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ON GIACCARDI'S INEQUALITY AND ASSOCIATED FUNCTIONAL IN THE PLANE

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Abstract. In this paper the authors extend Giaccardi's inequality to coordinates in the plane. The authors consider the nonnegative associated functional due to Giaccardi's inequality in plane and discuss its properties for certain class of parametrized functions. Also the authors proved related mean value theorems.

1. Introduction

Let I be a real interval. A function $f : I \to \mathbb{R}$ is said to be convex on I if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$.

In [2], Dragomir gave the definition of convex functions on coordinates as follows.

Definition 1.1. *Let $\Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ and $f : \Delta \to \mathbb{R}$ be a mapping. Define partial mappings*

$$f_y : [a,b] \to \mathbb{R} \textit{ by } f_y(u) = f(u,y) \tag{1.1}$$

and

$$f_x : [c,d] \to \mathbb{R} \textit{ by } f_x(v) = f(x,v). \tag{1.2}$$

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Then f is said to be convex on coordinates (or coordinated convex) in Δ if f_y and f_x are convex on $[a,b]$ and $[c,d]$ respectively for all $x \in [a,b]$ and $y \in [c,d]$. A mapping f is said to be strictly convex on coordinates (or strictly coordinated convex) in Δ if f_y and f_x are strictly convex on $[a,b]$ and $[c,d]$ respectively for all $x \in [a,b]$ and $y \in [c,d]$.

Now we define another important subclass of convex functions i.e. log convex functions.

Definition 1.2. *A function $f : I \to \mathbb{R}_+$ is called log convex on I if*

$$f(\alpha x + \beta y) \leq f^{\alpha}(x) f^{\beta}(y)$$

where $\alpha + \beta = 1$ $\alpha, \beta \geq 0$ and $x, y \in I$.

Log-convex functions have excellent closure properties. The sum and product of two log-convex functions is convex. If f is convex function and g is log-convex function then the functional composition $g \circ f$ is also log-convex. Many authors consider this function e.g. see [12], in which some of the properties of log convex functions has been discussed (also see [6, 7, 9] and references therein). In the following defintion, we define log convex function on coordinates.

Definition 1.3. *A function $f : \Delta \to \mathbb{R}_+$ is called log convex on coordinates in Δ if partial mappings defined in (1.1) and (1.2) are log convex on $[a,b]$ and $[c,d]$ respectively for all $x \in [a,b]$ and $y \in [c,d]$.*

Remark 1.1. *Every log convex function is log convex on coordinates but the converse is not true in general. For example, $f : [0,1]^2 \to [0,\infty)$ defined by $f(x,y) = e^{xy}$ is convex on coordinates but not convex.*

Giaccardi's inequality is stated as follows (see [8, page 153, 155] or [10]).

Theorem 1.1. *Let $[0,a) \subset \mathbb{R}$, $(x_1, ..., x_n) \in [0,a)^n$ and $(p_1, ..., p_n)$ be nonnegative $n-$tuples such that $(x_i - x_0)(\tilde{x}_n - x_i) \geqslant 0$ for $i = 1, \ . \ . \ . \ , n$ and $\tilde{x}_n \neq x_0$, where $x_0 \in [0,a)$ and $\tilde{x}_n = \sum_{k=1}^{n} p_k x_k$.*

If f is convex, then the inequality

$$\sum_{k=1}^{n} p_k f(x_k) \leqslant A f(\tilde{x}_n) + B \left(\sum_{k=1}^{n} p_k - 1 \right) f(x_0) \tag{1.3}$$

is valid, where

$$A = \frac{\sum_{k=1}^{n} p_k (x_k - x_0)}{\tilde{x}_n - x_0} \quad B = \frac{\tilde{x}_n}{\tilde{x}_n - x_0}.$$

Remark 1.2. *Condition that f is convex, can be replaced with $\frac{f(x) - f(x_0)}{x - x_0}$ is an increasing function, then inequality (1.3) is also valid.*

Remark 1.3. *If f is strictly convex, then strict inequality holds in (1.3) unless $x_1 = ... = x_n$ and $\sum_{i=1}^{n} p_i = 1$.*

Remark 1.4. *For $p_i = 1$ $(i = 1, ..., n)$, the above inequality becomes*

$$\sum_{i=1}^{n} f(x_i) \leqslant f\left(\sum_{i=1}^{n} x_i\right) + (n-1)\, f(x_0). \tag{1.4}$$

Remark 1.5. *If we put $x_0 = 0$ in above inequality, we get Petrović's inequality for convex functions on real line.*

In this paper we extend Giaccardi's inequality to coordinates in the plane. We consider functionals due to Giaccardi's inequality in plane and discuss its properties for certain class of coordinated log-convex functions. Also we proved related mean value theorems.

2. Main results

In the following theorem we give our first result that is Giaccardi's inequality for coordinated convex functions.

