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POSITIVE SOLUTIONS FOR MULTI-ORDER NONLINEAR FRACTIONAL SYSTEMS

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Abstract. In this paper, we study the existence of positive solutions for a class of multi-order systems of fractional differential equations with nonlocal conditions. The main tool used is Schauder fixed point theorem and upper and lower solutions method. The results obtained are illustrated by a numerical example.

1. Introduction

Recently, the investigation of fractional differential equations attracted more attention since it has many applications in several fields of sciences such as in engineering, physics, chemistry, biology, etc ... [8], [10].

In this work, we use the method of upper and lower solutions to prove the existence of positive solutions for a system of multi-order fractional differential equations with nonlocal boundary conditions, where each equation has an order that may be different from the order of the other equations, that is :

$$(P)\begin{cases} D_{0^+}^{\alpha}u(t)+f(t,u(t))=0,\ 0<t<1, \\ u(0)=u'(0)=0,\ Au(1)=Bu'(1), \end{cases}$$

where the function $u=(u_1,u_2,...,u_n)$, $u_i:[0,1]\to\mathbb{R}$,

$$D_{0^+}^{\alpha}u(t)=\left(D_{0^+}^{\alpha_1}u_1(t),D_{0^+}^{\alpha_2}u_2(t),...,D_{0^+}^{\alpha_n}u_n(t)\right),$$

$D_{0^+}^{\alpha_i}$ denotes the Reimann-Liouville fractional derivative of order α_i, $2<\alpha_i<3$, $i\in\{1,..,n\}$, $n\geq 2$, the function f is such that

$$\begin{aligned} f(t,u) &= (f_1(t,u),...,f_n(t,u)), \\ u &= (u_1,u_2,...,u_n), \end{aligned}$$

$f_i\in C([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$, $A=(a_1,...,a_n)$, $B=(b_1,...,b_n)\in\mathbb{R}^n$.

Fractional differential systems can arise from sciences problems such population problems, dielectric polarization, electromagnetic waves,...see [3]. Many methods are used for the investigation of fractional differential equations, such fixed point theory, lower and upper solutions method, Mawhin theory,...see [1], [2], [4], [5], [6], [7], [9], [11].

This paper is organized as follows: in the second Section, we state some preliminary materials that will be used later. In section three, we use the upper and lower solutions method to prove the existence of positive solutions for problem (P). Finally, we give an example illustrating the obtained results.

2. Preliminaries

In this section, we recall the basic definitions and lemmas from fractional calculus theory and the details can be found in [7], [10].

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Definition 2.1. *The Riemann-Liouville fractional integrals of order α of a function h is defined as*

$$I_{0^+}^{\alpha}h\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\frac{h\left(s\right)}{\left(t-s\right)^{1-\alpha}}ds,$$

where $\Gamma\left(\alpha\right)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt$ is the Gamma function, $\alpha>0$.

Definition 2.2. *The Riemann-Liouville derivative of fractional order $\alpha>0$ for a function h is defined as*

$$D_{0^+}^{\alpha}h\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\left(\frac{d}{dt}\right)^n\int_0^t\left(t-s\right)^{n-\alpha-1}h\left(s\right)ds,$$

where $n=[\alpha]+1$ ($[\alpha]$ denotes the integer part of the real number α).

Lemma 2.1. *For $\alpha>0$, the solution of the homogeneous equation*

$$D_{0^+}^{\alpha}h\left(t\right)=0,$$

is given by

$$h(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},$$

where $c_i, i=1,2,...,n$, are real constants.

Lemma 2.2. *Let p, $q\geq0$, $h\in L_1\left[0,1\right]$. Then*

$$I_{0^+}^{p}I_{0^+}^{q}h\left(t\right)=I_{0^+}^{p+q}h\left(t\right)=I_{0^+}^{q}I_{0^+}^{p}h\left(t\right).$$

3. Main results

Lemma 3.1. *Let $y_i\in C\left(\left[0,1\right],\mathbb{R}\right)$, $i\in\{1,..,n\}$. Assume that $a_i>0$ and $b_i<0$, then for $i\in\{1,..,n\}$, the linear nonhomogeneous problem*

$$\left(S_i\right)=\begin{cases}D_{0^+}^{\alpha_i}u_i\left(t\right)=-y_i\left(t\right), \ 0<t<1,\\ u_i\left(0\right)=u_i'\left(0\right)=0,\\ a_iu_i\left(1\right)=b_iu_i'\left(1\right),\end{cases}\tag{3.1}$$

