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GENERALIZED BETA-CONVEX FUNCTIONS AND INTEGRAL INEQUALITIES

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Abstract. In this paper, we introduce the concept of generalized beta-convex functions. This new class of convex functions includes several new and previous known classes of convex functions as special cases. We derive some integral inequalities of Hermite-Hadamard type via generalized beta-convex functions. Some special cases are also discussed. Results proved in this paper can be viewed as significant new contributions in this dynamic field.

1. Introduction and Preliminaries

Convexity theory had played a pivotal role in the development of every branch of pure and applied sciences. Closely related to this theory is inequality theory. In fact, it is known that every function is a convex function, if and if only, if satisfies an integral inequality. These type of integral inequalities are known as Hermite-Hadanard, Simpson, Trapeziodal and Newton. The integral inequalities are used to find the lower and upper bounds of natural phenomena. Due to their important applications in various branches of pure and applied science, the concept of convexity has been generalized and generalized using some interesting and novel techniques and ideas, see [1–4, 8–10, 13–15, 17–20, 23–25, 27, 30–32]. These developments played an crucial role to establish integral inequalities via various classes of convex functions and their variant forms. See [3–7, 11–20, 23–26, 28–30] and the references therein.
Motivated and inspired by the research going on in these fields, we introduced and consider a new class of convex functions, which is called generalized beta-convex functions. We show that this class of generalized beta-convex functions includes several other classes of convex functions. We also derive some new integral inequalities via beta-convex functions. Several special cases are considered which cab be obtained from our main results. Our results can be viewed as a significant refinement and improvement of the of the known results. Techniques and ideas of this paper may stimulate further research.

We now recall some known basic results and concepts, which are needed to obtain the main results.

Definition 1.1 ([32]). *An interval I is said to be a p-convex set if*

$$M_p(x,y;t) = [tx^p + (1-t)y^p]^{\frac{1}{p}} \in I$$

for all $x, y \in I, t \in [0,1]$, where $p = 2k+1$ or $p = \frac{n}{m}, n = 2r+1, m = 2t+1$ and $k, r, t \in \mathbb{N}$.

For $p = 1$, and $p = -1$, p-convex set reduces to convex set and harmonic convex set, respectively.

Definition 1.2 ([32]). *Let I be a p-convex set. A function $f : I \to \mathbb{R}$ is said to be p-convex function or belongs to the class $PC(I)$, if*

$$f(M_p(x,y;t)) \leq tf(x) + (1-t)f(y), \ \forall x, y \in I, t \in [0,1].$$

It is very much obvious that for $p = 1$ Definition 1.2 reduces to the definition for classical convex functions.
Note that for $p = -1$, we have the definition of harmonically convex functions.

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Definition 1.3 ([10]). *A function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonically convex function, if*

$$f\left(\frac{xy}{(1-t)x+ty}\right) \le tf(x) + (1-t)f(y),\ \forall x, y \in I, t \in [0,1].$$

Also note that for $t = \frac{1}{2}$ in Definition 1.2, we have Jensen p-convex functions or mid p-convex functions.

$$f(M_p(x, y; 1/2)) \le \frac{f(x) + f(y)}{2},\ \forall x, y \in I, t \in [0,1].$$

We now define the concept of generalized bet-convex functions, which is the main motivation of this paper.

Definition 1.4. *Let I be a p-convex set. A function $f : I \to \mathbb{R}$ is said to be a generalized beta-convex function, if*

$$f(M_p(x, y; t)) \le t^{\theta_1}(1-t)^{\theta_2} f(x) + (1-t)^{\theta_1} t^{\theta_2} f(y),\ \forall x, y \in I, t \in [0,1], \theta_1, \theta_2 \in (0,1].$$

For $p = 1$, we have beta-convex functions.

$$f(tx + (1-t)y) \le t^{\theta_1}(1-t)^{\theta_2} f(x) + (1-t)^{\theta_1} t^{\theta_2} f(y),\ \forall x, y \in \mathbb{R}, t \in [0,1], \theta_1, \theta_2 \in (0,1].$$

For $p = -1$, we have harmonic beta-convex functions, which were introduced and studies by Noor et. al [21, 22].

$$f\left(\frac{xy}{(1-t)x+ty}\right) \le t^{\theta_1}(1-t)^{\theta_2} f(x) + (1-t)^{\theta_1} t^{\theta_2} f(y),\ \forall x, y \in \mathbb{R}, t \in [0,1], \theta_1, \theta_2 \in (0,1].$$

We now consider some results, which are useful in obtaining our results.

