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TWO STEP MODIFIED ISHIKAWA ITERATION SCHEME FOR MULTI-VALUED MAPPINGS IN CAT(0) SPACE

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ABSTRACT. The aim of this paper is to prove some strong convergence theorems for the modified Ishikawa iteration scheme involving quasi-nonexpansive multi-valued mappings in the framework of CAT(0) spaces.

1. INTRODUCTION

Let K be a nonempty convex subset of a Banach space $X = (X, \|\cdot\|)$. The set K is called *proximal* if for each $x \in X$, there exists an element $y \in K$ such that $\|x - y\| = d(x, K)$, where $d(x, K) = \inf\{\|x - z\| : z \in K\}$. Let $CB(K), \mathcal{K}(K)$ and $P(K)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximal bounded subsets of K respectively. The *Hausdorff metric* on $CB(K)$ is defined by

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for $A, B \in CB(K)$. A single-valued mapping $T : K \to K$ is called *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for $x, y \in K$. A multi-valued mapping $T : K \to CB(K)$ is said to be *nonexpansive* if $H(T(x), T(y)) \leq \|x - y\|$ for all $x, y \in K$. An element $p \in K$ is called a *fixed point* of $T : K \to K$ (respectively, $T : K \to CB(K)$) if $p = T(p)$ (respectively, $p \in T(p)$). The set of fixed points of T is denoted by $F(T)$. The mapping $T : K \to CB(K)$ is called *quasi-nonexpansive* [27] if $F(T) \neq \phi$ and $H(T(x), T(p)) \leq \|x - p\|$ for all $x \in K$ and all $p \in F(T)$. It is clear that every nonexpansive multi-valued mapping T with $F(T) \neq \phi$ is quasi-nonexpansive. But there exists quasi-nonexpansive mappings that are not nonexpansive.

Example 1.1. *Let $K = [0, \infty)$ with the usual metric and $T : K \to CB(K)$ be defined by*

$$T(x) = \begin{cases} \{0\}, \text{if} \quad x \leq 1; \\ [x - \frac{3}{4}, x - \frac{1}{3}], \text{if} \quad x > 1 \end{cases}$$

Then clearly $F(T) = \{0\}$ and for any x we have $H(T(x), T(0)) \leq \|x - 0\|$, hence T is quasi-nonexpansive. However, if $x = 2, \quad y = 1$ we get $H(T(x), T(y)) > |x - y| = 1$ and hence not nonexpansive.

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The mapping $T : K \to CB(K)$ is called *hemi-compact* if, for any sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$. We note that if K is compact, then every multi-valued mapping $T : K \to CB(X)$ is hemi-compact .
$T : K \to CB(K)$ is said to satisfy Condition (I)[24], if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, \quad f(r) > 0$ for $r \in (0, \infty)$ such that $d(x, T(x)) \geq f(d(x, F(T)))$ for all $x \in K$.
Iterative techniques for approximating fixed points of nonexpansive single-valued mappings have been studied by various authors (see; [24],[30],[11],[22]) using the Mann iteration scheme or the Ishikawa iteration scheme. For details on the subject, we refer the reader to Berinde [2].
Sastry and Babu [23] studied the Mann and Ishikawa iteration schemes for multi-valued mappings and proved that these schemes for a multi-valued map T with a fixed point p converges to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. Panyanak [21] extended the result of Sastry and Babu [23] by modifying the iteration schemes of Sastry and Babu [23] in the setting of uniformly convex Banach spaces but the domain of T remains compact.
Song and Wang [28,29] noted that there was a gap in the proof of Theorem 3.1 of [21] and Theorem 5 of [23]. Because the iteration x_n depends on a fixed point $p \in F(T)$ as well as T. If $q \in F(T)$ and $q \neq p$, then the iteration x_n defined by q is different from the one defined by p. Therefore, one cannot derive the monotonicity of sequence $\{\|x_n - q\|\}$ from the monotonicity of $\{\|x_n - p\|\}$. So the conclusion of Theorem 3.1 [21] and Theorem 5 [23] are very dubious. They further solved/revised the gap and also gave the affirmative answer to the above question using the following Ishikawa iteration scheme.

$$y_n = \beta_n z_n + (1 - \beta_n) x_n,$$
$$x_{n+1} = \alpha_n z_n^{'} + (1 - \alpha_n) x_n$$

where $\|z_n - z_n^{'}\| \leq H(T(x_n), T(y_n)) + \gamma_n$ and $\|z_{n+1} - z_n^{'}\| \leq H(T(x_{n+1}), T(y_n)) + \gamma_n$ for $z_n \in T(x_n)$ and $z_n^{'} \in T(y_n)$.