Theorem 2.1. *Let $\Delta = [0, a) \times [0, b) \subset \mathbb{R}^2$, $(x_1, ..., x_n) \in [0, a)^n$, $(y_1, ..., y_n) \in [0, b)^n$, $(p_1, ..., p_n)$, $(q_1, ..., q_n)$ be non-negative $n-$tuples and $\sum_{i=1}^{n} p_i x_i = \tilde{x}_n$, $\sum_{j=1}^{n} q_j y_j = \tilde{y}_n$ such that*

$$(x_i - x_0)(\tilde{x}_n - x_i) \geqslant 0 \text{ for } i = 1, ..., n,\ \tilde{x}_n \neq x_0 \text{ and } \sum_{i=1}^{n} p_i \geq 1, \tag{2.1}$$

and

$$(y_j - y_o)(\tilde{y}_n - y_j) \geqslant 0 \text{ for } j = 1, ..., n \text{ and } \tilde{y}_n \neq y_o. \tag{2.2}$$

If f is coordinated convex function, then

$$\begin{aligned}\sum_{i,j=1}^{n} p_i q_j f(x_i, y_j) \leqslant A_1 &\left[A_2 f(\tilde{x}_n, \tilde{y}_n) + B_2\left(\sum_{j=1}^{n} q_j - 1\right) f(\tilde{x}_n, y_o)\right] + \\ &B_1\left(\sum_{i=1}^{n} p_i - 1\right)\left[A_2 f(x_0, \tilde{y}_n) + B_2\left(\sum_{j=1}^{n} q_j - 1\right) f(x_0, y_o)\right],\end{aligned} \tag{2.3}$$

holds where

$$A_1 = \frac{\sum_{i=1}^{n} p_i (x_i - x_0)}{\tilde{x}_n - x_0}, \quad B_1 = \frac{\tilde{x}_n}{\tilde{x}_n - x_0}. \tag{2.4}$$

and

$$A_2 = \frac{\sum_{j=1}^{n} q_j (y_j - y_0)}{\tilde{y}_n - y_0}, \quad B_2 = \frac{\tilde{y}_n}{\tilde{y}_n - y_0} \tag{2.5}$$

Proof. Let $f_x : [0, b) \to \mathbb{R}$ and $f_y : [0, a) \to \mathbb{R}$ be mappings such that $f_x(v) = f(x, v)$ and $f_y(u) = f(u, y)$. Since f is coordinated convex on Δ, therefore f_y is convex on $[0, a)$. By Theorem 1.1, one has

$$\sum_{i=1}^{n} p_i f_y(x_i) \leqslant A_1 f_y(\tilde{x}_n) + B_1 \left(\sum_{i=1}^{n} p_i - 1\right) f_y(x_0),$$

where A_1 and B_1 are defined in (2.4). We write

$$\sum_{i=1}^{n} p_i f(x_i, y) \leqslant A_1 f(\tilde{x}_n, y) + B_1 \left(\sum_{i=1}^{n} p_i - 1\right) f(x_0, y).$$

By setting $y = y_j$, we have

$$\sum_{i=1}^{n} p_i f(x_i, y_j) \leqslant A_1 f(\tilde{x}_n, y_j) + B_1 \left(\sum_{i=1}^{n} p_i - 1\right) f(x_0, y_j),$$

this gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j f(x_i, y_j) \leqslant A_1 \sum_{j=1}^{n} q_j f(\tilde{x}_n, y_j) + B_1 \left(\sum_{i=1}^{n} p_i - 1\right) \sum_{j=1}^{n} q_j f(x_0, y_j). \tag{2.6}$$

Again using Theorem 1.1 on terms of right hand side for second coordinates, we have

$$\sum_{j=1}^{n} q_j f(\tilde{x}_n, y_j) \leqslant A_2 f(\tilde{x}_n, \tilde{y}_n) + B_2 \left(\sum_{j=1}^{n} q_j - 1\right) f(\tilde{x}_n, y_0)$$

and

$$\left(\sum_{i=1}^{n} p_i - 1\right) \sum_{j=1}^{n} q_j f(x_0, y_j) \leqslant A_2 \left(\sum_{i=1}^{n} p_i - 1\right) \left[f(x_0, \tilde{y}_n) + B_2 \left(\sum_{j=1}^{n} q_j - 1\right) f(x_0, y_0)\right].$$

where A_2 and B_2 are defined in (2.5) Using above inequalities in (2.6), we get

$$\sum_{i,j=1}^{n} p_i q_j f(x_i, y_j) \leqslant A_1 \left[A_2 f(\tilde{x}_n, \tilde{y}_n) + B_2 \left(\sum_{j=1}^{n} q_j - 1\right) f(\tilde{x}_n, y_o)\right] +$$
$$B_1 \left(\sum_{i=1}^{n} p_i - 1\right) \left[A_2 f(x_0, \tilde{y}_n) + B_2 \left(\sum_{j=1}^{n} q_j - 1\right) f(x_0, y_o)\right],$$

which is the required result. □

Remark 2.1. *If f is strictly coordinated convex then above inequality is strict unless all x_i's and y_i's are not equal or $\sum_{i=1}^{n} p_i \neq 1$ and $\sum_{j=1}^{n} q_j \neq 1$.*

Remark 2.2. *If we take $y_j = 0$ and $q_j = 1$, $(i, j = 1, ..., n)$ with $f(x_i, 0) \mapsto f(x_i)$, then we get inequality* (1.3).

The following corollary is particular case of Theorem 2.1, which is stated in [11, Theorem 2].