has the following solution

$$u_i\left(t\right)=\int_0^1G_i\left(t,s\right)y_i\left(s\right)ds, 0\leq t\leq1, \forall i\in\{1,..,n\}\tag{3.2}$$

where

$$G_i\left(t,s\right)=\begin{cases}\dfrac{-\left(t-s\right)^{\alpha_i-1}}{\Gamma\left(\alpha_i\right)}+\dfrac{t^{\alpha_i-1}\left(1-s\right)^{\alpha_i-2}}{\left(a_i-b_i\left(\alpha_i-1\right)\right)\Gamma\left(\alpha_i-1\right)}\left(\dfrac{a_i\left(1-s\right)}{\alpha_i-1}-b_i\right),\\ \qquad s\leq t,\\ \dfrac{t^{\alpha_i-1}\left(1-s\right)^{\alpha_i-2}}{\left(a_i-b_i\left(\alpha_i-1\right)\right)\Gamma\left(\alpha_i-1\right)}\left(\dfrac{a_i\left(1-s\right)}{\alpha_i-1}-b_i\right), s\geq t.\end{cases}$$

Proof. By assuming that u_i is a solution of the fractional boundary value problem (P) and using Lemma 2.1, we obtain

$$u_i\left(t\right)=-I_{0^+}^{\alpha_i}y_i\left(t\right)+c_1t^{\alpha_i-1}+c_2t^{\alpha_i-2}+c_3t^{\alpha_i-3},\tag{3.3}$$

According to conditions $u_i\left(0\right)=0$ and $u_i'\left(0\right)=0$, we obtain $c_2=c_3=0$. Using the nonlocal condition $a_iu_i\left(1\right)=b_iu_i'\left(1\right)$, it yields

$$c_1=\frac{1}{a_i-b_i\left(\alpha_i-1\right)}\left(a_iI_{0^+}^{\alpha_i}y\left(1\right)-b_iI_{0^+}^{\alpha_i-1}y\left(1\right)\right).\tag{3.4}$$

Substituting c_1 in Equation 3.3, we get what follows

$$u_i\left(t\right)=\int_0^1G_i\left(t,s\right)y_i\left(s\right)ds,\tag{3.5}$$

where G_i is given above. □

Lemma 3.2. *If $a_i>0$ and $b_i<0$, $i\in\{1,..,n\}$, then the functions G_i are nonnegative, continuous and*

$$0\leq G_i\left(t,s\right)\leq\frac{1}{\Gamma\left(\alpha_i\right)}, 0\leq s,t\leq1, \forall i\in\{1,..,n\},$$

Proof. The proof is direct, we omit it. $\square$

Let X be the Banach space $\underbrace{C([0,1],\mathbb{R})\times ...\times C([0,1],\mathbb{R})}_{n \text{ times}}$, equipped with the norm $\|u\| = \sum_{i=1}^{i=n} \max_{t\in[0,1]} |u_i(t)|$.
Define the integral operator $T : X \to X$ by $Tu = (T_1u, T_2u, ..., T_nu)$ where

$$T_iu(t) = \int_0^1 G_i(t,s) f_i(s,u(s))\,ds,\ \forall i \in \{1,..,n\} \tag{3.6}$$

Let $C = (c_1, ..., c_n)$, $D = (d_1, ..., d_n) \in \mathbb{R}^n_+$ such that $D > C$. We recall that for $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ then $x \le y$ means $x_i \le y_i$, for all $i \in \{1,..,n\}$ and $[C,D] = \{x = (x_1,...,x_n), c_i \le x_i \le d_i, \forall i \in \{1,..,n\}\}$.
We define the upper and lower control operators $\overline{U} = \left(\overline{U}_1, ..., \overline{U}_n\right)$, $\underline{U} = (\underline{U}_1, ..., \underline{U}_n)$ respectively by

$$\begin{aligned}
\overline{U}_i(t,x) &= \sup\{f_i(t,y), C \le y \le x\}, \\
\underline{U}_i(t,x) &= \inf\{f_i(t,y), x \le y \le D\}, 0 \le t \le 1.
\end{aligned}$$

From the definition of $\overline{U}_i$ and $\underline{U}_i$ we have $\underline{U}_i(t,x) \le f_i(t,x) \le \overline{U}_i(t,x)$, $x \in [C,D]$, $0 \le t \le 1$, $i \in \{1,..,n\}$.