Recently, Shahzad and Zegeye [26] introduced the modified Ishikawa iteration schemes as follows:
(SZ1): Let K be a nonempty convex subset of a Banach space X and $\alpha_n, \beta_n \in [0, 1]$. The sequence of Ishikawa iterates is defined by $x_0 \in K$,

$$y_n = \beta_n z_n + (1 - \beta_n) x_n, \quad n \geq 0$$

where $z_n \in T(x_n)$, and

$$x_{n+1} = \alpha_n z_n^{'} + (1 - \alpha_n) x_n, \quad n \geq 0$$

where $z_n^{'} \in T(y_n)$

They also proved some interesting results on the strong convergence of the sequence defined by (SZ1).
Motivated and inspired by the above work, we introduced the following modified Ishikawa iteration schemes and prove some strong convergence theorems for these schemes in the setting of CAT(0) space.

Modified Ishikawa Iteration Scheme:
(PS1): Let K be a nonempty convex subset of a complete CAT(0) space X and $\alpha_n, \beta_n \in [0,1]$. The sequence of Ishikawa iterates is defined by $x_0 \in K$,

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \quad n \geq 0$$

where $z_n \in T(x_n)$, and

$$x_{n+1} = \alpha_n z_n^{'} + (1-\alpha_n)z_n, \quad n \geq 0$$

where $z_n^{'} \in T(y_n)$

The aim of this paper, is to prove strong convergence theorems of the modified Ishikawa iteration scheme (PS1) in the setting of CAT(0) space.

2. Preliminaries

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well-known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, $\mathbb{R}$-trees (see [3]), Euclidean buildings (see [4]), the complex Hilbert ball with a hyperbolic metric (see [10]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [3].
Fixed point theory in a CAT(0) space was first studied by Kirk (see [12] and [15]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed and much papers have appeared (see, e.g., [9],[17],[25],[26],[6]-[8], [12]-[13]).
Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x, \quad c(l) = y,$ and $d(c(t), c(t')) = |t-t'|$ for all $t, t' \in [0,l]$.In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.
A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points x_1,x_2,x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for the geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1,x_2,x_3) := \Delta(\bar{x_1},\bar{x_2},\bar{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x_i},\bar{x_j}) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$.
A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.
CAT(0): Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) *inequality* if for all $x,y \in \Delta$ and all comparison points $\bar{x},\bar{y} \in \bar{\Delta}$,

$$d(x,y) \leq d_{\mathbb{E}^2}(\bar{x},\bar{y})$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2 \tag{2.1}$$

This is the (CN) inequality of Bruhat and Tits [5]. In fact (cf. [11], p.163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

In the sequel, we need the following lemmas which will be used frequently in the proofs of our main results.

Lemma 2.1. *([21]) Let $\{\alpha_n\}, \{\beta_n\}$ be two real sequences such that*

(1) $0 \leq \alpha_n, \beta_n < 1$;
(2) $\beta_n \to 0$ *as* $n \to \infty$;
(3) $\sum \alpha_n \beta_n = \infty$

Let $\{\gamma_n\}$ be a nonnegative real sequence such that $\sum \alpha_n \beta_n (1 - \beta_n)\gamma_n$ is bounded. Then $\{\gamma_n\}$ has a subsequence which converges to zero.

Lemma 2.2. *([3, Proposition 2.4]) Let (X, d) be a CAT(0) space.Let K be a subset of X which is complete in the induced metric. Then, for every $x \in X$, there exists a unique point $P(x) \in K$ such that $d(x, P(x)) = \inf\{d(x, y) : y \in K\}$. Moreover, the map $x \mapsto P(x)$ is a nonexpansive retract from X into K.*

Lemma 2.3. *([8, Lemma 2.1(iv)]) Let (X, d) be a CAT(0) space. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \quad \textit{and} \quad d(y, z) = (1 - t)d(x, y) \tag{2.2}$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (2.2).

Lemma 2.4. *([8, Lemma 2.4]) Let Let (X, d) be a CAT(0) space. For $x, y, z \in X$ and $t \in [0, 1]$, we have*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$$

Lemma 2.5. *([8, Lemma 2.5]) Let Let (X, d) be a CAT(0) space. For $x, y, z \in X$ and $t \in [0, 1]$, we have*

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2$$

3. Main Results

Lemma 3.1. *Let K be a nonempty compact convex subset of a complete CAT(0) space X, and let $T : K \to CB(K)$ be a quasi-nonexpansive multi-valued mapping. Suppose that*

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0$$

for some sequence $\{x_n\}$ in K. Then T has a fixed point. Moreover, if $\{d(x_n, y)\}$ converges for each $y \in F(T)$, then $\{x_n\}$ strongly converges to a fixed point of T.