Corollary 2.1. *Let* $\Delta = [0, a) \times [0, b) \subset \mathbb{R}^2$, $(x_1, ..., x_n) \in [0, a)^n$, $(y_1, ..., y_n) \in [0, b)^n$, $(p_1, ..., p_n)$, $(q_1, ..., q_n)$ *be non-negative* $n-$*tuples and* $\sum_{i=1}^{n} p_i x_i = \tilde{x}_n$, $\sum_{j=1}^{n} q_j y_j = \tilde{y}_n$ *such that* $\tilde{x}_n \geq x_j$ *and* $\tilde{y}_n \geq y_j$ *for* $j = 1, ..., n$. *Also let that* $\tilde{x}_n \in [0, a)$, $\sum_{i=1}^{n} p_i \geq 1$ *and* $\tilde{y}_n \in [0, b)$. *If* $f : \Delta \to \mathbb{R}$ *is coordinated convex function, then*

$$\begin{aligned}\sum_{i,j=1}^{n} p_i q_j f(x_i, y_j) \leqslant f(\tilde{x}_n, \tilde{y}_n) + \left(\sum_{j=1}^{n} q_j - 1\right) f(\tilde{x}_n, 0) + \\ \left(\sum_{i=1}^{n} p_i - 1\right)\left[f(0, \tilde{y}_n) + \left(\sum_{j=1}^{n} q_j - 1\right) f(0, 0)\right],\end{aligned} \tag{2.7}$$

holds.

Proof. If we put $x_0 = y_0 = 0$ in Theorem 2, conditions (2.4) and (2.5) becomes $A_1 = A_2 = B_1 = B_2 = 1$, so (2.3) takes the form

$$\begin{aligned}\sum_{i,j=1}^{n} p_i q_j f(x_i, y_j) \leqslant f(\tilde{x}_n, \tilde{y}_n) + \left(\sum_{j=1}^{n} q_j - 1\right) f(\tilde{x}_n, 0) + \\ \left(\sum_{i=1}^{n} p_i - 1\right)\left[f(0, \tilde{y}_n) + \left(\sum_{j=1}^{n} q_j - 1\right) f(0, 0)\right],\end{aligned}$$

as required. □

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Then for distinct points $u_i \in I, i = 0, 1, 2$. The divided differences of first and second order are defined as follows.

$$[u_i, u_{i+1}, f] = \frac{f(u_{i+1}) - f(u_i)}{u_{i+1} - u_i}, (i = 0, 1) \tag{2.8}$$

$$[u_0, u_1, u_2, f] = \frac{[u_1, u_2, f] - [u_0, u_1, f]}{u_2 - u_0}. \tag{2.9}$$

The values of the divided differences are independent of the order of the points u_0, u_1, u_2 and may be extended to include the cases when some or all points are equal, that is

$$[u_0, u_0, f] = \lim_{u_1 \to u_0} [u_0, u_1, f] = f'(u_0) \tag{2.10}$$

provided that f' exists. Now passing the limit $u_1 \to u_0$ and replacing u_2 by u in second order divided difference, we have

$$[u_0, u_0, u, f] = \lim_{u_1 \to u_0} [u_0, u_1, u, f] = \frac{f(u) - f(u_0) - (u - u_0) f'(u_0)}{(u - u_0)^2}, u \neq u_0 \tag{2.11}$$

provided that f' exists. Also passing to the limit $u_i \to u$ $(i = 0, 1, 2)$ in second order divided difference, we have

$$[u, u, u, f] = \lim_{u_i \to u} [u_0, u_1, u_2, f] = \frac{f''(u)}{2} \tag{2.12}$$

provided that f'' exists.

One can note that, if for all $u_0, u_1 \in I$, $[u_0, u_1, f] \geq 0$, then f is increasing on I and if for all $u_0, u_1, u_2 \in I$, $[u_0, u_1, u_2, f] \geq 0$, then f is convex on I.

Now we define some families of parametric functions which we use in sequal.

Let $I = [0, a)$ and $J = [0, b)$ be intervals and let for $t \in (c, d) \subseteq \mathbb{R}$, $f_t : I \times J \to \mathbb{R}$ be a mapping. Then we define functions

$$f_{t,y} : I \to \mathbb{R} \text{ by } f_{t,y}(u) = f_t(u, y)$$

and

$$f_{t,x} : J \to \mathbb{R} \text{ by } f_{t,x}(v) = f_t(x, v),$$

where $x \in I$ and $y \in J$.

Suppose $\mathcal{M}_1$ denotes the class of functions $f_t : I \times J \to \mathbb{R}$ for $t \in (c, d)$ such that

$$t \mapsto [u_0, u_1, u_2, f_{t,y}] \quad \forall\ u_0, u_1, u_2 \in I$$

and

$$t \mapsto [v_0, v_1, v_2, f_{t,x}] \quad \forall\ v_0, v_1, v_2 \in J$$

are *log* convex functions in Jensen sense on (c, d) for all $x \in I$ and $y \in J$.

Under the assumptions of Theorem 2.1 we define linear functional $\mathcal{G}(f; x_0, y_0)$ as a non negative difference of inequality (2.3)

$$\begin{aligned} \mathcal{G}(f; x_0, y_0) = A_1 \left[A_2 f\left(\tilde{x}_n, \tilde{y}_n\right) + B_2 \left(\sum_{j=1}^{n} q_j - 1 \right) f\left(\tilde{x}_n, y_0\right) \right] + \\ B_1 \left(\sum_{i=1}^{n} p_i - 1 \right) \left[A_2 f\left(x_0, \tilde{y}_n\right) + B_2 \left(\sum_{j=1}^{n} q_j - 1 \right) f(x_0, y_0) \right] - \sum_{i,j=1}^{n} p_i q_j f(x_i, y_j) \end{aligned} \tag{2.13}$$

where A_1, B_1 and A_2, B_2 are defined in (2.4) and (2.5) respectively.

