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FUZZY HYPERIDEALS OF LEFT ALMOST SEMIHYPERGROUPS

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Abstract. This paper explores the foundations of fuzzy left (resp. right) hyperideals of left almost semihypergroups (briefly, LA-semihypergroups). We investigate the properties of fuzzy left hyperideals and fuzzy right hyperideals in regular and intra-regular LA-semihypergroups. We also characterize regular and intra-regular LA-semihypergroups in terms of fuzzy hyperideals.

1. Introduction

The idea of generalization of a commutative semigroup, (known as left almost semigroup) was introduced by Kazim and Naseeruddin in 1972 (see [1]). A groupoid $(S, \cdot)$ is called an AG-groupoid if it satisfies the left invertive law:

$$(ab)c = (cb)a \text{ for all } a, b, c \in S.$$

This structure is closely related with a commutative semigroup because if an AG-groupoid contains right identity then it becomes a commutative monoid. An AG-groupoid may or may not contain a left identity. Some other names have also been used in literature for left almost semigroups. Cho et al. [2] studied this structure under the name of right modular groupoid. Holgate [3] studied it as left invertive groupoid. Similarly, Stevanovic and Protic [4] called this structure an Abel-Grassmann groupoid (or simply LA-semigroup), which is the primary name under which this structure is known nowadays. There are many important applications of AG-groupoids in the theory of flocks [5]. The concept of a fuzzy set was introduced by Zadeh [9], in 1965. Since its inception, the theory has developed in many directions and found applications in a wide variety of fields. Many researchers published high-quality research articles on fuzzy sets in a variety of international journals. The study of fuzzy set in algebraic structure has been started in the definitive paper of Rosenfeld 1971 [15], in which he defined fuzzy subgroup and gave its important properties. In 1981, Kuroki introduced the concept of fuzzy ideals and fuzzy bi-ideals in semigroups in his paper [16].

The theory of hyperstructures was introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians [20]. Marty introduced hypergroups as a generalization of groups. He published some papers on hypergroups, using them in different contexts as algebraic functions, rational fractions, non commutative groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. In [17] Corsini and Leoreanu-Fotea collected numerous applications of algebraic hyperstructures such as: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence, and probabilities. Especially, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Nowadays many scholars have studied different aspects of semihypergroups see [18,19,21,22]. Recently, Hila and Dine [12] introduced the notion of LA-semihypergroups. They investigated several properties of hyperideals of LA-semihypergroup and defined the topological space and study the topological structure of LA-semihypergroups using hyperideal theory. In [13], Yaqoob, Corsini and Yousafzai have characterized intra-regular LA-semihypergroups by using the properties of their left and right hyperideals,

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and investigated some useful conditions for an LA-semihypergroup to become an intra-regular LA-semihypergroup. This non-associative hyper structure has been further explored in [14], by Yousafzai and Corsini.

In this paper, we introduce the notion of fuzzy left (resp. right) hyperideals in LA-semihypergroups and present some related examples of these concepts. We characterize regular and intra-regular LA-semihypergroups in terms of fuzzy hyperideals.

2. Preliminaries

A hypergroupoid is a nonempty set S equipped with a hyperoperation $\circ$, that is a map $\circ : S \times S \longrightarrow P^*(S)$, where $P^*(S)$ denotes the set of all nonempty subsets of S (see [20]). We shall denote by $x \circ y$, the hyperproduct of elements x, y of S. Let A, B be two nonempty subsets of S. Then the hyperproduct of A and B is defined as $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$. We shall write $A \circ x$ instead of $A \circ \{x\}$ and $x \circ A$ for $\{x\} \circ A$.

A hypergroupoid $(S, \circ)$ is called an LA-semihypergroup [12], if it satisfies the left invertive law:

$$(a \circ b) \circ c = (c \circ b) \circ a \text{ for all } a, b, c \in S.$$

Every LA-semihypergroup satisfies the medial law [12]. That is,

$$(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w) \text{ for all } w, x, y, z \in S.$$

Definition 2.1. *(see [14]). Let $(S, \circ)$ be an LA-semihypergroup then an element $e \in S$ is called*

(i) left identity (resp. pure left identity) if $\forall\, a \in S$, $a \in e \circ a$ (resp. $a = e \circ a$);
(ii) right identity (resp. pure right identity) if $\forall\, a \in S$, $a \in a \circ e$ (resp. $a = a \circ e$);
(iii) identity (resp. pure identity) if $\forall\, a \in S$, $a \in e \circ a \cap a \circ e$ (resp. $a = e \circ a \cap a \circ e$).
An LA-semihypergroup $(S, \circ)$ with pure left identity e, paramedial law holds. That is

$$(x \circ y) \circ (z \circ w) = (w \circ z) \circ (y \circ x) \text{ for all } w, x, y, z \in S.$$

An LA-semihypergroup $(S, \circ)$ with pure left identity e, satisfies the following law

$$x \circ (y \circ z) = y \circ (x \circ z) \qquad (1).$$

A nonempty subset A of an LA-semihypergroup $(S, \circ)$ is called *an LA-subsemihypergroup* of S if $A \circ A \subseteq A$.
A nonempty subset A of an LA-semihypergroup $(S, \circ)$ is a called *left* (resp. *right*) hyper*ideal* of S if $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$).

If A is both a left hyperideal and a right hyperideal of S then it is called a *two-sided hyperideal* or simply a hyperideal of S.

An LA-semihypergroup S is called [13];
(i) regular if for all $a \in S$, there exist $x \in S$ such that $a \in (a \circ x) \circ a$.
(ii) intra-regular if for all $a \in S$, there exist $x, y \in S$ such that $a \in (x \circ a^2) \circ y$.

3. Fuzzy concepts in LA-semihypergroups

Let S be an LA-semihypergroup. A function f from a nonempty set X to the unit interval $[0, 1]$ is called a fuzzy subset of S.

Let S be an LA-semihypergroup and f be a fuzzy subset of S. Then for every $t \in (0, 1]$ the set

$$U(f; t) = \{x \mid x \in S,\ f(x) \geq t\},$$

is called the level set of f.

For $x \in S$, define

$$A_x = \{(y, z) \in S \times S : x \in y \circ z \text{ or } x = y \circ z\}.$$

We denote by $F(S)$ the set of all fuzzy subsets of S.

Let S be an LA-semihypergroup and f, g are any two fuzzy subsets of S. We define the product $f * g$ of f and g as follows:

$$(f * g)(x) = \bigvee_{(y,z)\in A_x} \{f(y) \wedge g(z)\}.$$

The fuzzy subsets defined by $\mathcal{S} : S \longrightarrow [0,1], x \longrightarrow \mathcal{S}(x) = 1$ and $0 : S \longrightarrow [0,1], x \longrightarrow 0(x) = 0$ for all $x \in S$ are the greatest and least elements of $F(S)$.