Proof. By the compactness of K, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q \in K$. Thus

$$\begin{aligned} d(q, Tq) &\leq d(q, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, q) \\ &\to 0 \qquad \text{as}, k \to \infty \end{aligned}$$

This implies that q is a fixed point of T. Since the limit of $\{d(x_n, q)\}$ exists and $\lim_{k\to\infty} d(x_{n_k}, q) = 0$, we have $\lim_{k\to\infty} d(x_n, q) = 0$. This shows that the sequence $\{x_n\}$ converges strongly to a fixed point of $q \in K$. □

Theorem 3.2. *Let K be a nonempty compact convex subset of a complete CAT(0) space X, and let $T : K \to CB(K)$ be a quasi-nonexpansive multi-valued mapping and $F(T) \neq \phi$ satisfying $T(p) = \{p\}$ for any fixed point $p \in F(T)$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by* ***PS1***. *Assume that*

(1) $\alpha_n, \beta_n \in [0, 1)$;

(2) $\lim_{n\to\infty} \beta_n = 0$;

(3) $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$.

Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.

Proof. Let $p \in F(T)$. Then by Lemma 2.5, we have

$$\begin{aligned}
d(x_{n+1}, p)^2 &= d((1-\alpha_n)z_n \oplus \alpha_n z_n^{'}, p)^2 \\
&\leq (1-\alpha_n)d(z_n, p)^2 + \alpha_n d(z_n^{'}, p)^2 - \alpha_n(1-\alpha_n)d(z_n, z_n^{'})^2 \\
&\leq (1-\alpha_n)(H(Tx_n, Tp))^2 + \alpha_n(H(Ty_n, Tp))^2 - \alpha_n(1-\alpha_n)d(z_n, z_n^{'})^2 \\
&\leq (1-\alpha_n)d(x_n, p)^2 + \alpha_n d(y_n, p)^2 - \alpha_n(1-\alpha_n)d(z_n, z_n^{'})^2 \\
&\leq (1-\alpha_n)d(x_n, p)^2 + \alpha_n d(y_n, p)^2
\end{aligned}$$

Also

$$\begin{aligned}
d(y_n, p) &= d(\beta_n z_n \oplus (1-\beta_n)x_n, p)^2 \\
&\leq (1-\beta_n)d(x_n, p)^2 + \beta_n d(z_n, p)^2 - \beta_n(1-\beta_n)d(x_n, z_n)^2 \\
&\leq (1-\beta_n)d(x_n, p)^2 + \beta_n(H(Tx_n, Tp))^2 - \beta_n(1-\beta_n)d(x_n, z_n)^2 \\
&\leq (1-\beta_n)d(x_n, p)^2 + \beta_n d(x_n, p)^2 - \beta_n(1-\beta_n)d(x_n, z_n)^2 \\
&\leq d(x_n, p)^2 - \beta_n(1-\beta_n)d(x_n, z_n)^2
\end{aligned}$$

So

$$\begin{aligned}
d(x_{n+1}, p)^2 &\leq (1-\alpha_n)d(x_n, p)^2 + \alpha_n d(x_n, p)^2 - \alpha_n\beta_n(1-\beta_n)d(x_n, z_n)^2 \\
&\leq d(x_n, p)^2 - \alpha_n\beta_n(1-\beta_n)d(x_n, z_n)^2
\end{aligned}$$

This implies

$$d(x_{n+1}, p)^2 \leq d(x_n, p)^2 \tag{3.1}$$

$$\alpha_n\beta_n(1-\beta_n)d(x_n, z_n)^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2 \tag{3.2}$$

It follows from (3.1) that the sequence $\{d(x_n, p)\}$ is decreasing and hence $\lim_{n\to\infty} d(x_n, p)$ exists. On the other hand (3.2) implies

$$\sum_{n=0}^{\infty} \alpha_n\beta_n(1-\beta_n d(x_n, z_n)^2 \leq d(x_1, p)^2 \leq \infty$$