Remark 2.3. *Under the assumptions of Theorem 2.1, if f is coordinated convex in Δ, then $\mathcal{G}(f; x_0, y_0) \geq 0$.*

Remark 2.4. *As a special case, if we put $x_0 = y_0 = 0$, in (2.13), then we get*

$$\begin{aligned} \Upsilon(f) = f\left(\tilde{x}_n, \tilde{y}_n\right) + \left(\sum_{i=1}^{n} q_j - 1 \right) f\left(\tilde{x}_n, 0\right) + \left(\sum_{i=1}^{n} p_i - 1 \right) \\ \left[f(0, \tilde{y}_n) + \left(\sum_{i=1}^{n} q_j - 1 \right) f(0, 0) \right] - \sum_{i,j=1}^{n} p_i, q_j f(x_i, y_j), \end{aligned} \tag{2.14}$$

that is $\mathcal{G}(f; 0, 0) = \Upsilon(f)$.

Remark 2.5. *If we put $y_j = 1$ for $j = 1, ..., n$ in (2.13) then we get functional*

$$\mathcal{P}(f) = f(\tilde{x}_n) - \sum_{i=1}^{n} p_i f(x_i) - \left(1 - \sum_{i=1}^{n} p_i\right) f(0) \tag{2.15}$$

defined in [1].

The following lemmas are given in [9].

Lemma 2.1. *Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to (0, \infty)$ is log-convex in Jensen sense on I, that is, for each $r, t \in I$*

$$f(r)f(t) \geq f^2\left(\frac{t+r}{2}\right)$$

if and only if the relation

$$m^2 f(t) + 2mnf\left(\frac{t+r}{2}\right) + n^2 f(r) \geq 0$$

holds for each $m, n \in \mathbb{R}$ and $r, t \in I$.

Lemma 2.2. *If f is convex function on interval I then for all $x_1, x_2, x_3 \in I$ for which $x_1 < x_2 < x_3$, the following inequality is valid:*

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0.$$

In [11], authors has given some important properties related to the functional defined for Petrović's inequality on coordinates. Our next result comprises similar properties of functional defined in (2.13).

Theorem 2.2. *Suppose $f_t \in \mathcal{M}_1$ and $\mathcal{G}$ be a functional defined in (2.13). Then $\mathcal{G}(f_t, x_0, y_0)$ is log-convex function in Jensen sense for all $t \in (c, d)$.*

Proof. Let

$$h(u, v) = m^2 f_t(u, v) + 2mnf_{\frac{t+r}{2}}(u, v) + n^2 f_r(u, v),$$

where $m, n \in \mathbb{R}$ and $t, r \in (c, d)$. We can assume that

$$h_y(u) = m^2 f_{t,y}(u) + 2mnf_{\frac{t+r}{2},y}(u) + n^2 f_{r,y}(u)$$

and

$$h_x(v) = m^2 f_{t,x}(v) + 2mnf_{\frac{t+r}{2},x}(v) + n^2 f_{r,x}(v).$$

Since divided differences satisfy the linearity property, therefore we can have

$$[u_0, u_1, u_2, h_y] = m^2[u_0, u_1, u_2, f_{t,y}] + 2mn[u_0, u_1, u_2, f_{\frac{t+r}{2},y}] + n^2[u_0, u_1, u_2, f_{r,y}].$$

Since we have given that $[u_0, u_1, u_2; h_y]$ is log-convex in Jensen sense, therefore using $f_t = [u_0, u_1, u_2; h_y]$ in Lemma 2.1, we get that

$$[u_0, u_1, u_2, h_y] = m^2[u_0, u_1, u_2, f_{t,y}] + 2mn[u_0, u_1, u_2, f_{\frac{t+r}{2},y}] + n^2[u_0, u_1, u_2, f_{r,y}] \geq 0$$

which is equivalent to write

$$[u_0, u_1, u_2; h_y] \geq 0.$$

This shows that h_y is convex on interval I. In the similar way, one can prove that h_x is convex on J. This concludes that h is coordinated convex on Δ. By Remark 2.3, we have

$$\mathcal{G}(h, x_0, y_0) \geq 0,$$

that is,

$$m^2\mathcal{G}(f_t, x_0, y_0) + 2mn\mathcal{G}(f_{\frac{t+r}{2}}, x_0, y_0) + n^2\mathcal{G}(f_r, x_0, y_0) \geq 0.$$

Thus by Lemma 2.1 we have that $\mathcal{G}(f_t, x_0, y_0)$ is log-convex in Jensen sense on (c, d). □

Corollary 2.2. *Let the functional Υ defined in (2.14) and $f_t \in \mathcal{M}_1$. Then the function $t \mapsto \Upsilon(f_t)$ is log convex in Jensen sense on (c, d)*

Proof. On putting $x_0 = y_0 = 0$ in above theorem, we get $\mathcal{G}(f_t; 0, 0) = \Upsilon(f_t)$, hence the required result follows. □

Theorem 2.3. *Suppose f_t is from class $\mathcal{M}_1$ and $\mathcal{G}$ be a functional defined in (2.13), If $\mathcal{G}(f_t, x_0, y_0)$ is continuous for all $t \in (c, d)$, then $\mathcal{G}(f_t, x_0, y_0)$ is log convex for all $t \in (c, d)$.*

Proof. Since we know that if a function is log convex in Jensen sense and continuous, then it is log convex. From Theorem 2.2, if $f_t \in \mathcal{M}_1$, then $\mathcal{G}(f_t, x_0, y_0)$ is log convex in Jensen sense and we have given that it is continuous, hence $\mathcal{G}(f_t, x_0, y_0)$ is log convex for all $t \in (c, d)$. □

