; \mathbb{H})$ by

$$\begin{aligned} Df &= \frac{\partial f}{\partial x_0} + e_1\frac{\partial f}{\partial x_1} + e_2\frac{\partial f}{\partial x_2}, \\ \overline{D}f &= \frac{\partial f}{\partial x_0} - e_1\frac{\partial f}{\partial x_1} - e_2\frac{\partial f}{\partial x_2}. \end{aligned}$$

The equation $Df = 0$ has the solutions for all $x \in \Omega$, are called left-hyperholomorphic functions and they are generalized of the analytic function classes from the functions in one complex variable theory. For more details about monogenic function classes and general Clifford analysis, we refer to [2, 6, 11] and others.

Let $\mathbb{B}$ be the unit ball in $\subset \mathbb{R}^3$, with boundary $\mathbb{S} = \partial\mathbb{B}$. The class $\mathcal{M}(\mathbb{B})$ consists of all monogenic functions on $\mathbb{B}$. For $r > 0$ and $a \in \mathbb{R}^3$, let $\mathbb{B}(a, r)$ denote by the ball with center a and radius r. Also, for $a \in \mathbb{B}$ and $0 < R < 1$, an Euclidean ball $U(a, R) = \{x : |\varphi_a(x)| < R\}$, with center and radius, respectively, $\frac{(1-R^2)a}{1-R^2|a|^2}$ and $\frac{(1-|a|^2)R}{1-R^2|a|^2}$, is called the pseudo-hyperbolic ball. Where $\varphi_a(x) : \mathbb{B} \to \mathbb{B}$ is defined by $\varphi_a(x) = (a - x)/(1 - \bar{a}x)$, for $a \in \mathbb{B}$.

Let $\alpha > 0$, the quaternion α-Bloch space $\mathcal{B}^\alpha$ (see [4, 9]) defined by :

$$\mathcal{B}^\alpha = \{f \in \mathcal{M}(\mathbb{B}) : \|f\|_{\mathcal{B}^\alpha} = \sup_{x\in\mathbb{B}} |\overline{D}f(x)|(1 - |x|^2)^\alpha < \infty\}.$$

If $\alpha = \frac{3}{2}$, we have the standard quaternion Bloch space $\mathcal{B}$. The space $\mathcal{B}_0^\alpha$ is called the quaternion little α-Bloch, which consists of all $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|x|\to 1^-} |\overline{D}f(x)|(1-|x|^2)^\alpha = 0.$$

For $f \in \mathcal{M}(\mathbb{B})$, the weighted quaternion Dirichlet space $\mathcal{D}_{p,q}, (0 < p < \infty, -2 < q < \infty)$, is given by:

$$\mathcal{D}_{p,q} = \Big\{ f \in \mathcal{M}(\mathbb{B}) : \|f\|_{\mathcal{D}_{p,q}} = \int_{\mathbb{B}} |\overline{D}f(x)|^p (1-|x|^2)^q dx < \infty \Big\}.$$

If $q = 0$, we have the space $\mathcal{D}_{2,0}$ (the quaternion Dirichlet space $\mathcal{D}$).

Through this work, we let $K(t), 0 < t < \infty$, be a non-decreasing and non-negative (righ-continuous) function, which is not equal to 0 identically. For $0 < p < \infty, -2 < q < \infty$ and $f \in \mathcal{M}(\mathbb{B})$, define $J_{K,p,q}f : \mathbb{B} \to [0,\infty)$ by

$$J_K^{p,q} f(a) = \int_{\mathbb{B}} |\overline{D}f(x)|^p (1-|x|^2)^q K(1-|\varphi_a(x)|^2) dx, \qquad a \in \mathbb{B}.$$

The set $\mathcal{Q}_K(p,q)$ given by

$$\mathcal{Q}_K(p,q) := \Big\{ f \in \mathcal{M}(\mathbb{B}) : \|f\|_{\mathcal{Q}_K(p,q)} = \sup_{a\in\mathbb{B}} J_K^{p,q} f(a) < \infty \Big\},$$

and the little quaternion $\mathcal{Q}_{K,0}(p,q)$ is defined by

$$\mathcal{Q}_{K,0}(p,q) := \Big\{ f \in \mathcal{M}(\mathbb{B}) : \|f\|_{\mathcal{Q}_{K,0}(p,q)} = \lim_{|a|\to 1^-} J_K^{p,q} f(a) = 0 \Big\}.$$