Lemma 3.3. *The function $u \in X$ is a solution of the system (P) if and only if $T_iu(t) = u_i(t)$, for all $t \in [0,1]$, $\forall i \in \{1,...,n\}$.*

Consequently, the existence of solutions for system (P) can be turned into a fixed point problem in X for the operator T. Define the cone

$$K = \{u \in X, u(t) \ge 0, 0 \le t \le 1\}.$$

Let us make the following hypothesis:

(H) There exist two functions $\overline{\theta} = \left(\overline{\theta}_1, ..., \overline{\theta}_n\right)$, $\underline{\theta} = (\underline{\theta}_1, ..., \underline{\theta}_n) \in K$, such that $C \le \underline{\theta}(t) \le \overline{\theta}(t) \le D$, $0 \le t \le 1$ and

$$\begin{cases}
\overline{\theta}_i(t) \ge \int_0^1 G_i(t,s)\,\overline{U}_i\left(s,\overline{\theta}(s)\right) ds, i \in \{1,..,n\} \\
\underline{\theta}_i(t) \le \int_0^1 G_i(t,s)\,\underline{U}_i\left(s,\underline{\theta}(s)\right) ds, i \in \{1,..,n\}
\end{cases}$$

The functions $\overline{\theta}$ and $\underline{\theta}$ are called respectively upper and lower solutions for problem (P).

Now we are ready to give the main result for problem (P).

Theorem 3.1. *Assume that hypothesis (H) holds and $f(t,0) \ne 0$, $0 \le t \le 1$, then the fractional boundary value problem (P) has at least one positive solution $u \in K$ satisfying $\underline{\theta}(t) \le u(t) \le \overline{\theta}(t)$, $0 \le t \le 1$.*

Proof. Clearly, the continuity of the operator T follows from the continuity of f. Set

$$\Omega = \left\{u \in K : \underline{\theta}(t) \le u(t) \le \overline{\theta}(t), 0 \le t \le 1\right\},$$

then Ω is a nonempty, closed and convex subset of X. Firstly, we show that $T(\Omega) \subset \Omega$. In fact, let $u \in \Omega$, then by the definition of the control functions and hypothesis (H), it yields

$$\begin{aligned}
T_iu(t) &= \int_0^1 G_i(t,s) f_i(s,u(s))\,ds \\
&\le \int_0^1 G_i(t,s)\,\overline{U}_i\left(s,\overline{\theta}(s)\right) ds \le \overline{\theta}_i(t), i \in \{1,..,n\}
\end{aligned}$$

thus

$$Tu(t) \le \overline{\theta}(t), 0 \le t \le 1.$$

Similarly, we get

$$\begin{aligned}
T_iu(t) &= \int_0^1 G_i(t,s) f_i(s,u(s))\,ds \\
&\ge \int_0^1 G_i(t,s)\,\underline{U}_i\left(s,\underline{\theta}(s)\right) ds \ge \underline{\theta}_i(t), i \in \{1,..,n\}
\end{aligned}$$

from which follows

$$Tu(t) \geq \underline{\theta}(t), 0 \leq t \leq 1$$

thus $T(\Omega) \subset \Omega$. Now, we prove that $T : \Omega \to X$ is completely continuous operator. Set

$$M_i = \max\{f_i(t, u(t)), 0 \leq t \leq 1, u \in \Omega\},$$

then we have

$$\begin{aligned} |T_i u(t)| &\leq \int_0^1 G_i(t,s) f_i(s, u(s))\, ds \\ &\leq \frac{M_i}{\Gamma(\alpha_i)}. \end{aligned}$$

Taking the supremum over $[0, 1]$, then summing the obtained inequalities according to i from 1 to n, we get

$$\|Tu\| \leq \sum_{i=1}^{n} \frac{M_i}{\Gamma(\alpha_i)},$$

which implies that $T(\Omega)$ is uniformly bounded.
Let us show that (Tu) is equicontinuous. Indeed, let $u \in \Omega$ and $0 \leq t_1 < t_2 \leq 1$, then

$$\begin{aligned} |T_i u(t_1) - T_i u(t_2)| &\leq \int_0^1 |G_i(t_1, s) - G_i(t_2, s)| f_i(s, u(s))\, ds \\ &\leq M_i \left[\int_0^{t_1} |G_i(t_1, s) - G_i(t_2, s)|\, ds \right. \\ &\quad + \int_{t_1}^{t_2} |G_i(t_1, s) - G_i(t_2, s)|\, ds \\ &\quad \left. + \int_{t_2}^{1} |G_i(t_1, s) - G_i(t_2, s)|\, ds \right] \end{aligned}$$