Corollary 2.3. *Let the functional Υ defined in (2.14) and $f_t \in \mathcal{M}_1$. If the function $t \mapsto \Upsilon(f_t)$ is continuous on (c,d), then it is log convex on (c, d)*

Proof. On putting $x_0 = y_0 = 0$ in above theorem, we get $\mathcal{G}(f_t; 0, 0) = \Upsilon(f_t)$, hence the required result follows. □

Theorem 2.4. *Suppose $f_t \in \mathcal{M}_1$ and $\mathcal{G}$ be a functional defined in (2.13). If $\mathcal{G}(f_t; x_0, y_0)$ is positive, then for $r, s, t \in (c, d)$ such that $r < s < t$, one has*

$$[\mathcal{G}(f_s; x_0, y_0)]^{t-r} \leq [\mathcal{G}(f_r; x_0, y_0)]^{t-s} [\mathcal{G}(f_t; x_0, y_0)]^{s-r}. \tag{2.16}$$

Proof. By taking $f = \log \mathcal{G}(f_t, x_0, y_0)$ in Lemma 2.2, we have for $t \neq r$, $u \neq v$,

$$(t-s)\log \mathcal{G}(f_r; x_0, y_0) + (r-t)\log \mathcal{G}(f_s; x_0, y_0) + (s-r)\log \mathcal{G}(f_t; x_0, y_0) \geq 0,$$

which is equivalent to

$$[\mathcal{G}(f_s; x_0, y_0)]^{t-r} \leq [\mathcal{G}(f_r; x_0, y_0)]^{t-s} [\mathcal{G}(f_t; x_0, y_0)]^{s-r} \tag{2.17}$$

that is our required result. □

Corollary 2.4. *Let the functional Υ defined in (2.14) and $f_t \in \mathcal{M}_1$. If $\Upsilon(f_t)$ is positive, then for some $r < s < t$, where $r, s, t \in (c, d)$, one has*

$$[\Upsilon(f_s)]^{t-r} \leq [\Upsilon(f_r)]^{t-s} [\Upsilon(f_t)]^{s-r}. \tag{2.18}$$

Proof. On putting $x_0 = y_0 = 0$ in above theorem, we get $\mathcal{G}(f_t; 0, 0) = \Upsilon(f_t)$, hence the required result follows. □

The following Lemma is equivalent to the definition of convex function (see [5, Page 2]).

Lemma 2.3. *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is convex if and only if for all $t, r, u, v \in I$ such that $t \leq u, r \leq v, t \neq r, u \neq v$, one has*

$$\frac{f(t) - f(r)}{t - r} \leq \frac{f(u) - f(v)}{u - v}.$$

Theorem 2.5. *Let $\mathcal{G}(f_t; x_0, y_0)$ be the linear functional defined in (2.13), where $f_t \in \mathcal{M}_1$. If the function $t \mapsto \mathcal{G}(f_t; x_0, y_0)$ is derivable on (c, d), then for $t, r, u, v \in (c, d)$ such that $t \leqslant u, r \leqslant v$, we have*

$$\mathcal{C}_1(t, r) \leqslant \mathcal{C}_1(u, v),$$

where

$$\mathcal{C}_1(t, r) = \begin{cases} \left(\frac{\mathcal{G}(f_t; x_0, y_0)}{\mathcal{G}(f_r; x_0, y_0)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{dt}(\mathcal{G}(f_t; x_0, y_0))}{\mathcal{G}(f_t; x_0, y_0)}\right), & t = r. \end{cases} \tag{2.19}$$

Proof. By taking $f = \mathcal{G}(f_t, x_0, y_0)$ in Lemma 2.3, we have for $t \neq r, u \neq v$,

$$\frac{\log \mathcal{G}(f_t; x_0, y_0) - \log \mathcal{G}(f_r; x_0, y_0)}{t - r} \leqslant \frac{\log \mathcal{G}(f_u; x_0, y_0) - log\mathcal{G}(f_v; x_0, y_0)}{u - v}.$$

This gives

$$\mathcal{C}_1(t, r) \leqslant \mathcal{C}_1(u, v), \qquad t \neq r, u \neq v.$$

For $t = r, u = v$, we consider limiting cases in above inequality, when $r \to t$ and $v \to u$.

□

The following corollaries that are stated in [11], are special cases of Theorem 2.5.

Corollary 2.5. *Under the assumptions of Theorem (2.5), let $\Upsilon(f_t)$ be the linear functional defined in (2.14) then $\mathcal{E}(t,r,f_t) \leqslant \mathcal{E}(u,v,f_t)$, where*

$$\mathcal{E}(t,r,f_t) = \begin{cases} \left(\frac{\Upsilon(f_t)}{\Upsilon(f_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{dt}(\Upsilon(f_t))}{\Upsilon(f_t)}\right), & t = r. \end{cases} \tag{2.20}$$

Proof. On putting $x_0 = y_0 = 0$ in Theorem (2.5), we get $\mathcal{G}(f_t; x_0, y_0) = \Upsilon(f_t)$, hence the required result follows. □

Corollary 2.6. *Under the assumptions of Theorem (2.5), let $\mathcal{P}(f_t)$ be the linear functional defined in (2.15) then $\mathcal{T}(t,r,f_t) \leqslant \mathcal{T}(u,v,f_t)$, where*

$$\mathcal{T}(t,r,f_t) = \begin{cases} \left(\frac{\mathcal{P}(f_t)}{\mathcal{P}(f_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{dt}(\mathcal{P}(f_t))}{\mathcal{P}(f_t)}\right), & t = r. \end{cases} \tag{2.21}$$