Definition 3.1. *Let S be an LA-semihypergroup and $\emptyset \neq A \subseteq S$. Then the characteristic function χ_A of A is defined as:*

$$\chi_A : S \longrightarrow [0,1], \longrightarrow \chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Definition 3.2. *Let S be an LA-semihypergroup and f be a fuzzy subset of S. Then f is called a fuzzy LA-subsemihypergroup of S if:*

$$(\forall x, y \in S) \quad \bigwedge_{\alpha\in x\circ y} f(\alpha) \geq f(x) \wedge f(y).$$

Definition 3.3. *Let S be an LA-semihypergroup and f be a fuzzy subset of S. Then f is called a fuzzy left (resp. right) hyperideal of S if:*

$$(\forall x, y \in S) \quad \bigwedge_{\alpha\in x\circ y} f(\alpha) \geq f(y) \quad (\text{resp. } \bigwedge_{\alpha\in x\circ y} f(\alpha) \geq f(x)).$$

Definition 3.4. *A fuzzy hyperideal f of an LA-semihypergroup S is called idempotent if*

$$f * f = f.$$

Example 3.1. *Let us consider an LA-semihypergroup $S = \{a, b, c\}$ in the following cayley's table*

$\circ$	a	b	c
a	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a, c\}$
c	$\{a\}$	$\{a\}$	$\{a\}$

Let us define a fuzzy subset $f : S \longrightarrow [0,1]$ as follows

$$f(x) = \begin{cases} 0.9 \text{ if } x = a \\ 0.7 \text{ if } x = b \\ 0.5 \text{ if } x = c \end{cases}$$

Then it is easy to observe that f is a fuzzy LA-subsemihypergroup of S.

Example 3.2. *Let us consider an LA-semihypergroup $S = \{e_1, e_2, e_3\}$ in the following cayley's table*

$\circ$	e_1	e_2	e_3
e_1	$\{e_1\}$	$\{e_1\}$	$\{e_1\}$
e_2	$\{e_1\}$	$\{e_1\}$	$\{e_1, e_3\}$
e_3	$\{e_1\}$	$\{e_1\}$	$\{e_1\}$

Let us define a fuzzy subset $f : S \longrightarrow [0,1]$ as follows

$$f(x) = \begin{cases} 0.8 \text{ if } x = e_1 \\ 0.4 \text{ if } x = e_2 \\ 0.6 \text{ if } x = e_3 \end{cases}$$

Then it is easy to see that f is a fuzzy hyperideal of LA-semihypergroup S.

Example 3.3. *Let us consider an LA-semihypergroup $S = \{e_1, e_2, e_3\}$ in the following cayley's table*

$\circ$	e_1	e_2	e_3
e_1	$\{e_1, e_3\}$	$\{e_2\}$	$\{e_2, e_3\}$
e_2	$\{e_2, e_3\}$	$\{e_2, e_3\}$	$\{e_2, e_3\}$
e_3	$\{e_2, e_3\}$	$\{e_2, e_3\}$	$\{e_2, e_3\}$

Let us define a fuzzy subset $f : S \longrightarrow [0,1]$ as follows

$$f(x) = \begin{cases} 0.5 \text{ if } x = e_1 \\ 0.7 \text{ if } x = e_2 \\ 0.7 \text{ if } x = e_3 \end{cases}$$

Then it is easy to see that f is a fuzzy hyperideal of LA-semihypergroup S.

Proposition 3.1. *The set $(F(S), *)$ is an LA-semihypergroup.*

Proof. Clearly $F(S)$ is closed. Let f, g and h be in $F(S)$. If $A_x = \emptyset$ for any $x \in S$. Then $((f*g)*h)(x) = 0 = ((h*g)*f)(x)$. Let $A_x \neq \emptyset$, then there exist y and z in S such that $(y,z) \in A_x$. Therefore by using left invertive law, we have

$$\begin{aligned} ((f*g)*h)(x) &= \bigvee_{(y,z)\in A_x} \{(f*g)(y) \wedge h(z)\} \\ &= \bigvee_{(y,z)\in A_x} \left\{ \bigvee_{(p,q)\in A_y} \{f(p) \wedge g(q)\} \wedge h(z) \right\} \\ &= \bigvee_{x\in((p\circ q)\circ z)} \{f(p) \wedge g(q) \wedge h(z)\} \\ &= \bigvee_{x\in((z\circ q)\circ p)} \{h(z) \wedge g(q) \wedge f(p)\} \\ &= \bigvee_{(w,p)\in A_x} \left\{ \bigvee_{(z,q)\in A_w} (h(z) \wedge g(q)) \wedge f(p) \right\} \\ &= \bigvee_{(w,p)\in A_x} \{(h*g)(w) \wedge f(p)\} \\ &= ((h*g)*f)(x). \end{aligned}$$

Hence $(F(S), *)$ is an LA-semihypergroup. □

Lemma 3.1. *Let S be an LA-semihypergroup. Then the medial law holds in $F(S)$.*

Proof. Let f, g, h and k be the arbitrary elements of $F(S)$. By successive use of left invertive law, $(f*g)*(h*k) = ((h*k)*g)*f = ((g*k)*h)*f = (f*h)*(g*k)$. □

Proposition 3.2. *An LA-semihypergroup with $F(S) = (F(S))^2$ is a commutative semihypergroup if and only if*

$$(f*g)*h = f*(h*g)$$

holds for all fuzzy subsets $f, g, h \in F(S)$.

Proof. Let S be a commutative semihypergroup. For any fuzzy subsets $f, g, h \in F(S)$. If $A_x = \emptyset$ then $((f*g)*h)(x) = 0 = (f*(h*g))(x)$. Let $A_x \neq \emptyset$ then $(s,t) \in A_x$, therefore by the use of left invertive law and commutative law, we get

$$
\begin{aligned}
((f*g)*h)(x) &= \bigvee_{(s,t)\in A_x} \{(f*g)(s)\wedge h(t)\} \\
&= \bigvee_{(s,t)\in A_x} \left\{ \bigvee_{(m,n)\in A_s} (f(m)\wedge g(n))\wedge h(t) \right\} \\
&= \bigvee_{x\in((m\circ n)\circ t)} \{f(m)\wedge h(t)\wedge g(n)\} \\
&= \bigvee_{x\in((t\circ n)\circ m)} \{f(m)\wedge h(t)\wedge g(n)\} \\
&= \bigvee_{x\in(m\circ(t\circ n))} \{f(m)\wedge h(t)\wedge g(n)\} \\
&= \bigvee_{(m,p)\in A_x} \left\{ f(m)\wedge \bigvee_{(t,n)\in A_p} (h(t)\wedge g(n)) \right\} \\
&= \bigvee_{(m,p)\in A_x} \{f(m)\wedge (h*g)(p)\} \\
&= (f*(h*g))(x).
\end{aligned}
$$