Remark 1.1. *If we put $s < 3$ and $K(t) = t^s$, then $\mathcal{Q}_K(p,q) = F(p,q,s)$ (see [8]). If $p = 2, q = 0$, then $\mathcal{Q}_K(2,0) = \mathcal{Q}_K$ (see [1]). Also, if $K(t) = 1$, then $\mathcal{Q}_K(p,q) = \mathcal{D}_{p,q}$, the quaternion Dirichlit space.*

For $0 < p < \infty, -1 < q < \infty$, define the $\mathcal{D}_K(p,q)$ quaternion Dirichlet-type space as the set of $f \in \mathcal{M}(\mathbb{B})$ satisfying

$$J_K^{p,q} f(0) < \infty.$$

From the definition of $\mathcal{Q}_K(p,q)$ spaces the following lemma become immediate with $a = 0$.

Lemma 1.1. *Let $0 \le p < \infty, -1 < q < \infty$, then $\mathcal{Q}_K(p,q) \subset \mathcal{D}_K(p,q)$.*

From now, we assume that

$$\int_0^1 (1-\rho^2)^q K\big(1-\rho^2\big)\rho^2 d\rho < \infty. \tag{1.1}$$

Otherwise, $\mathcal{Q}_K(p,q)$ contains only constant functions.

Fact 1

Let $0 \le p < \infty, -1 < q < \infty$, and let $f \in \mathcal{M}(\mathbb{B})$ be a non-constant function. If (1.1) does not hold, then

$f \notin \mathcal{Q}_K(p,q)$.

Proof. Let $f \in \mathcal{Q}_K(p,q)$ be a non constant function. Then, there is $x_0 \in \mathbb{B}$ and $0 < R < 1$ such that $|\overline{D}f(x)| > 0$ for each $x \in \mathbb{B}(x_0,R)$. Thus by Lemma 1.1 and subharmonicity of $|\overline{D}f|^p$ where $A(x_0,R) = \mathbb{B}\backslash\mathbb{B}(0,|x_0|-R)$, we obtain

$$\begin{aligned}
\infty &> J_K^{p,q}f(0) \\
&\geq \int\limits_{A(x_0,R)} |\overline{D}f(x)|^p(1-|x|^2)^q K\big(1-|x|^2\big)dx \\
&\geq \int\limits_{|x_0|}^{1} (1-\rho^2)^q K\big(1-\rho^2\big)\rho^2 \int\limits_{\mathbb{S}} |\overline{D}f(\rho\zeta)|^p d\sigma(\zeta)d\rho \\
&\geq \int\limits_{\mathbb{S}} |\overline{D}f(|x_0|\zeta)|^p d\sigma(\zeta) \int\limits_{|x_0|}^{1} (1-\rho^2)^q K\big(1-\rho^2\big)\rho^2 d\rho = \infty,
\end{aligned}$$

where $d\sigma$ denotes the normalized surface element in $\mathbb{S}$. This is a contradiction; therefor f is constant and the fact is proved.

In this work, we introduce a classes of $\mathbb{H}$-valued functions on $\mathbb{R}^3$. These classes are so called $\mathcal{Q}_K(p,q)$ spaces of monogenic function. We will study these classes and their relations to the quaternion α-Bloch space. We shall prove some basic properties concerning $\mathcal{Q}_K(p,q)$ and $\mathcal{Q}_{K,0}(p,q)$ spaces in hyperholomorphic functions. Our results in this work are extensions of our results in [1] and the results due to Essén and Wulan (see [3]) in hyperholomorphic functions case. For simplicity we restricted us to$\mathbb{R}^3$ the lowest non-commutative case and quaternion-valued functions. Next, the hyperholomorphic function spaces were the aim of many works as [1, 4, 8] and [9].