Proof. On putting, $y_j = 1$ for $j = 1, ..., n$ in Corollary 2.5, we get our required result. □

Example 2.1. *Let $t \in (0,\infty)$ and $\varphi_t : [0,\infty)^2 \to \mathbb{R}$ be a function defined as*

$$\varphi_t(u,v) = \begin{cases} \frac{u^t v^t}{t(t-1)}, & t \neq 1, \\ uv(\log u + \log v), & t = 1. \end{cases} \tag{2.22}$$

Define partial mappings

$$\varphi_{t,v} : [0,\infty) \to \mathbb{R} \text{ by } \varphi_{t,v}(u) = \varphi_t(u,v)$$

and

$$\varphi_{t,u} : [0,\infty) \to \mathbb{R} \text{ by } \varphi_{t,u}(v) = \varphi_t(u,v).$$

As we have

$$[u,u,u,\varphi_{t,v}] = \frac{\partial^2 \varphi_{t,v}}{\partial u^2} = u^{t-2}v^t \geq 0 \quad \forall\, t \in (0,\infty).$$

This gives $t \mapsto [u_0,u_0,u_0,\varphi_{t,v}]$ is log convex in Jensen sense. Similarly one can deduce that $t \mapsto [v_0,v_0,v_0,\varphi_{t,u}]$ is also log-convex in Jensen sense. If we choose $f_t = \varphi_t$ in Theorem 2.2, we get log convexity of the functional $\mathcal{G}(\gamma_t)$.

In special case, if we choose $\varphi_t(u,v) = \varphi_t(u,1)$, then we get [1, Example 3].

Example 2.2. *Let $t \in [0,\infty)$ and $\delta_t : [0,\infty)^2 \to \mathbb{R}$ be a function defined as*

$$\delta_t(u,v) = \begin{cases} \frac{uve^{uvt}}{t}, & t \neq 0, \\ u^2v^2, & t = 0. \end{cases} \tag{2.23}$$

Define partial mappings

$$\delta_{t,v} : [0,\infty) \to \mathbb{R} \text{ by } \delta_{t,v}(u) = \delta_t(u,v)$$

and

$$\delta_{t,u} : [0,\infty) \to \mathbb{R} \text{ by } \delta_{t,u}(v) = \delta_t(u,v)$$

for all $u, v \in [0,\infty)$.

As we have

$$[u,u,u,\delta_{t,v}] = \frac{\partial^2 \delta_{t,v}}{\delta u^2} = e^{uvt}(2v^2 + uv^2) \geq 0 \quad \forall\ t \in (0,\infty).$$

This gives $t \mapsto [u_0,u_0,u_0,\delta_{t,v}]$ *is log convex in Jensen sense. Similarly one can deduce that* $t \mapsto [v_0,v_0,v_0,\delta_{t,u}]$ *is also log-convex in Jensen sense. If we choose* $f_t = \delta_t$ *in Theorem 2.2, we get log convexity of the functional* $\mathcal{G}(\delta_t)$.

In special case, if we choose $\delta_t(u,v) = \delta_t(u,1)$*, then we get [1, Example 8].*

Example 2.3. *Let* $t \in [0,\infty)$ *and* $\gamma_t : [0,\infty)^2 \to \mathbb{R}$ *be a function defined as*

$$\gamma_t(u,v) = \begin{cases} \frac{e^{uvt}}{t}, & t \neq 0, \\ uv, & t = 0. \end{cases} \tag{2.24}$$

Define partial mappings

$$\gamma_{t,v} : [0,\infty) \to \mathbb{R} \text{ by } \gamma_{t,v}(u) = \gamma_t(u,v)$$

and

$$\gamma_{t,u} : [0,\infty) \to \mathbb{R} \text{ by } \gamma_{t,u}(v) = \gamma_t(u,v).$$

As we have

$$[u,u,u,\gamma_{t,v}] = \frac{\partial^2 \gamma_{t,v}}{\partial u^2} = tv^2 e^{uvt} \geq 0 \quad \forall\ t \in (0,\infty).$$

This gives $t \mapsto [u_0,u_0,u_0,\gamma_{t,v}]$ *is log convex in Jensen sense. Similarly one can deduce that* $t \mapsto [v_0,v_0,v_0,\gamma_{t,u}]$ *is also log-convex in Jensen sense. If we choose* $f_t = \gamma_t$ *in Theorem 2.2, we get log convexity of the functional* $\mathcal{G}(\gamma_t)$.

In special case, if we choose $\gamma_t(u,v) = \gamma_t(u,1)$*, then we get [1, Example 9].*

Example 2.4. *Let* $t \in [0,\infty)$ *and* $\lambda_t : [0,\infty)^2 \to \mathbb{R}$ *be a function defined as*

$$\lambda_t(u,v) = \frac{ue^{u\sqrt{t}}}{\sqrt{t}} \tag{2.25}$$

Define partial mappings

$$\lambda_{t,v} : [0,\infty) \to \mathbb{R} \text{ by } \lambda_{t,v}(u) = \lambda_t(u,v)$$

and

$$\lambda_{t,u} : [0,\infty) \to \mathbb{R} \text{ by } \lambda_{t,u}(v) = \lambda_t(u,v).$$

As we have

$$[u, u, u, \lambda_{t,v}] = \frac{\partial^2 \lambda_{t,v}}{\partial u^2} = v^2 e^{uv\sqrt{t}} \left(2 + uv\sqrt{t}\right) \geq 0 \quad \forall\ t \in (0, \infty).$$

This gives $t \mapsto [u_0, u_0, u_0, \lambda_{t,v}]$ *is log convex in Jensen sense. Similarly one can deduce that* $t \mapsto [v_0, v_0, v_0, \lambda_{t,u}]$ *is also log-convex in Jensen sense. If we choose* $f_t = \lambda_t$ *in Theorem 2.2, we get log convexity of the functional* $\mathcal{G}(\lambda_t)$.