Conversely, let $(f*g)*h = f*(h*g)$ holds for all fuzzy subsets $f,g,h\in F(S)$. We have to show that $F(S)$ is a commutative semihypergroup. Let f and g be any fuzzy subsets of S. If $A_x=\emptyset$ for any $x\in S$, then $(f*g)(x)=0=(g*f)(x)$. Let $A_x\neq\emptyset$. Then $(s,t)\in A_x$. Since $F(S)=(F(S))^2$. So $f=(h*k)$ where h and k are any fuzzy subsets of S. Now by using left invertive law, we have

$$
\begin{aligned}
(f*g)(x) = ((h*k)*g)(x) &= \bigvee_{(s,t)\in A_x} \{(h*k)(s)\wedge g(t)\} \\
&= \bigvee_{(s,t)\in A_x} \left\{ \bigvee_{(m,n)\in A_s} (h(m)\wedge k(n))\wedge g(t) \right\} \\
&= \bigvee_{x\in((m\circ n)\circ t)} \{h(m)\wedge k(n)\wedge g(t)\} \\
&= \bigvee_{x\in((t\circ n)\circ m)} \{g(t)\wedge k(n)\wedge h(m)\} \\
&= \bigvee_{(p,m)\in A_x} \left\{ \bigvee_{(t,n)\in A_p} (g(t)\wedge k(n))\wedge h(m) \right\} \\
&= \bigvee_{(p,m)\in A_x} \{(g*k)(p)\wedge h(m)\} \\
&= ((g*k)*h)(x) \\
&= (g*(h*k))(x).
\end{aligned}
$$

This shows that $f*g = g*(h*k) = g*f$. Thus commutative law holds in $F(S)$.

Now if $A_x = \emptyset$. Then $((f * g) * h)(x) = 0 = (f * (h * g))(x)$. Let $A_x \neq \emptyset$. Then $(s,t) \in A_x$. Therefore by the use of commutative law and left invertive law we get

$$\begin{aligned}
((f * g) * h)(x) &= \bigvee_{(s,t)\in A_x} \{(f * g)(s) \wedge h(t)\} \\
&= \bigvee_{(s,t)\in A_x} \left\{ \bigvee_{(m,n)\in A_s} (f(m) \wedge g(n)) \wedge h(t) \right\} \\
&= \bigvee_{x\in((m\circ n)\circ t)} \{f(m) \wedge g(n) \wedge h(t)\} \\
&= \bigvee_{x\in((t\circ n)\circ m)} \{f(m) \wedge g(n) \wedge h(t)\} \\
&= \bigvee_{x\in(m\circ(t\circ n))} \{f(m) \wedge g(n) \wedge h(t)\} \\
&= \bigvee_{x\in(m\circ(n\circ t))} \{f(m) \wedge g(n) \wedge h(t)\} \\
&= \bigvee_{(m,p)\in A_x} \left\{ f(m) \wedge \bigvee_{(n,t)\in A_p} (g(n) \wedge h(t)) \right\} \\
&= \bigvee_{(m,p)\in A_x} \{f(m) \wedge (g * h)(p)\} \\
&= (f * (g * h))(x).
\end{aligned}$$

□

Theorem 3.1. *If S has a pure left identity then the following properties holds in $F(S)$.*

(1) $f * (g * h) = g * (f * h)$ for all f, g and $h \in F(S)$.
(2) $(f * g) * (h * k) = (k * h) * (g * f)$ for all f, g, h and $k \in F(S)$.

Proof. (1). Let $x \in S$. If $A_x = \emptyset$. Then $(f * (g * h))(x) = 0 = (g * (f * h))(x)$. Let $A_x \neq \emptyset$. Then $(y,z) \in A_x$. Now by using medial law with pure left identity, we have

$$\begin{aligned}
(f * (g * h))(x) &= \bigvee_{(y,z)\in A_x} \{f(y) \wedge (g * h)(z)\} \\
&= \bigvee_{(y,z)\in A_x} \left\{ f(y) \wedge \bigvee_{(p,q)\in A_z} (g(p) \wedge h(q)) \right\} \\
&= \bigvee_{x\in(y\circ(p\circ q))} \{f(y) \wedge g(p) \wedge h(q)\} \\
&= \bigvee_{x\in(p\circ(y\circ q))} \{g(p) \wedge f(y) \wedge h(q)\} \\
&= \bigvee_{(p,w)\in A_x} \left\{ g(p) \wedge \bigvee_{(y,q)\in A_w} (f(y) \wedge h(q)) \right\} \\
&= \bigvee_{(p,w)\in A_x} \{g(p) \wedge (f * h)(w)\} \\
&= (g * (f * h))(x).
\end{aligned}$$

Thus $(f * (g * h))(x) = (g * (f * h))(x)$ for all $x \in S$.

(2). If $A_x = \emptyset$ for $x \in S$, then $((f * g) * (h * k))(x) = 0 = ((k * h) * (g * f))(x)$. Let $A_x \neq \emptyset$ then there exist y and z in S such that $(y, z) \in A_x$. Therefore by using paramedial law, we have

$$\begin{aligned}
((f * g) * (h * k))(x) &= \bigvee_{(y,z)\in A_x} \{(f * g)(y) \wedge (h * k)(z)\} \\
&= \bigvee_{(y,z)\in A_x} \left\{ \bigvee_{(p,q)\in A_y} \{f(p) \wedge g(q)\} \wedge \bigvee_{(u,v)\in A_z} \{(h(u) \wedge k(v))\} \right\} \\
&= \bigvee_{x\in((p\circ q)\circ(u\circ v))} \{f(p) \wedge g(q) \wedge h(u) \wedge k(v)\} \\
&= \bigvee_{x\in((v\circ u)\circ(q\circ p))} \{k(v) \wedge h(u) \wedge g(q) \wedge f(p)\} \\
&= \bigvee_{(m,n)\in A_x} \left[\bigvee_{(v,u)\in A_m} \{k(v) \wedge h(u)\} \bigvee_{(q,p)\in A_n} \{g(q) \wedge f(p)\} \right] \\
&= \bigvee_{(m,n)\in A_x} \{(k * h)(m) \wedge (g * f)(n)\} \\
&= ((k * h) * (g * f))(x).
\end{aligned}$$

Thus $(f * g) * (h * k) = (k * h) * (g * f)$ for all $x \in S$. □

Theorem 3.2. *Let S be an LA-semihypergroup. Then $L = \{f \mid f \in F(S),\ f * h = f$ where $h = h * h\}$ is a commutative monoid in S.*

Proof. The fuzzy subset L of S is nonempty since $h * h = h$, which implies that h is in L. Let f and g be the fuzzy subsets of S in L, then $f * h = f$ and $g * h = g$. If $A_x = \emptyset$ for $x \in S$, then $(f * g)(x) = 0 = ((f * g) * h)(x)$. Let $A_x \neq \emptyset$. Then by using medial law, we have

$$\begin{aligned}
(f * g)(x) &= \bigvee_{(y,z)\in A_x} \{(f * h)(y) \wedge (g * h)(z)\} \\
&= \bigvee_{(y,z)\in A_x} \left[\bigvee_{(p,q)\in A_y} \{f(p) \wedge h(q)\} \wedge \bigvee_{(u,v)\in A_z} \{g(u) \wedge h(v)\} \right] \\
&= \bigvee_{x\in((p\circ q)\circ(u\circ v))} \{f(p) \wedge h(q) \wedge g(u) \wedge h(v)\} \\
&= \bigvee_{x\in((p\circ u)\circ(q\circ v))} \{f(p) \wedge g(u) \wedge h(q) \wedge h(v)\} \\
&= \bigvee_{(m,n)\in A_x} \left[\bigvee_{(p,u)\in A_m} \{f(p) \wedge g(u)\} \bigvee_{(q,v)\in A_n} \{h(q) \wedge h(v)\} \right] \\
&= \bigvee_{(m,n)\in A_x} \{(f * g)(m) \wedge (h * h)(n)\} \\
&= ((f * g) * (h * h))(x).
\end{aligned}$$

Thus $f * g = (f * g) * (h * h) = (f * g) * h$ which implies that L is closed.