In particular, we will need the following results for quaternion sense in the sequel:

Lemma 1.2. [5] *Let $1 \leq p < \infty, f \in \mathcal{M}(\mathbb{B})$ and let $0 < R < 1$. Then, we have*

$$|\overline{D}f(0)|^p \leq \frac{3}{4\pi R^2}\int_{U(a,R)} |\overline{D}f(x)|^p dx, \quad \textit{for all } a \in \mathbb{B}. \tag{1.2}$$

Lemma 1.3. [9] *Let $1 < p < \infty, f \in \mathcal{M}(\mathbb{B})$ and let $0 < R < 1$. Then, for every $a \in \mathbb{B}$, we have*

$$|\overline{D}f(a)|^p \leq \frac{C4^p}{R^3(1-R^2)^{2p}(1-|a|^2)^3}\int_{U(a,R)} |\overline{D}f(x)|^p dx, \quad \textit{for all } a \in \mathbb{B}, \tag{1.3}$$

where $C = \frac{48}{\pi}$.

Remark 1.2. *The problem in quaternion sense is that, $\overline{D}f(x)$ is monogenic, but $\overline{D}f(\varphi_a(w))$ is not monogenic. From [10] we know that $\frac{1-\bar{w}a}{|1-\bar{a}w|^3}\overline{D}f(\varphi_a(w))$ is hyperholomorphic. So, by the Jacobian determinant $\left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^3$, which has no singularities we can solve this problem.*

Lemma 1.4. [8] *Let $f \in \mathcal{M}(\mathbb{B}), f_a = f \circ \varphi_a$ and let $\Psi_{f_a} : \mathbb{B} \to \mathbb{H}$ given by*

$$\Psi_{f_a}(x) = \frac{1-\overline{x}a}{|1-\bar{a}x|^3}\overline{D}f(\varphi_a(x)). \tag{1.4}$$

Then $\Psi_{f_a} \in \mathcal{M}(\mathbb{B})$ and $|\Psi_{f_a}|$ is a subharmonic function.

2. Characterizations of $\mathcal{Q}_K(p,q)$ classes

In this part, we prove some essential properties of quaternion $\mathcal{Q}_K(p,q)$ spaces as basic scale properties.

Proposition 2.1. *Let K satisfy (1.1) and let $f \in \mathcal{M}(\mathbb{B}), 1 \le p < \infty$, and $-2 < q < \infty$. Then, we have*

$$(1-|a|^2)^{q+3}|\overline{D}f(a)|^p \le \frac{4^{q-p+3}}{C(R)} J_K^{p,q} f(a), \quad \textit{for } 0 < R < 1.$$

Proof. Since $0 < R < 1$, by Lemma 1.4 after the change of variable $x = \varphi_a(w)$ then we deduce that

$$\begin{aligned} J_K^{p,q} f(a) &\ge \int_{U(a,R)} |\overline{D}f(x)|^p (1-|x|^2)^q K\big(1-|\varphi_a(w)|^2\big)dx \\ &\ge \frac{(1-|a|^2)^{q+3}}{4^{q-p+3}} \int_{\mathbb{B}(a,R|1-a|)} |\Psi_{f_a}(w)|^p (1-|w|^2)^q K\big(1-|w|^2\big)dw \\ &\ge \frac{1}{4^{q-p+3}}(1-|a|^2)^{q+3}|\overline{D}f(a)|^p \int_0^R (1-\rho^2)^q K\big(1-\rho^2\big)\rho^2 d\rho \\ &\ge \frac{C(R)}{4^{q-p+3}}(1-|a|^2)^{q+3}|\overline{D}f(a)|^p, \end{aligned}$$

with $\Psi_{f_a}(w) = \frac{1-\bar{w}a}{|1-\bar{a}w|^3}\overline{D}f(\varphi_a(w))$. Which implies that

$$(1-|a|^2)^{q+3}|\overline{D}f(a)|^p \le \frac{4^{q-p+3}}{C(R)} J_K^{p,q} f(a).$$