In special case, if we choose $\lambda_t(u, v) = \lambda_t(u, -1)$*, then we get [1, Example 6].*

3. Mean value theorems

If a function is twice differentiable on an interval I, then it is convex on I if and only if its second order derivative is nonnegative. If a function $f(X) := f(x, y)$ has continuous second order partial derivatives on interior of Δ then it is convex on Δ if the Hessian matrix

$$H_f(X) := \begin{pmatrix} \frac{\partial^2 f(X)}{\partial x^2} & \frac{\partial^2 f(X)}{\partial y \partial x} \\ \frac{\partial^2 f(X)}{\partial x \partial y} & \frac{\partial^2 f(X)}{\partial y^2} \end{pmatrix}$$

is nonnegative definite, that is, $\mathbf{v} H_f(X) \mathbf{v}^t$ is nonnegative for all real nonnegative vector $\mathbf{v}$. It is easy to see that $f : \Delta \to \mathbb{R}$ is coordinated covex on Δ iff

$$f''_x(y) = \frac{\partial^2 f(x, y)}{\partial y^2} \text{ and } f''_y(x) = \frac{\partial^2 f(x, y)}{\partial x^2}$$

are nonnegative for all interior points (x, y) in Δ.

Lemma 3.1. *Let* $f : \Delta \to \mathbb{R}$ *be a function such that*

$$\mathcal{M}_1 \leq \frac{\partial^2 f(x, y)}{\partial x^2} \leq \mathcal{M}_1$$

and

$$m_2 \leq \frac{\partial^2 f(x, y)}{\partial y^2} \leq M_2$$

for all interior points (x, y) *in* Δ^2*. Consider the function* $\psi_1, \psi_2 : \Delta \to \mathbb{R}$ *defined as*

$$\psi_1 = \frac{1}{2} \max\{\mathcal{M}_1, M_2\}(x^2 + y^2) - f(x, y)$$

$$\psi_2 = f(x, y) - \frac{1}{2} \min\{\mathcal{M}_1, m_2\}(x^2 + y^2),$$

then ψ_1, ψ_2 *are convex on coordinates in* Δ.

Proof. Since

$$\frac{\partial^2 \psi_1(x, y)}{\partial x^2} = \max\{\mathcal{M}_1, M_2\} - \frac{\partial^2 f(x, y)}{\partial x^2} \geq 0$$

and

$$\frac{\partial^2 \psi_1(x, y)}{\partial y^2} = \max\{\mathcal{M}_1, M_2\} - \frac{\partial^2 f(x, y)}{\partial y^2} \geq 0$$

for all interior points (x, y) in Δ, ψ_1 is convex on coordinates in Δ. Similarly one can prove that ψ_2 is also convex on coordinates in Δ. □

In [3] and [4], we have given mean value theorems of Lagrange type and Cauchy type for certain functional. Here we give a theorem similar to those but for functional introduced in (2.13).

Theorem 3.1. *Let $\tilde{\Delta} = [0, a_1] \times [0, b_1] \subset \Delta$ and $f : \tilde{\Delta} \to \mathbb{R}$ which has continuous partial derivatives of second order in $\tilde{\Delta}$ and $\varphi(x, y) := x^2 + y^2$. Then there exist (β_1, γ_1) and (β_2, γ_2) in the interior of $\tilde{\Delta}$ such that*

$$\mathcal{G}(f; x_0, y_0) = \frac{1}{2}\frac{\partial^2 f(\beta_1, \gamma_1)}{\partial x^2}\Upsilon(\varphi)$$

and

$$\mathcal{G}(f; x_0, y_0) = \frac{1}{2}\frac{\partial^2 f(\beta_2, \gamma_2)}{\partial y^2}\Upsilon(\varphi)$$

provided that $\mathcal{G}(\varphi; x_0, y_0)$ is non-zero.

Proof. Since f has continuous partial derivatives of second order in $\tilde{\Delta}$ and $\tilde{\Delta}$ is compact, there exist real numbers $\mathcal{M}_1, m_2, \mathcal{M}_1$ and M_2 such that

$$\mathcal{M}_1 \leq \frac{\partial^2 f(x, y)}{\partial x^2} \leq \mathcal{M}_1 \quad \text{and} \quad m_2 \leq \frac{\partial^2 f(x, y)}{\partial y^2} \leq M_2,$$

for all $(x, y) \in \tilde{\Delta}$.