Now if $A_x = \emptyset$. Then $(f * g)(x) = 0 = (g * f)(x)$. Let $A_x \neq \emptyset$ then $(y, z) \in A_x$. Therefore by using left invertive law, we have

$$\begin{aligned}
(f * g)(x) &= \bigvee_{(y,z)\in A_x} \{(f * h)(y) \wedge g(z)\} \\
&= \bigvee_{(y,z)\in A_x} \left\{ \bigvee_{(p,q)\in A_y} (f(p) \wedge h(q)) \wedge g(z) \right\} \\
&= \bigvee_{x\in((p\circ q)\circ z)} \{f(p) \wedge h(q) \wedge g(z)\} \\
&= \bigvee_{x\in((z\circ q)\circ p)} \{g(z) \wedge h(q) \wedge f(p)\} \\
&= \bigvee_{(t,p)\in A_x} \left\{ \bigvee_{(z,q)\in A_t} (g(z) \wedge h(q)) \wedge f(p) \right\} \\
&= \bigvee_{(t,p)\in A_x} \{(g * h)(t) \wedge f(p)\} \\
&= ((g * h) * f)(x).
\end{aligned}$$

Thus $f * g = (g * h) * f = g * f$, which implies that commutative law holds in L and associative law holds in L due to commutativity. Since for any fuzzy subset f in L, we have $f * h = f$ (where h is fixed) implies that h is pure right identity in $F(S)$ and hence an identity. □

Lemma 3.2. *Let S be an LA-semihypergroup. If S has a pure left identity then*

$$\mathcal{S} * \mathcal{S} = \mathcal{S}.$$

Proof. Every x in S can be written as $x = e \circ x$, where e is the pure left identity in S. Therefore

$$\begin{aligned}
(\mathcal{S} * \mathcal{S})(x) &= \bigvee_{(y,z)\in A_x} \{\mathcal{S}(y) \wedge \mathcal{S}(z)\} \\
&\geq \{\mathcal{S}(e) \wedge \mathcal{S}(x)\} \\
&= 1 = \mathcal{S}(x).
\end{aligned}$$

Hence $\mathcal{S} * \mathcal{S} = \mathcal{S}$. □

Theorem 3.3. *Let χ_A and χ_B be fuzzy subsets of an LA-semihypergroup S, where A and B are nonempty subsets of S. Then the following properties hold:*

(1) If $A \subseteq B$ then $\chi_A \subseteq \chi_B$.
(2) $\chi_A \cap \chi_B = \chi_{A\cap B}$.
(3) $\chi_A * \chi_B = \chi_{A\circ B}$.

Proof. (1). It is obvious.

(2). Let $x \in S$. If $x \in A \cap B$, then $x \in A$ and $x \in B$. So $\chi_A(x) = 1$ and $\chi_B(x) = 1$. Thus we have $(\chi_A \cap \chi_B)(x) = \chi_A(x) \wedge \chi_B(x) = 1 = \chi_{A\cap B}$. If $x \notin A \cap B$, then $x \notin A$ and $x \notin B$. So $\chi_A(x) = 0$ and $\chi_B(x) = 0$. Thus we have $(\chi_A \cap \chi_B)(x) = \chi_A(x) \wedge \chi_B(x) = 0 = \chi_{A\cap B}$. Thus

$$\chi_A \cap \chi_B = \chi_{A\cap B}.$$

(3). For any $x \in S$. If $x \notin A \circ B$, then

$$\chi_{A\circ B}(x) = 0 \qquad \text{(i)}$$

This means that there does not exist $y \in A$ and $z \in B$ such that $x \in y \circ z$.

If $A_x = \emptyset$ then

$$(\chi_A * \chi_B)(x) = 0 \qquad \text{(ii)}$$

If $A_x \neq \emptyset$ and $(y, z) \in A_x$ then $x \in y \circ z$. Then $y \notin A$ or $z \notin B$. Thus either $\chi_A(y) = 0$ or $\chi_B(z) = 0$. So we have, $\chi_A(y) \wedge \chi_B(z) = 0$. Hence $(\chi_A * \lambda_B)(x) = 0$.

Let $x \in A \circ B$, then $\chi_{A\circ B}(x) = 1$. Thus $x \in a \circ b$, for some $a \in A$ and $b \in B$, so $(a, b) \in A_x$. Since $A_x \neq \emptyset$, we have

$$\begin{aligned}(\chi_A * \chi_B)(x) &= \bigvee_{(y,z)\in A_x} \{\chi_A(y) \wedge \chi_B(z)\} \\ &\geq \chi_A(a) \wedge \chi_B(b) = 1.\end{aligned}$$

Thus $(\chi_A * \chi_B)(x) = 1$. Hence $\chi_A * \chi_B = \chi_{A\circ B}$. □

Theorem 3.4. *A fuzzy subset f of an LA-semihypergroup S is a fuzzy LA-subsemihypergroup of S if and only if*

$$f * f \subseteq f.$$

Proof. Assume that f is a fuzzy LA-subsemihypergroup of S. If $A_a = \emptyset$. Then $(f * f)(a) = 0 = f(a)$. If $A_a \neq \emptyset$, then there exist x and y in S such that $(x, y) \in A_a$. Then for any $\alpha \in x \circ y$, we have $a \in \alpha$. Since f is a fuzzy LA-subsemihypergroup of S, we have

$$\begin{aligned}(f * f)(a) &= \bigvee_{(x,y)\in A_a} \{f(x) \wedge f(y)\} \\ &\leq \bigvee_{(x,y)\in A_a} f(\alpha) \\ &\leq \bigvee_{(x,y)\in A_a} f(a) \\ &= f(a).\end{aligned}$$

Thus $f * f \subseteq f$.