Lemma 1.1. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, b]$. Then*

$$\int_a^b (b^p - x^p)^\alpha (x^p - a^p)^\beta \left(\frac{f(x)}{x^{1-p}}\right) \mathrm{d}x = (b^p - a^p)^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta f(M_p(x, y; t)) \mathrm{d}t.$$

Proof. The proof follows from simple calculations. □

Lemma 1.2 ([18]). *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{I}^0$ (the interior of $\mathcal{I}$) with $a < b$. If $f' \in \mathscr{L}[a, b]$, then, we have*

$$\begin{aligned} R_f(a, b; p) =& \frac{f(a) + f(b)}{2} - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} \mathrm{d}x \\ =& \frac{b^p - a^p}{2p} \int_0^1 [ta^p + (1-t)b^p]^{1-\frac{1}{p}} (1-2t) f'([ta^p + (1-t)b^p]^{\frac{1}{p}}) \mathrm{d}t. \end{aligned}$$

2. Main Results

In this section, we derive our main results.

Theorem 2.1. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a generalized beta-convex function. If $f \in \mathscr{L}[a, b]$, then*

$$2f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} \mathrm{d}x \le [f(a) + f(b)]\mathbb{B}(\theta_1 + 1, \theta_2 + 1).$$

Proof. Let f be a generalized beta-convex function. Then

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{1}{4}\Big[f\big([ta^p + (1-t)b^p]^{\frac{1}{p}}\big) + f\big([(1-t)a^p + tb^p]^{\frac{1}{p}}\big)\Big].$$

Integrating both sides of above inequality with respect to t on $[0,1]$, we have

$$2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\mathrm{d}x. \tag{2.1}$$

Also

$$f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) \leq t^{\theta_1}(1-t)^{\theta_2}f(x)+(1-t)^{\theta_1}t^{\theta_2}f(y).$$

Integrating both sides of above inequality with respect to t on $[0,1]$, we have

$$\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\mathrm{d}x \leq [f(a)+f(b)]\mathbb{B}(\theta_1+1,\theta_2+1). \tag{2.2}$$

On summation of inequalities (2.1) and (2.2) the proof is complete. □

We now discuss a new special case of Theorem 2.1.

If $\theta_1=\theta=\theta_2$ in Theorem 2.1, then we have following new result for Brecker type of generalized tgs-convex functions.

Corollary 2.1. *Let $f:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be Brecker type of tgs-convex function. If $f\in\mathscr{L}[a,b]$, then*

$$2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\mathrm{d}x \leq [f(a)+f(b)]\mathbb{B}(\theta+1,\theta+1).$$

If $\theta_1=-\theta=\theta_2$ in Theorem 2.1, then we have following new result for Godunova-Levin-Dragomir type generalized tgs-convex functions.

Corollary 2.2. *Let $f:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be Godunova-Levin-Dragomir generalized tgs-convex function. If $f\in\mathscr{L}[a,b]$, then*

$$2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\mathrm{d}x \leq [f(a)+f(b)]\mathbb{B}(1-\theta,1-\theta).$$

If $p=-1$ in Theorem 2.1, then we have following new result for harmonic beta-convex functions.

Corollary 2.3. *Let $f:\mathcal{I}\setminus\{0\}\subset\mathbb{R}\to\mathbb{R}$ be a harmonic beta-convex function. If $f\in\mathscr{L}[a,b]$, then, we have*

$$2f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq [f(a)+f(b)]\mathbb{B}(\theta_1+1,\theta_2+1).$$

We now derive a lower bound for Hermite-Hadamard's inequality via product of two generalized beta-convex functions.

Theorem 2.2. *Let $f,g:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be two generalized beta-convex functions. If $fg\in\mathscr{L}[a,b]$, then*

$$\begin{aligned}
&8f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\\
&\qquad-\mathbb{B}(\theta_1+\theta_2+1,\theta_1+\theta_2+1)M(a,b)+\mathbb{B}(2\theta_1+1,2\theta_2+1)N(a,b)\\
&\leq \frac{p}{b^p-a^p}\int_0^1 \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x\\
&\leq \mathbb{B}(2\theta_1+1,2\theta_2+1)M(a,b)+\mathbb{B}(\theta_1+\theta_2+1,\theta_1+\theta_2+1)N(a,b),
\end{aligned}$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b), \tag{2.3}$$

and

$$N(a,b)=f(a)g(b)+f(b)g(a), \tag{2.4}$$

respectively.