Theorem 2.1. *Suppose that K satisfy (1.1), $1 \le p < \infty, -2 < q < \infty$ and $f \in \mathcal{M}(\mathbb{B})$. When $f_n \to f$, assume that $\overline{D}f_n \to \overline{D}f$ uniformly on compact set $M \subset \mathbb{B}$, as $n \to \infty$. Then, the space $\mathcal{Q}_K(p,q)$ under the norm $\|f\|_K = |\overline{D}f(0)| + \|f\|_{\mathcal{Q}_K(p,q)}$ is a Banach space.*

Proof. Since $1 \le p < \infty$, t is easy to prove that $\|.\|_K$ is a norm. To show the completeness of $(\|.\|_K, \mathcal{Q}_K(p,q))$, fix $0 < R < 1$. Applying Proposition 2.1, we obtain

$$(1-|a|^2)^{q+3}|\overline{D}f(a)|^p \le \frac{4^{q-p+3}}{C(R)} J_K^{p,q} f(a),$$

which gives

$$\|f\|_{\mathcal{B}^{\frac{q+3}{p}}} \le \frac{4^{q-p+3}}{C(R)} \|f\|_{\mathcal{Q}_K(p,q)}.$$

By the fact that $(1-|x|^2)^3 \approx |U(a,R)|$, from Lemma 1.2 and Lemma 1.3, we get

$$\begin{aligned}
|\overline{D}f(a)-\overline{D}f(0)| &\leq \left(\frac{4^{q-p+3}}{C(R)} - (\frac{3}{4\pi R^3})^{\frac{1}{p}}\right)\left(\int_{U(a,R)} |\overline{D}f(x)|^p dx\right)^{\frac{1}{p}} \\
&\leq C(R,p,q)\|f\|_{\mathcal{B}^{\frac{q+3}{p}}}\left(\int_{U(a,R)} \frac{1}{(1-|x|^2)^{q+3}} dx\right)^{\frac{1}{p}} \\
&\leq C(R,p,q)\|f\|_{\mathcal{B}^{\frac{q+3}{p}}} \frac{1}{|U(a,R)|^{\frac{q+3}{3p}}}\left(\int_{U(a,R)} dx\right)^{\frac{1}{p}} \\
&\leq C_1(R,p,q)\|f\|_{\mathcal{B}^{\frac{q+3}{p}}},
\end{aligned}$$

where a positive constant $C_1(R,p,q)$ is depending on R, p and q, which implies that

$$|\overline{D}f(a)| \leq |\overline{D}f(0)| + C_1(R,p,q)\|f\|_{\mathcal{B}^{\frac{q+3}{p}}}.$$

So, fore each compact set $M \subset \mathbb{B}$, there is a constant $C \in M$ such that

$$|\overline{D}f(a)| \leq |\overline{D}f(0)| + C_1(R,p,q)\|f\|_{\mathcal{Q}_K(p,q)} \leq C\|f\|_K, \tag{2.1}$$

where the constant $C = \max\left\{1, C(M, C_1(R,p,q), \frac{q+3}{p})\right\}$.

Now, we let $\{f_n\}$ be a Cauchy sequence in $\mathcal{Q}_K(p,q)$ spaces. From (2.1) we deduce that $\{f_n\}$ is also a Cauchy sequence in the topology of uniform convergence on compact sets. Thus thereis a function $f \in \mathcal{M}(\mathbb{B})$ such that $f_n \to f$ also, $\overline{D}f_n \to \overline{D}f$ uniformly on compact subsets of $\mathbb{B}$, as $n \to \infty$. To show that $\|f_n - f\|_K \to 0$ as $n \to \infty$, we give $\varepsilon > 0$. Since, $\{f_n\}$ is a Cauchy sequence, there is an $N > 0$ such that $\|f_k - f_n\|_K < \frac{\varepsilon}{2}$ and $|\overline{D}f_n(0) - \overline{D}f(0)| < \frac{\varepsilon}{2}$ for all $n, k \in \geq N$.