Then by Lemma 2.1, there exists a subsequence $\{d(x_{n_k}, z_{n_k})\}$ of $d(x_n, z_n)$ such that

$$\lim_{k\to\infty} d(x_{n_k}, z_{n_k}) = 0$$

This implies

$$\lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = 0$$

By Lemma 3.1, $\{x_{n_k}\}$ converges to a point $q \in F(T)$. Since the limit of $\{d(x_n, q)\}$ exists, it must be the case that $\lim_{n\to\infty} d(x_n, q) = 0$ and hence the conclusion follows. □

Theorem 3.3. *Let K be a nonempty compact convex subset of a complete CAT(0) space X, and let $T : K \to CB(K)$ be a quasi-nonexpansive multi-valued mapping that satisfying Condition (I). Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by **PS1**. Assume that $F(T) \neq \phi$ satisfying $T(p) = \{p\}$ for any fixed point $p \in F(T)$ and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Similar to the proof of Theorem 3.2, we obtain $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and

$$\alpha_n\beta_n(1-\beta_n)d(x_n, z_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2 \tag{3.3}$$

Then

$$a^2(1-b)d(x_n, z_n)^2 \le \alpha_n\beta_n(1-\beta_n)d(x_n, z_n)^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2 \tag{3.4}$$

This implies

$$\sum_{n=0}^{\infty} a^2(1-b)d(x_n, z_n)^2 \le d(x_1, p)^2 < \infty \tag{3.5}$$

Thus, $\lim_{n\to\infty} d(x_n, z_n)^2$. Since $z_n \in T(x_n)$,

$$d(x_n, T(x_n)) \le d(x_n, z_n) \tag{3.6}$$

Therefore $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Furthermore,

$$\lim_{n\to\infty} d(x_n, F(T)) = 0 \tag{3.7}$$

The proof of remaining part closely follows the proof of [21, Theorem 3.8], simply replacing $\|\cdot\|$ with $d(\cdot,\cdot)$. □

Corollary 3.4. *Let K be a nonempty compact convex subset of a complete CAT(0) space X, and let $T : K \to CB(K)$ be a nonexpansive multi-valued mapping that satisfying Condition (I). Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by **PS1**. Assume that $F(T) \neq \phi$ satisfying $T(p) = \{p\}$ for any fixed point $p \in F(T)$ and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Theorem 3.5. *Let K be a nonempty compact convex subset of a complete CAT(0) space X, and let $T : K \to CB(K)$ be a quasi-nonexpansive multi-valued mapping and $F(T) \neq \phi$ satisfying $T(p) = \{p\}$ for any fixed point $p \in F(T)$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by **PS1**. Assume that T is hemicompact and continuous, and*

(1) $\alpha_n, \beta_n \in [0, 1)$;

(2) $\lim_{n\to\infty} \beta_n = 0$;

(3) $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$.

Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.

Proof. Let $p \in F(T)$. Then, from 3.2

$$\alpha_n\beta_n(1-\beta_n)d(x_n,z_n)^2 \le d(x_n,p)^2 - d(x_{n+1},p)^2$$

which implies that

$$\sum_{n=0}^{\infty}\alpha_n\beta_n(1-\beta_n d(x_n,z_n)^2 \le d(x_1,p)^2 < \infty$$

Thus, $\lim_{n\to\infty} d(x_n,z_n) = 0$. Since $d(x_n,T(x_n)) \le d(x_n,z_n) \to 0$ as $n\to\infty$ and T is hemicompact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q$ for some $q \in K$. Since T is continuous, we have $d(x_{n_k},T(x_{n_k})) \to d(q,T(q))$. Thus, we have $d(q,T(q)) = 0$ and so $q \in F(T)$. By Theorem 3.2 $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F(T)$, it follows that $\{x_n\}$ converges strongly to q. This completes the proof of the theorem. □

Corollary 3.6. *Let K be a nonempty compact convex subset of a complete CAT(0) space X, and let $T : K \to CB(K)$ be a nonexpansive multi-valued mapping and $F(T) \neq \phi$ satisfying $T(p) = \{p\}$ for any fixed point $p \in F(T)$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by* ***PS1****. Assume that T is hemicompact and continuous, and*

(1) $\alpha_n, \beta_n \in [0,1)$;

(2) $\lim_{n\to\infty}\beta_n = 0$;

(3) $\sum_{n=0}^{\infty}\alpha_n\beta_n = \infty$.

Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.

References

[1] N.A. Assad and W.A. Kirk, *Fixed point theorems for set-valued mappings of contractive type*, Pacific J. Math., **43(3)**(1972),553-562.

[2] V. Berinde, *Iterative approximation of fixed points*, in: Lecture Notes in Mathematics, Vol. 1912, Springer, Berlin, 2007.