by computation, we get

$$\begin{aligned} & |T_i u(t_1) - T_i u(t_2)| \\ \leq\ & M_i \left(\frac{(t_2 - t_1)(\alpha_i - 1)}{\Gamma(\alpha_i)} \right. \\ & \left. + \frac{(t_2 - t_1)^{\alpha_i - 1}}{\Gamma(\alpha_i)} + \frac{3\left(t_2^{\alpha_i - 1} - t_1^{\alpha_i - 1}\right)}{a_i - b_i(\alpha_i - 1)} \left(\frac{a_i}{\Gamma(\alpha_i)} + \frac{b_i}{\Gamma(\alpha_i - 1)} \right) \right). \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. By Ascoli-Arzela theorem, we conclude that the operator $T : \Omega \to \Omega$ is completely continuous. Finally, Schauder fixed point theorem implies that T has at least one fixed point $u \in \Omega$ and then problem (P) has at least one positive solution in Ω.

As direct consequence of Theorem 3.1, we get the following Corollary. □

Corollary 3.1. *Assume that f_i are continuous, nonnegative, $f_i(t, 0) \neq 0$, $0 \leq t \leq 1$ and there exist two positive constants l_i and L_i such that*

$$0 < l_i \leq f_i(t, x) \leq L_i,\ x \geq 0,\ 0 \leq t \leq 1,\ i \in \{1, .., n\}, \tag{3.7}$$

then problem (P) has at at least one positive solution $u \in X$. Furthermore the solution satisfies

$$\begin{aligned} 0 &< l_i \int_0^1 G_i(t,s)\, ds \leq u_i(t) \leq L_i \int_0^1 G_i(t,s)\, ds, \\ 0 &\leq t \leq 1,\ \forall i \in \{1, .., n\}. \end{aligned}$$

Proof. From equation 3.7 we have

$$\overline{U}_i(t,x) \leq L_i,\ \underline{U}_i(t,x) \geq l_i,\ 0 \leq t \leq 1,\ x \geq 0.$$

Let us choose

$$\begin{cases} \overline{\theta}_i(t) = L_i \int_0^1 G_i(t,s)\,ds = L_i \dfrac{t^{\alpha_i - 1}}{\alpha_i \Gamma(\alpha_i)} \left(1 + 1 + \frac{1}{\alpha_i}\right) \\ \quad \geq \int_0^1 G_i(t,s)\,\overline{U}_i\left(s, \overline{\theta}(s)\right) ds,\ i \in \{1,..,n\} \\ \underline{\theta}_i(t) = l_i \int_0^1 G_i(t,s)\,ds = l_i \dfrac{t^{\alpha_i - 1}}{\alpha_i \Gamma(\alpha_i)} \left(1 + 1 + \frac{1}{\alpha_i}\right) \\ \quad \leq \int_0^1 G_i(t,s)\,\underline{U}_i\left(s, \underline{\theta}(s)\right) ds,\ i \in \{1,..,n\}, \end{cases}$$

then the conclusion follows from Theorem 3.1. □

Now, we give an examples to illustrate the usefulness of our main results.

Example 3.1. *Consider the following two-dimensional fractional order system*

$$(S) = \begin{cases} D^{\frac{5}{2}} u_1(t) + \left(1 + \frac{1}{1+u_1+u_2}\right) = 0, \quad D^{\frac{8}{3}} u_2(t) + \left(1 + e^{-u_1}\right) = 0, \\ u_1(0) = 0,\ u_1'(0) = 0,\ u_1(1) - u_1'(0) = 0, \\ u_2(0) = 0,\ u_2'(0) = 0,\ \ u_2(1) - u_2'(0) = 0. \end{cases}$$

We have $\alpha = \left(\frac{5}{2}, \frac{8}{3}\right)$, $a_1 = a_2 = 1$, $b_1 = b_2 = -1$, $f_1(t,u_1,u_2) = 1 + \frac{1}{1+u_1+u_2}$, $f_2(t,u_1,u_2) = 1 + e^{-u_1}$, $f_i \in C\left([0,1] \times \mathbb{R}^2, \mathbb{R}_+\right)$, $f_i(t,0) \neq 0$, *and*

$$1 \leq f_i(t,u_1,u_2) \leq 2.$$

From Corollary 3.1, we conclude the system (S) has at at least one positive solution $u \in X$. *Furthermore, the solution* u *satisfies*

$$\begin{aligned} 0.722\,15 t^{\frac{3}{2}} &\leq u_1(t) \leq 1.444\,3 t^{\frac{3}{2}} \\ 0.591\,95 t^{\frac{5}{3}} &\leq u_2(t) \leq 1.183\,9 t^{\frac{5}{3}}. \end{aligned}$$

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