For each $a \in \mathbb{B}$ and $n \geq N$, by applying Fatou's lemma, we obtain

$$\begin{aligned}
&\int_{\mathbb{B}} |\overline{D}f(x) - \overline{D}f_n(x)|^p (1-|x|^2)^q K\big(1-|\varphi_a(x)|^2\big) dx \\
= &\int_{\mathbb{B}} \lim_{k\to\infty} |\overline{D}f_k(x) - \overline{D}f_n(x)|^p (1-|x|^2)^q K\big(1-|\varphi_a(x)|^2\big) dx \\
\leq &\lim_{k\to\infty} \int_{\mathbb{B}} |\overline{D}f_k(x) - \overline{D}f_n(x)|^p (1-|x|^2)^q K\big(1-|\varphi_a(x)|^2\big) dx \\
= &\lim_{k\to\infty} \|f_k - f_n\|^p_{\mathcal{Q}_K(p,q)} < \left(\frac{\varepsilon}{2}\right)^p.
\end{aligned}$$

Thus, for all $n \geq N$,

$$\begin{aligned}
\|f_n - f\|_K &= |\overline{D}f_n(0) - \overline{D}f(0)| \\
&\quad + \left(\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\overline{D}f(x) - \overline{D}f_n(x)|^p (1-|x|^2)^q K\big(1-|\varphi_a(x)|^2\big) dx\right)^{\frac{1}{p}} \\
&\leq \varepsilon,
\end{aligned}$$

which implies that $f_n \to f$ in $\mathcal{Q}_K(p,q)$. Hence, the norm $\|.\|_K$ is complete, therefore $\mathcal{Q}_K(p,q)$ spaces is a Banach space in Clifford setting.

3. The Quaternion Bloch and $\mathcal{Q}_K(p,q)$ Spaces

In this part of the paper, we consider the relations between $\mathcal{Q}_K(p,q)$ and α-Bloch spaces in quaternion sense. We characterize the quaternion α-Bloch spaces by the help of integral norms of quaternion $\mathcal{Q}_K(p,q)$ spaces. Our results extend the results due to Wulan and Zhou [12] in quaternion sense.

Theorem 3.1. *Let $f \in \mathcal{M}(\mathbb{B})$ and let $1 \leq p < \infty, -2 < q < \infty$. Then*

(i): $\mathcal{Q}_K(p,q) \subset \mathcal{B}^{\frac{q+3}{p}}$,

(ii): $\mathcal{Q}_K(p,q) = \mathcal{B}^{\frac{q+3}{p}}$; *if*

$$\int_0^1 (1-\rho^2)^{-3} K(1-\rho^2)\rho^2 d\rho < \infty. \tag{3.1}$$

Proof. **(i)** Let $0 < R < 1$ be fixed and $a \in \mathbb{B}$. From Proposition 2.1, we acquire

$$(1-|a|^2)^{q+3}|\overline{D}f(a)|^p \leq \frac{4^{q-p+3}}{C(R)} J_K^{p,q} f(a).$$

If $f \in \mathcal{Q}_K(p,q)$, then by estimate above we have $f \in \mathcal{B}^{\frac{q+3}{p}}$.

(ii) Let $f \in \mathcal{B}^{\frac{q+3}{p}}$ be non constant. Then, there is $M > 0$ constant such that

$$(1-|a|^2)^{\frac{q+3}{p}}|\overline{D}f(a)| \leq M, \quad \text{for all } x \in \mathbb{B}.$$

Now we change the variable $x = \varphi_a(w)$, then we acquire

$$\begin{aligned}
J_K^{p,q} f(a) &\leq \int_{\mathbb{B}} M^p (1-|x|^2)^{-3} K\big(1-|\varphi_a(x)|^2\big) dx \\
&\leq M^p \int_{\mathbb{B}} (1-|\varphi_a(w)|^2)^{-3} K(1-|w|^2) \frac{(1-|a|^2)^3}{|1-\bar{a}w|^6} dw \\
&\leq M^p \int_{\mathbb{B}} (1-|w|^2)^{-3} K(1-|w|^2) dw \\
&\leq M^p \int_0^1 (1-\rho^2)^{-3} K(1-\rho^2)\rho^2 d\rho < \infty.
\end{aligned}$$

Thus, $f \in \mathcal{Q}_K(p,q)$. This show that $\mathcal{B}^{\frac{q+3}{p}} \subset \mathcal{Q}_K(p,q)$.