[3] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Vol. 319 of Fundamental principles of Mathematical Sciences, Springer-Verlag, Berlin, Germany, 1999.

[4] K. S. Brown, Buildings, Springer-Verlag, New York, 1989.

[5] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Publications Mathematiques de LTHES,**41(1)**1972,5-251.

[6] P. Chaoha and A. Phon-on, *A note on fixed sets in CAT(0) spaces*, J. Math. Anal. Appl., **320(2)**(2006), 983-987.

[7] S. Dhompongsa, A. Kaewkhao and B. Panyanak, *Lim's theorems for multi-valued mappings in CAT(0) spaces*, J. Math. Anal. Appl., **312(2)**(2005), 478-487.

[8] S. Dhompongsa and B. Panyanak, *On $\Delta-$convergence theorems in CAT(0) spaces*, Comput. Math. Appl., **56**(2008), 2572-2579.

[9] H. Fujiwara, K. Nagano and T. Shioya, *Fixed point sets of parabolic isometries of CAT(0) spaces*, Commentarii Mathematici Helvetici, **81(2)**(2006), 305-335.

[10] K. Goebel and R. Reich, *Uniform convexity, Hyperbolic Geometry and Nonexpansive mappings*, vol. 83 of Monographs and Textbooks in Pure & Applied Mathematics, Marcel Dekker Inc., New York, 1984.

[11] S. Ishikawa, *Fixed point and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc., **59**(1976), 65-71.

[12] W. A. Kirk, *Geodesic geometry and fixed point theory*, In Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), vol. **64**(2003) of Colecc. Abierta, pp. 195-225, Univ. Sevilla Serc. Publ., Seville.

[13] W.A. Kirk, *Fixed point theorems in CAT(0) spaces and* $\mathbb{R}$*-trees*, Fixed Point Theory & Application, vol. **2004(4)**(2004),309-316.

[14] W.A. Kirk and B. Panyanak, *A concept of convergence in Geodesic spaces*, Nonlinear Analysis: TMA, **68(12)**, 3689-3696.

[15] W.A. Kirk, *Geodesic geometry and fixed point theory II*, International conference on Fixed Point Theory & Applications, Yokohama Publ. Japan, (2004), 113-142.

[16] W. Laowang and B. Panyanak, *Srong and* $\Delta-$*convergence theorems for multi-valued mappings in CAT(0) spaces*, J.Inequal. Appl., Vol. 2009, Article ID 730132, 16 pages.

[17] L. Leustean, *A quadratic rate of asymptotic regularity for CAT(0) spaces*, J. Math. Anal. Appl., **325(1)**(2007), 386-399.

[18] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4**(1953), 506-510.

[19] S. B. Nadler Jr., *Multi-valued contraction mappings*, Pacific J. Math., **30**(1969), 475-487.

[20] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc.,**73**(1967), 591-597.

[21] B. Panyanak, *Mann and Ishikawa iterative processes for multi-valued mappings in Banach spaces*, Comput. Math. Appl., **54(6)**(2007), 872-877.

[22] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **67**(1979), 274-276.

[23] K.P.R. Sastry and G.V.R. Babu, *Convergence of Ishikawa iterates for a multivalued mapping with fixed point*, Czechoslovak Math. J., **55(1)**(2005), 817-399.

[24] H.F. Senter and W.G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., **44**(1974), 375-380.

[25] N. Shahzad and J. Markin, *Invariant approximations for commuting mappings in CAT(0) and hyperconvex spaces*, J. Math. Anal. Appl., **337(2)**(2008), 1457-1464.

[26] N. Shahzad and H. Zegeye, *On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces*, Nonlinear Analysis, **71**(2009), no. 3-4, 838-844.

[27] C. Shiau, K.K. Tan and C.S. Wang, *Quasi-nonexpansive multi-valued maps and selections*, Fund. Math., **87**(1975), 109-119.

[28] Y. Song and H. Wang, *Convergence of iterative algorithms for multivalued mappings in Banach spaces*, Nonlinear Analysis: Theory, Method & Applications, **70(4)**(2009), 1547-1556.

[29] Y. Song and H. Wang, *Erratum to "Mann and Ishikawa iterative processes for multi-valued mappings in Banach spaces"[Comput. Math. Appl. 54(2007),872-877]*, Comput. Math. Appl., **55(12)**(2008), 2999-3002.

[30] K.K. Tan and H.K. Xu, *Approximating fixed points of nonexpansive mappings by Ishikawa iteration process*, J. Math. Anal. Appl., **178**(1993), 301-308.

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