Conversely, assume that $f * f \subseteq f$. Let $x, y \in S$ and $\alpha \in x \circ y$. We have,

$$\begin{aligned}f(\alpha) &\geq (f * f)(\alpha) \\ &= \bigvee_{(x,y)\in A_\alpha} \{f(x) \wedge f(y)\} \\ &\geq \{f(x) \wedge f(y)\} \\ f(\alpha) &\geq \{f(x) \wedge f(y)\}.\end{aligned}$$

Thus $\bigwedge_{\alpha\in x\circ y} f(\alpha) \geq \{f(x) \wedge f(y)\}$. Thus f is a fuzzy LA-subsemihypergroup of S. □

Theorem 3.5. *A nonempty subset A of an LA-semihypergroup S is an LA-subsemihypergroup if and only if the characteristic fuzzy set χ_A is a fuzzy LA-subsemihypergroup.*

Proof. Let A be a nonempty subset of an LA-semihypergroup S, x and y be arbitrary elements of S. Let A be an LA-subsemihypergroup of S. Let $x, y \in A$, then $x \circ y \subseteq A$. For any $\alpha \in x \circ y$, we have, $\chi_A(x) = 1$ and $\chi_A(y) = 1$. Hence $\bigwedge_{\alpha\in x\circ y} \chi_A(\alpha) = 1 = \chi_A(x) \wedge \chi_A(y)$. Now let $x \in A$ and $y \notin A$, then $\chi_A(x) = 1$ and $\chi_A(y) = 0$, so we have $\bigwedge_{\alpha\in x\circ y} \chi_A(\alpha) \geq 0 = \chi_A(x) \wedge \chi_A(y)$. Now let both x and y are not in A, then $\chi_A(x) = 0$ and $\chi_A(y) = 0$, so we have $\bigwedge_{\alpha\in x\circ y} \chi_A(\alpha) \geq 0 = \chi_A(x) \wedge \chi_A(y)$. Thus for all $x, y \in S$, we have $\bigwedge_{\alpha\in x\circ y} \chi_A(\alpha) \geq \chi_A(x) \wedge \chi_A(y)$. Thus χ_A is a fuzzy LA-subsemihypergroup of S.

Conversely, Let χ_A be a fuzzy LA-subsemihypergroup of S. If the elements x and y are in A, then $\chi_A(x) = 1 = \chi_A(y)$. But $\bigwedge_{\alpha\in x\circ y} \chi_A(\alpha) \geq \chi_A(x) \wedge \chi_A(y) = 1$, which implies that $\chi_A(\alpha) \geq 1$ for any $\alpha \in x \circ y$. Hence for any $\alpha \in x \circ y$, $\chi_A(\alpha) = 1$, i.e., $\alpha \in A$. It thus follows that $x \circ y \subseteq A$. Hence A is an LA-subsemihypergroup of S. □

Theorem 3.6. *Let S be an LA-semihypergroup and for a nonempty subset A of S the following statements are equivalent:*

(1) A is left (resp. right) hyperideal of S.
(2) The characteristic fuzzy set χ_A is a fuzzy left (resp. right) hyperideal of S.

Proof. $(1) \Longrightarrow (2)$. Assume that A is a left hyperideal of S. Let $x, y \in S$ be such that both x and y are in A. Then since A is left hyperideal of S, $x \circ y \subseteq A$. For any $\alpha \in x \circ y$, we have, $\chi_A(x) = 1$ and $\chi_A(y) = 0$. Hence $\bigwedge\limits_{\alpha \in x \circ y} \chi_A(\alpha) = 1 = \chi_A(y)$. Now let $x \in A$ and $y \notin A$, then $\chi_A(x) = 1$ and $\chi_A(y) = 0$, so we have $\bigwedge\limits_{\alpha \in x \circ y} \chi_A(\alpha) \geq 0 = \chi_A(y)$. Now let both x and y are not in A, then $\chi_A(x) = 0$ and $\chi_A(y) = 0$, so we have $\bigwedge\limits_{\alpha \in x \circ y} \chi_A(\alpha) \geq 0 = \chi_A(y)$. Thus for all $x, y \in S$, we have $\bigwedge\limits_{\alpha \in x \circ y} \chi_A(\alpha) \geq \chi_A(y)$. Thus χ_A is a fuzzy left hyperideal of S.

$(2) \Longrightarrow (1)$. Let χ_A be a fuzzy left hyperideal of S. If the elements x and y are in A, then $\chi_A(x) = 1 = \chi_A(y)$. But $1 = \chi_A(y) \leq \bigwedge\limits_{\alpha \in x \circ y} \chi_A(\alpha)$, which implies that $\chi_A(\alpha) \geq 1$ for any $\alpha \in x \circ y$. Hence for any $\alpha \in x \circ y$, $\chi_A(\alpha) = 1$, i.e., $\alpha \in A$. It thus follows that $S \circ A \subseteq A$. Therefore A is left hyperideal of S. Similarly we can prove that χ_A is a fuzzy right hyperideal of S when A is right hyperideal of S. $\square$

Theorem 3.7. *A fuzzy subset f of an LA-semihypergroup S is a fuzzy left (resp. right) hyperideal of S if and only if for each $t \in (0, 1]$, $U(f; t) \neq \phi$ is a left (resp. right) hyperideal of S.*

Proof. Suppose f be a fuzzy left hyperideal of S and $x \in U(f; t)$ and $y \in S$. Then $f(x) \geq t$. Since f is a fuzzy left hyperideal of S, so $f(x) \leq \bigwedge\limits_{\alpha \in y \circ x} f(\alpha)$. Hence $f(\alpha) \geq t$ for all $\alpha \in y \circ x$, this implies $\alpha \in U(f; t)$ that is $y \circ x \subseteq U(f; t)$. Hence $U(f; t)$ is a fuzzy left hyperideal of S.

Conversely, assume that $U(f; t) \neq \emptyset$ is a left hyperideal of S. Let $x \in S$ such that $f(x) > \bigwedge\limits_{\alpha \in y \circ x} f(\alpha)$ for all $y \in S$. Select $t \in (0, 1]$ such that $f(x) = t > \bigwedge\limits_{\alpha \in y \circ x} f(\alpha)$. Then $x \in U(f; t)$ but $y \circ x \nsubseteq U(f; t)$, a contradiction. Hence $f(x) \leq \bigwedge\limits_{\alpha \in y \circ x} f(\alpha)$, that is f is a fuzzy left hyperideal of S. $\square$

Proposition 3.3. *Let S be an LA-semihypergroup then the following properties hold.*

(1) Let f and g be two fuzzy LA-subsemihypergroups of S. Then $f \cap g$ is also fuzzy LA-subsemihypergroup of S.

(2) The intersection of any family of fuzzy left (resp. right, two sided) hyperideals of S is a fuzzy left (resp. right, two sided) hyperideal of S.

Proof. (1). Let f and g be two fuzzy LA-subsemihypergroups of S. Let x,$y \in S$. Then for any $\alpha \in x \circ y$, we have $\bigwedge\limits_{\alpha \in x \circ y} f(\alpha) \geq f(x) \wedge f(y)$ and $\bigwedge\limits_{\alpha \in x \circ y} g(\alpha) \geq g(x) \wedge g(y)$. Hence $f(\alpha) \geq f(x) \wedge f(y)$ and $g(\alpha) \geq g(x) \wedge g(y)$. Thus

$$\begin{aligned}(f \cap g)(\alpha) = f(\alpha) \wedge g(\alpha) &\geq f(x) \wedge f(y) \wedge g(x) \wedge g(y) \\ &= f(x) \wedge g(x) \wedge f(y) \wedge g(y) \\ &= (f \cap g)(x) \wedge (f \cap g)(y).\end{aligned}$$

Hence $\bigwedge\limits_{\alpha \in x \circ y} (f \cap g)(\alpha) \geq (f \cap g)(x) \wedge (f \cap g)(y)$. Therefore $f \cap g$ is a fuzzy LA-subsemihypergroup of S.