Combining Theorem 3.1, we deduce the following corollary:

Corollary 3.1. *Let $1 \leq p < \infty, -2 < q < \infty$ and let $f \in \mathcal{M}(\mathbb{B})$. Then*

(i): $\mathcal{Q}_{K,0}(p,q) \subset \mathcal{B}_0^{\frac{q+3}{p}}$,

(ii): $\mathcal{Q}_{K,0}(p,q) = \mathcal{B}_0^{\frac{q+3}{p}}$; *if (3.1) holds.*

Proposition 3.1. *Let* $1 \leq p < \infty, -2 < q < \infty$ *and let* $f \in \mathcal{M}(\mathbb{B})$*. Then* $f \in \mathcal{B}^{\frac{q+3}{p}}$ *if and only if there is an* $0 < R < 1$*, such that*

$$\sup_{a\in\mathbb{B}} \int_{U(a,R)} |\overline{D}f(x)|^p(1-|x|^2)^q K\big(1-|\varphi_a(x)|^2\big)dx < \infty. \tag{3.2}$$

Proof. If $f \in \mathcal{B}^{\frac{q+3}{p}}$, and $a \in \mathbb{B}$. Then for any $0 < R < 1$, we deduce

$$\begin{aligned}
&\int_{U(a,R)} |\overline{D}f(x)|^p(1-|x|^2)^q K\big(1-|\varphi_a(x)|^2\big)dx \\
\leq\ & \int_{\mathbb{B}(0,R)} |\Psi_{f_a}(w)|^p(1-|\varphi_a(w)|^2)^q K(1-|w|^2)\frac{(1-|a|^2)^3}{|1-\bar{a}w|^{6-2p}}dw \\
\leq\ & \|f\|_{\mathcal{B}^{\frac{q+3}{p}}} \int_{\mathbb{B}(0,R)} (1-|w|^2)^{-3} K\big(1-|w|^2\big)dw \\
\leq\ & C\|f\|_{\mathcal{B}^{\frac{q+3}{p}}}.
\end{aligned}$$

Conversely, let (3.2) holds then, we deduce

$$\begin{aligned}
&\int_{U(a,R)} |\overline{D}f(x)|^p(1-|x|^2)^q K\big(1-|\varphi_a(x)|^2\big)dx \\
\geq\ & K\big(1-R^2\big)\int_{\mathbb{B}(0,R)} |\overline{D}f(x)|^p(1-|x|^2)^q dx \\
\geq\ & \frac{C(R)K\big(1-R^2\big)}{4^{q-p+3}}(1-|a|^2)^{q+3}|\overline{D}f(a)|^p,
\end{aligned}$$

which shows that $f \in \mathcal{B}^{\frac{q+3}{p}}$.

Corollary 3.2. *Let* $K : (0, \infty) \to [0, \infty)$ *and* $f \in \mathcal{M}(\mathbb{B})$*. Then* $f \in \mathcal{B}_0^{\frac{q+3}{p}}$ *if and only if there is an* $0 < R < 1$*, such that*

$$\lim_{|a|\to 1} \int_{U(a,R)} |\overline{D}f(x)|^p(1-|x|^2)^q K\big(1-|\varphi_a(x)|^2\big)dx = 0.$$

Conclusion. Our results in this work will be of important uses in the study of operator theory at the interface of monogenic function spaces. This work is a try to synthesize the achievements in the properties of monogenic $\mathcal{Q}_K(p,q)$ function spaces. The problem in quaternion sense is that, $\overline{D}f(x)$ is monogenic, but $\overline{D}f(\phi(x))$ is not monogenic, where $\phi : \mathbb{B} \to \mathbb{B}$ is a monogenic function. The following question is open problem: What properties of operators act between this classes of monogenic functions, like $F(p,q,s)$ and $\mathcal{Q}_K(p,q)$ classes?

In quaternion case, several authors have studied function spaces and classes like Q_p, $\mathcal{Q}_K$ classes and $F(p,q,s)$ spaces, see [1,3,8] and others.

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