Now consider functions ψ_1 and ψ_2 defined in Lemma 3.1. As ψ_1 is convex on coordinates in Δ,

$$\mathcal{G}(\psi_1; x_0, y_0) \geq 0,$$

that is

$$\mathcal{G}\left(\frac{1}{2}\max\{\mathcal{M}_1, M_2\}\varphi(x, y) - f(x, y)\right) \geq 0,$$

this leads us to

$$2\mathcal{G}(f; x_0, y_0) \leq \max\{\mathcal{M}_1, M_2\}\mathcal{G}(\varphi; x_0, y_0). \tag{3.1}$$

On the other hand for function ψ_2, one has

$$\min\{\mathcal{M}_1, m_2\}\mathcal{G}(\varphi; x_0, y_0) \leq 2\mathcal{G}(f; x_0, y_0). \tag{3.2}$$

As $\mathcal{G}(\varphi; x_0, y_0) \neq 0$, combining inequalities (3.1) and (3.2), we get

$$\min\{\mathcal{M}_1, m_2\} \leq \frac{2\mathcal{G}(f; x_0, y_0)}{\mathcal{G}(\varphi; x_0, y_0)} \leq \max\{\mathcal{M}_1, M_2\}.$$

Then there exist (β_1, γ_1) and (β_2, γ_2) in the interior of Δ such that

$$\frac{2\mathcal{G}(f; x_0, y_0)}{\mathcal{G}(\varphi; x_0, y_0)} = \frac{\partial^2 f(\beta_1, \gamma_1)}{\partial x^2}$$

and
$$\frac{2\mathcal{G}(f;x_0,y_0)}{\mathcal{G}(\varphi;x_0,y_0)} = \frac{\partial^2 f(\beta_2,\gamma_2)}{\partial y^2},$$
hence the required result follows. □

The following corollary is particular case of Theorem 3.1, which is stated in [11, Theorem 4].

Corollary 3.1. *Under the assumptions of above theorem, let $\Upsilon(f)$ be the linear functional defined in (2.14), then*
$$\Upsilon(f) = \frac{1}{2}\frac{\partial^2 f(\beta_1,\gamma_1)}{\partial x^2}\Upsilon(\varphi)$$
and
$$\Upsilon(f) = \frac{1}{2}\frac{\partial^2 f(\beta_2,\gamma_2)}{\partial y^2}\Upsilon(\varphi)$$
provided that $\Upsilon(f)$ is non-zero.

Proof. On putting $x_0 = y_0 = 0$ in Theorem 3.1, we get $\mathcal{G}(f;x_0,y_0) = \Upsilon(f)$, hence the required result follows. □

Theorem 3.2. *Let $\psi_1,\psi_2 : \tilde{\Delta} \to \mathbb{R}$ be mappings which have continuous partial derivatives of second order in $\tilde{\Delta}$. Then there exists (η_1,ξ_1) and (η_2,ξ_2) in $\tilde{\Delta}$ such that*
$$\frac{\mathcal{G}(\psi_1;x_0,y_0)}{\mathcal{G}(\psi_2;x_0,y_0)} = \frac{\frac{\partial^2\psi_1(\eta_1,\xi_1)}{\partial x^2}}{\frac{\partial^2\psi_2(\eta_1,\xi_1)}{\partial x^2}} \tag{3.3}$$
and
$$\frac{\mathcal{G}(\psi_1;x_0,y_0)}{\mathcal{G}(\psi_2;x_0,y_0)} = \frac{\frac{\partial^2\psi_1(\eta_2,\xi_2)}{\partial y^2}}{\frac{\partial^2\psi_2(\eta_2,\xi_2)}{\partial y^2}}. \tag{3.4}$$

Proof. We define the mapping $P : \tilde{\Delta} \to \mathbb{R}$ such that
$$P = k_1\psi_1 - k_2\psi_2,$$
where $k_1 = \mathcal{G}(\psi_2;x_0,y_0)$ and $k_2 = \mathcal{G}(\psi_1;x_0,y_0)$.

Using Theorem 3.1 with $f = P$, we have
$$2\mathcal{G}(P;x_0,y_0) = 0 = \left\{k_1\frac{\partial^2\psi_1}{\partial x^2} - k_2\frac{\partial^2\psi_2}{\partial x^2}\right\}\mathcal{G}(\varphi;x_0,y_0)$$
and
$$2\mathcal{G}(P;x_0,y_0) = 0 = \left\{k_1\frac{\partial^2\psi_1}{\partial y^2} - k_2\frac{\partial^2\psi_2}{\partial y^2}\right\}\mathcal{G}(\varphi;x_0,y_0).$$
Since $\mathcal{G}(\varphi;x_0,y_0) \neq 0$, we have
$$\frac{k_2}{k_1} = \frac{\frac{\partial^2\psi_1(\eta_1,\xi_1)}{\partial x^2}}{\frac{\partial^2\psi_2(\eta_1,\xi_1)}{\partial x^2}}$$
and
$$\frac{k_2}{k_1} = \frac{\frac{\partial^2\psi_1(\eta_2,\xi_2)}{\partial y^2}}{\frac{\partial^2\psi_2(\eta_2,\xi_2)}{\partial y^2}},$$

which are equivalent to required results. □

Corollary 3.2. *Under the assumptions of above theorem, let $\Upsilon(f)$ be the linear functional defined in (2.14) then*

$$\frac{\Upsilon(\psi_1)}{\Upsilon(\psi_2)} = \frac{\frac{\partial^2 \psi_1(\eta_1,\xi_1)}{\partial x^2}}{\frac{\partial^2 \psi_2(\eta_1,\xi_1)}{\partial x^2}} \tag{3.5}$$

and

$$\frac{\Upsilon(\psi_1)}{\Upsilon(\psi_2)} = \frac{\frac{\partial^2 \psi_1(\eta_2,\xi_2)}{\partial y^2}}{\frac{\partial^2 \psi_2(\eta_2,\xi_2)}{\partial y^2}}. \tag{3.6}$$

Proof. On putting $x_0 = y_0 = 0$ in Theorem 3.2, we get $\mathcal{G}(f; x_0, y_0) = \Upsilon(f)$, hence the required result follows. □

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