(2). Let $g = \bigcap\limits_{i \in I} g_i$ be a family of fuzzy left hyperideals of S. Let $x, y \in S$. Then, since each g_i $(i \in I)$ is a fuzzy left hyperideals of S, so $\bigwedge\limits_{\alpha \in x \circ y} g_i(\alpha) \geq g_i(y)$. Thus for any $\alpha \in x \circ y$, $g_i(\alpha) \geq g_i(y)$, and we have

$$\begin{aligned}
g(\alpha) &= \left(\bigcap_{i\in I} g_i\right)(\alpha) = \bigwedge_{i\in I}(g_i(\alpha)) \\
&\geq \bigwedge_{i\in I} g_i(y) \\
&= \left(\bigcap_{i\in I} g_i\right)(y) \\
&= g(y).
\end{aligned}$$

Thus $\bigwedge_{i\in I} g(\alpha) \geq g(y)$. Therefore $g = \bigcap_{i\in I} g_i$ is a fuzzy left hyperideal of S. □

Proposition 3.4. *Let S is an LA-semihypergroup. If f is fuzzy left (resp. right or two-sided) hyperideal of S. Then f is a fuzzy LA-subsemihypergroup.*

Proof. Let f be a fuzzy left hyperideal of S. Let $x, y \in S$. Then $\bigwedge_{\alpha\in x\circ y} f(\alpha) \geq f(y) \geq f(x) \wedge f(y)$. Thus $\bigwedge_{\alpha\in x\circ y} f(\alpha) \geq f(x) \wedge f(y)$. Therefore f is a fuzzy LA-subsemihypergroup of S. □

Proposition 3.5. *A fuzzy subset f of an LA-semihypergroup S is a fuzzy left (resp. right) hyperideal of S if and only if $\mathcal{S} * f \subseteq f$ (resp. $f * \mathcal{S} \subseteq f$).*

Proof. Let f be a fuzzy left hyperideal of S and $x \in S$. Then

$$\begin{aligned}
(\mathcal{S} * f)(x) &= \bigvee_{x\in y\circ z} \{\mathcal{S}(y) \wedge f(z)\} \\
&= \bigvee_{x\in y\circ z} \{f(z)\} \quad (\because \mathcal{S}(y) = 1) \\
&\leq \bigvee_{x\in y\circ z} f(x), \text{ because } f(z) \leq \bigwedge_{\alpha\in y\circ z} \{f(\alpha)\} \leq f(\alpha) \text{ for each } \alpha \in y \circ z. \\
&= f(x).
\end{aligned}$$

Hence, $(\mathcal{S} * f)(x) \leq f(x)$. Thus $\mathcal{S} * f \subseteq f$.

Conversely, suppose that $\mathcal{S} * f \subseteq f$. We show that f is a fuzzy left hyperideal of S. Let $x \in S$. Then

$$\begin{aligned}
f(x) &\geq (\mathcal{S} * f))(x) \\
&= \bigvee_{x\in y\circ z} \{\mathcal{S}(y) \wedge f(z)\} \\
&= \bigvee_{x\in y\circ z} \{f(z)\}, \text{ (because } \mathcal{S}(y) = 1) \\
&\geq f(z), \text{ for each } z \text{ such that } x \in y \circ z.
\end{aligned}$$

Thus $\bigwedge_{x\in y\circ z} f(x) \geq f(z)$. Hence f is a fuzzy left hyperideal of S.

Similarly we can prove the case of fuzzy right hyperideal of S. □

Theorem 3.8. *If S is an LA-semihypergroup with pure left identity. Then every fuzzy right hyperideal is a fuzzy left hyperideal of S.*

Proof. Let S be an LA-semihypergroup with pure left identity e, and f be a fuzzy right hyperideal of S. Since f is a fuzzy right hyperideal of S, so $f * \mathcal{S} \subseteq f$. Thus by Lemma 3.2, and left invertive law, we have

$$\begin{aligned}\mathcal{S}*f &= (\mathcal{S}*\mathcal{S})*f \\ &= (f*\mathcal{S})*\mathcal{S} \\ &\subseteq f*\mathcal{S} \\ &\subseteq f.\end{aligned}$$

Thus, $\mathcal{S}*f \subseteq f$. Thus, f is a fuzzy left hyperideal of S. □

Proposition 3.6. *The product of two fuzzy left (resp. right) hyperideals of an LA-semihypergroup S with pure left identity is a fuzzy left (resp. right) hyperideal of S.*

Proof. Let f and g be any two fuzzy left hyperideals of S. Then by using (1), we have,

$$\mathcal{S}*(f*g) = f*(\mathcal{S}*g) \subseteq f*g.$$

Let f and g be any two fuzzy right hyperideals of S. Then by using medial law and 3.2, we have

$$(f*g)*\mathcal{S} = (f*g)*(\mathcal{S}*\mathcal{S}) = (f*\mathcal{S})*(g*\mathcal{S}) \subseteq f*g.$$

Therefore $f*g$ is a fuzzy hyperideal of S. □

Proposition 3.7. *In LA-semihypergroup with pure left identity for every fuzzy left hyperideal f of S, we have $\mathcal{S}*f = f$.*

Proof. It suffices to show that $f \subseteq \mathcal{S}*f$. Since every element $x \in S$ can be written as $x = e \circ x$, where e is the pure left identity in S,

$$\begin{aligned}(\mathcal{S}*f)(x) &= \bigvee_{(y,z)\in A_x} \{\mathcal{S}(y) \wedge f(z)\} \\ &\geq \{\mathcal{S}(e) \wedge f(x)\} \\ &= f(x).\end{aligned}$$

Hence $\mathcal{S}*f = f$. □

Proposition 3.8. *In an LA-semihypergroup S with pure left identity for every fuzzy right hyperideal h of S, we have $h*\mathcal{S} = h$.*

Proof. It suffices to show that $h \subseteq h*\mathcal{S}$. Since every element $a \in S$ can be written as $a = e \circ a = (e \circ e) \circ a = (a \circ e) \circ e$, then there exist $u \in a \circ e$ such that $a \in u \circ e$. Then $(u, e) \in A_a$, where e is the pure left identity in S,

$$\begin{aligned}(h*\mathcal{S})(a) &= \bigvee_{(x,y)\in A_a} \{h(x) \wedge \mathcal{S}(y)\} \\ &\geq \{h(u) \wedge \mathcal{S}(e)\}.\end{aligned}$$

Since h is a fuzzy right hyperideal of S. Then $\bigwedge_{u\in a\circ e} h(u) \geq h(a)$. Hence $h(u) \geq h(a)$. Thus

$$\begin{aligned}(h*\mathcal{S})(a) &\geq \{h(u) \wedge \mathcal{S}(e)\} \\ &\geq h(a) \wedge 1 \\ &= h(a).\end{aligned}$$

Hence $h*\mathcal{S} = h$. □

Proposition 3.9. *Let S be an LA-semihypergroup with pure left identity, f be a fuzzy subset and k be a fuzzy left hyperideal of S. Then for any fuzzy subset h and fuzzy left hyperideal g of S, $f*g = h*k$ implies that $g*f = k*h$.*

Proof. Since g and k are fuzzy left hyperideals of S, by Proposition 3.7, $\mathcal{S}*g = g$ and $\mathcal{S}*k = k$. Then,

$$g*f = (\mathcal{S}*g)*f = (f*g)*\mathcal{S} = (h*k)*\mathcal{S} = (\mathcal{S}*k)*h = k*h.$$

□

Proposition 3.10. *Every idempotent fuzzy left hyperideal of an LA-semihypergroup S is a fuzzy hyperideal of S.*

Proof. Let f be a fuzzy left hyperideal of S which is idempotent. Then

$$f * \mathcal{S} = (f * f) * \mathcal{S} = (\mathcal{S} * f) * f \subseteq f * f = f.$$

Hence f is a fuzzy right hyperideal of S and so f is a fuzzy hyperideal of S. □

Proposition 3.11. *If f is an idempotent element in an LA-semihypergroup S with pure left identity. Then $\mathcal{S} * f$ is an idempotent element.*

Proof. Let f be an idempotent element in an LA-semihypergroup S with pure left identity. Then by using medial law,

$$(\mathcal{S} * f) * (\mathcal{S} * f) = (\mathcal{S} * \mathcal{S}) * (f * f) = \mathcal{S} * f.$$

□

Proposition 3.12. *If f is an idempotent element in an LA-semihypergroup S with pure left identity. Then every fuzzy left hyperideal g of S commutes with f.*

Proof. Let f be an idempotent element in an LA-semihypergroup S with pure left identity. Then

$$f * g = (f * f) * g = (g * f) * f \subseteq (g * \mathcal{S}) * f \subseteq g * f.$$

Also,

$$g * f = g * (f * f) = f * (g * f) \subseteq f * (g * \mathcal{S}) \subseteq f * g.$$

□

Proposition 3.13. *If f is a fuzzy left hyperideal of an LA-semihypergroup S with pure left identity, then*

$$f \cup (f * \mathcal{S})$$

is a fuzzy hyperideal of S.

Proof. Assume that f is a fuzzy left hyperideal of S. Then

$$\begin{aligned}(f \cup (f * \mathcal{S})) * \mathcal{S} &= (f * \mathcal{S}) \cup ((f * \mathcal{S}) * \mathcal{S}) \\ &= (f * \mathcal{S}) \cup ((\mathcal{S} * \mathcal{S}) * f) \\ &= (f * \mathcal{S}) \cup (\mathcal{S} * f) \\ &= (f * \mathcal{S}) \cup f = f \cup (f * \mathcal{S}).\end{aligned}$$

Hence $f \cup (f * \mathcal{S})$ is a fuzzy right hyperideal of S. and by Theorem 3.8, it is a fuzzy hyperideal of S. □

Proposition 3.14. *If f is a fuzzy right hyperideal of an LA-semihypergroup S with pure left identity, then*

$$f \cup (\mathcal{S} * f).$$

is a fuzzy hyperideal of S.

Proof. Assume that f is a fuzzy right hyperideal of S. Then

$$\begin{aligned}(f \cup (\mathcal{S} * f)) * \mathcal{S} &= ((f * \mathcal{S}) \cup (\mathcal{S} * f) * \mathcal{S}) \\ &\subseteq f \cup (\mathcal{S} * f) * (\mathcal{S} * \mathcal{S}) \\ &= f \cup (\mathcal{S} * \mathcal{S}) * (f * \mathcal{S}) \\ &= f \cup (\mathcal{S} * (f * \mathcal{S})) \\ &= f \cup (f * (\mathcal{S} * \mathcal{S})) \\ &= f \cup (f * \mathcal{S}) \\ &= f \subseteq f \cup (\mathcal{S} * f).\end{aligned}$$

Also,

$$\begin{aligned}\mathcal{S}*(f\cup(\mathcal{S}*f)) &= (\mathcal{S}*f)\cup(\mathcal{S}*(\mathcal{S}*f))\\ &= (\mathcal{S}*f)\cup((\mathcal{S}*\mathcal{S})*(\mathcal{S}*f))\\ &= (\mathcal{S}*f)\cup((f*\mathcal{S})*(\mathcal{S}*\mathcal{S}))\\ &\subseteq (\mathcal{S}*f)\cup(f*(\mathcal{S}*\mathcal{S}))\\ &= (\mathcal{S}*f)\cup(f*\mathcal{S})\\ &\subseteq (\mathcal{S}*f)\cup f\\ &= f\cup(\mathcal{S}*f)\end{aligned}$$

Hence $f\cup(\mathcal{S}*f)$ is a fuzzy hyperideal of S. □

4. Characterizations of regular and intra-regular LA-semihypergroups in terms of fuzzy hyperideals

In this section, we characterize regular as well as intra-regular LA-semihypergroups in terms of fuzzy hyperideals.

Theorem 4.1. *Let S be a regular LA-semihypergroup. Then for every fuzzy right hyperideal f and every fuzzy left hyperideal g of S, we have*

$$f*g=f\cap g.$$

Proof. Let S be a regular LA-semihypergroup and f is a fuzzy right hyperideal and g a fuzzy left hyperideal of S. Then $f*g\subseteq f*\mathcal{S}\subseteq f$ and $f*g\subseteq \mathcal{S}*g\subseteq g$. This implies that $f*g\subseteq f\cap g$. Now let a be any element of S, then, since S is a regular LA-semihypergroup, so there exist an element $x\in S$ such that $a\in(a\circ x)\circ a$. Then there exist $u\in a\circ x$ such that $a\in u\circ a$. Then $(u,a)\in A_a$. Thus we have

$$\begin{aligned}(f*g)(a) &= \bigvee_{(y,z)\in A_x}\{f(y)\wedge g(z)\}\\ &\geq \{f(u)\wedge g(a)\}.\end{aligned}$$

Since f is fuzzy right hyperideal of S, $\bigwedge\limits_{u\in a\circ x} f(u)\geq f(a)$. Hence $f(u)\geq f(a)$. Thus

$$\begin{aligned}(f*g)(a) &\geq \{f(u)\wedge g(a)\}\\ &\geq \{f(a)\wedge g(a)\}\\ &= (f\cap g)(a).\end{aligned}$$

Thus $f*g\supseteq f\cap g$. Therefore $f*g=f\cap g$. □

Corollary 4.1. *Let S be a regular LA-semihypergroup. Then for every fuzzy hyperideal f and every fuzzy hyperideal g of S, we have*

$$f*g=f\cap g.$$

Proposition 4.1. *Let S be a regular LA-semihypergroup. Then for every fuzzy right hyperideal f of S is idempotent.*

Proof. Let S be a regular LA-semihypergroup and f is a fuzzy right hyperideal of S. Then $f*f\subseteq f*\mathcal{S}\subseteq f$. Next since S is regular so for any $a\in S$, there exist an element $x\in S$ such that $a\in(a\circ x)\circ a$. Then there exist $\alpha\in a\circ x$ such that $a\in\alpha\circ a$. Then $(\alpha,a)\in A_a$. Thus we have

$$\begin{aligned}(f*f)(a) &= \bigvee_{(y,z)\in A_x}\{f(y)\wedge f(z)\}\\ &\geq \{f(\alpha)\wedge f(a)\}.\end{aligned}$$

Since f is fuzzy right hyperideal of S, $\bigwedge_{\alpha\in a\circ x} f(\alpha)\geq f(a)$. Hence $f(\alpha)\geq f(a)$. Thus

$$\begin{aligned}(f*f)(a) &\geq \{f(\alpha)\wedge f(a)\}\\ &\geq f(a)\wedge f(a)\\ &= f(a).\end{aligned}$$

Hence $f\subseteq f*f$. Therefore $f*f=f$. □

Corollary 4.2. *Let S be a regular LA-semihypergroup. Then for every fuzzy hyperideal f of S is idempotent.*

Proposition 4.2. *If S is a regular LA-semihypergroup. Then every fuzzy right hyperideal is a fuzzy left hyperideal of S.*

Proof. Let S be a regular LA-semihypergroup and f be a fuzzy right hyperideal of S. Let $x,y\in S$. Since S is regular and $x\in S$, so there exist an element $a\in S$ such that $x\in(x\circ a)\circ x$. Thus we have $\bigwedge_{\alpha\in x\circ y} f(\alpha)=\bigwedge_{\alpha\in(((x\circ a)\circ x)\circ y)} f(\alpha)=\bigwedge_{\alpha\in((y\circ x)\circ(x\circ a))} f(\alpha)=\bigwedge_{\substack{\alpha\in u\circ v\\ u\in y\circ x, v\in x\circ a}} f(\alpha)\geq f(u)\geq\bigwedge_{u\in y\circ x} f(u)\geq f(y)$. Hence $\bigwedge_{\alpha\in x\circ y} f(\alpha)\geq f(y)$. Therefore f is a fuzzy left hyperideal of S. □

Proposition 4.3. *A fuzzy set of an intra-regular LA-semihypergroup S is a fuzzy right hyperideal if and only if it is a fuzzy left hyperideal of S.*

Proof. Let f be a fuzzy right hyperideal of S. Let $a,b\in S$. Since $a\in S$ and S is intra-regular LA-semihypergroup, so there exist $x,y\in S$ such that $a\in\left(x\circ a^2\right)\circ y$. Thus for any $\alpha\in a\circ b$, we have, $\bigwedge_{\alpha\in a\circ b} f(\alpha)=\bigwedge_{\alpha\in(((x\circ a^2)\circ y)\circ b)} f(\alpha)=\bigwedge_{\alpha\in((b\circ y)\circ(x\circ a^2))} f(\alpha)=\bigwedge_{\substack{\alpha\in u\circ v\\ u\in b\circ y, v\in x\circ a^2}} f(\alpha)\geq f(u)\geq\bigwedge_{u\in b\circ y} f(u)\geq f(b)$. Thus f is a fuzzy left hyperideal of S.

Conversely, assume that f is a fuzzy left hyperideal of S. Now for any $\alpha\in a\circ b$, we have, $\bigwedge_{\alpha\in a\circ b} f(\alpha)=\bigwedge_{\alpha\in(((x\circ a^2)\circ y)\circ b)} f(\alpha)=\bigwedge_{\alpha\in((b\circ y)\circ(x\circ a^2))} f(\alpha)=\bigwedge_{\substack{\alpha\in u\circ v\\ u\in b\circ y, v\in x\circ a^2}} f(\alpha)\geq f(v)\geq\bigwedge_{v\in x\circ a^2} f(v)\geq f\left(a^2\right)\geq\bigwedge_{\beta\in a\circ a} f(\beta)\geq f(a)$. Thus $\bigwedge_{\alpha\in a\circ b} f(\alpha)\geq f(a)$. Hence f is a fuzzy right hyperideal of S. □

Proposition 4.4. *Every fuzzy two-sided hyperideal of an intra-regular LA-semihypergroup S with pure left identity is idempotent.*

Proof. Assume that f is a fuzzy two-sided hyperideal of S. Then clearly $f*f\subseteq f*\mathcal{S}\subseteq f$. Since S is intra-regular, so for each $a\in S$, there exist $x,y\in S$ such that $a\in\left(x\circ a^2\right)\circ y$. So by using (1) and left invertive law, we have

$$a\in\left(x\circ a^2\right)\circ y=(x\circ(a\circ a))\circ y=(a\circ(x\circ a))\circ y=(y\circ(x\circ a))\circ a.$$

Then there exist $u\in(y\circ(x\circ a))$ such that $a\in u\circ a$. Then $(u,a)\in A_a$. Thus we have,

$$\begin{aligned}(f*f)(a) &= \bigvee_{(p,q)\in A_a}\{f(p)\wedge f(q)\}\\ &\geq\{f(u)\wedge f(a)\}.\end{aligned}$$

Since f is a fuzzy two-sided hyperideal of S, so we have $\bigwedge\limits_{u\in(y\circ(x\circ a))} f(u) = \bigwedge\limits_{\substack{u\in y\circ v\\ v\in x\circ a}} f(u) \geq f(v) \geq \bigwedge\limits_{v\in x\circ a} f(v) \geq f(a)$. Hence $f(u) \geq f(a)$. Thus

$$\begin{aligned}(f * f)(a) &\geq \{f(u) \wedge f(a)\}\\ &\geq f(a) \wedge f(a)\\ &= f(a).\end{aligned}$$

Hence $f * f = f$. □

Proposition 4.5. *If S is an intra-regular LA-semihypergroup with pure left identity. Then*

$$f = (\mathcal{S}*f)^2 \text{ for all fuzzy left hyperideal } f \text{ of } S.$$

Proof. Let S be an intra-regular LA-semihypergroup with pure left identity and f be a fuzzy left hyperideal of S. Then $\mathcal{S}*f \subseteq f$. Since $\mathcal{S}*f$ is a fuzzy left hyperideal of S, so it is idempotent. Thus

$$(\mathcal{S}*f)^2 = (\mathcal{S}*f) \subseteq f.$$

Moreover,

$$f = f * f \subseteq \mathcal{S}*f = (\mathcal{S}*f)^2.$$

Thus $f = (\mathcal{S}*f)^2$. □

5. Conclusion

In this paper, we have introduced and studied the notions of fuzzy LA-subsemihypergroups and fuzzy left (resp. right) hyperideals of LA-semihypergroups and their interrelations. We characterized regular and intra-regular LA-semihypergroups in terms of these notions. Some important directions for future work are

(1) To develop strategies for obtaining more valuable results.

(2) To define other fuzzy hyperideals in LA-semihypergroups.

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