Proof. Since f and g are generalized beta-convex functions respectively, so

$$\begin{aligned}
f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\le \frac{1}{4}\left[f\big([ta^p+(1-t)b^p]^{\frac{1}{p}}\big)+f\big([(1-t)a^p+tb^p]^{\frac{1}{p}}\big)\right]\\
&\quad\times\frac{1}{4}\left[g\big([ta^p+(1-t)b^p]^{\frac{1}{p}}\big)+g\big([(1-t)a^p+tb^p]^{\frac{1}{p}}\big)\right]\\
&=\frac{1}{16}\Big[f\big([ta^p+(1-t)b^p]^{\frac{1}{p}}\big)g\big([ta^p+(1-t)b^p]^{\frac{1}{p}}\big)\\
&\quad+f\big([(1-t)a^p+tb^p]^{\frac{1}{p}}\big)g\big([(1-t)a^p+tb^p]^{\frac{1}{p}}\big)\\
&\quad+f\big([ta^p+(1-t)b^p]^{\frac{1}{p}}\big)g\big([(1-t)a^p+tb^p]^{\frac{1}{p}}\big)\\
&\quad+f\big([(1-t)a^p+tb^p]^{\frac{1}{p}}\big)g\big([ta^p+(1-t)b^p]^{\frac{1}{p}}\big)\Big]\\
&\le\frac{1}{16}\Big[f([ta^p+(1-t)b^p]^{\frac{1}{p}})g([ta^p+(1-t)b^p]^{\frac{1}{p}})\\
&\quad+f([(1-t)a^p+tb^p]^{\frac{1}{p}})g([(1-t)a^p+tb^p]^{\frac{1}{p}})\\
&\quad+[2t^{\theta_1+\theta_2}(1-t)^{\theta_1+\theta_2}][f(a)g(a)+f(b)g(b)]\\
&\quad+[t^{2\theta_1}(1-t)^{2\theta_2}+t^{2\theta_2}(1-t)^{2\theta_1}][f(a)g(b)+f(b)g(a)]\Big].
\end{aligned}$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}
&f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\\
&\le\frac{1}{8}\Bigg[\frac{p}{b^p-a^p}\int_0^1\frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x\\
&\quad+\mathbb{B}(\theta_1+\theta_2+1,\theta_1+\theta_2+1)M(a,b)+\mathbb{B}(2\theta_1+1,2\theta_2+1)N(a,b)\Bigg].
\end{aligned}\tag{2.5}$$

Also since f and g are generalized beta-convex functions, then

$$f\big([ta^p+(1-t)b^p]^{\frac{1}{p}}\big)\le t^{\theta_1}(1-t)^{\theta_2}f(a)+(1-t)^{\theta_1}t^{\theta_2}f(b),$$

and

$$g\big([ta^p+(1-t)b^p]^{\frac{1}{p}}\big)\le t^{\theta_1}(1-t)^{\theta_2}g(a)+(1-t)^{\theta_1}t^{\theta_2}g(b).$$

Multiplying both sides of above inequality and then integrating it with respect to t on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f\big([ta^p+(1-t)b^p]^{\frac{1}{p}}\big)g\big([ta^p+(1-t)b^p]^{\frac{1}{p}}\big)\mathrm{d}t\\
&\le f(a)g(a)\int_0^1 t^{\theta_1}(1-t)^{\theta_2}t^{\theta_1}(1-t)^{\theta_2}\mathrm{d}t+f(b)g(b)\int_0^1 t^{\theta_2+\theta_2}(1-t)^{\theta_1+\theta_1}\mathrm{d}t\\
&\quad+[f(a)g(b)+f(b)g(a)]\int_0^1 t^{\theta_1}(1-t)^{\theta_2}t^{\theta_2}(1-t)^{\theta_1}\mathrm{d}t.
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{p}{b^p-a^p}\int\limits_a^b \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x \\
&\leq \mathbb{B}(2\theta_1, 2\theta_2+1)M(a,b)+\mathbb{B}(\theta_1+\theta_2+1, \theta_1+\theta_2+1)N(a,b). \qquad (2.6)
\end{aligned}$$

Combining (2.5) and (2.6) completes the proof. □

Next we discuss a new special case of Theorem 2.2.

If $\theta_1 = \theta = \theta_2$ in Theorem 2.2, then we have following new result for Brecker generalized tgs-convex functions.

Corollary 2.4. *Let $f, g : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be two Brecker type of tgs-convex functions. If $fg \in \mathscr{L}[a,b]$, then, we have*

$$\begin{aligned}
&8f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) - \mathbb{B}(2\theta+1, 2\theta+1)[M(a,b)+N(a,b)] \\
&\leq \frac{p}{b^p-a^p}\int\limits_0^1 \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x \\
&\leq \mathbb{B}(2\theta_1+1, 2\theta_2+1)M(a,b)+\mathbb{B}(\theta_1+\theta_2+1, \theta_1+\theta_2+1)N(a,b),
\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (2.3) and (2.4) respectively.

If $\theta_1 = -\theta = \theta_2$ in Theorem 2.2, then we have following new result for Godunova-Levin-Dragomir generalized tgs-convex functions.

Corollary 2.5. *Let $f, g : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be two Godunova-Levin-Dragomir generalized tgs-convex functions. If $fg \in \mathscr{L}[a,b]$, then*

$$\begin{aligned}
&8f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) - \mathbb{B}(1-2\theta, 1-2\theta)[M(a,b)+N(a,b)] \\
&\leq \frac{p}{b^p-a^p}\int\limits_0^1 \frac{f(x)g(x)}{x^{1-p}}\mathrm{d}x \\
&\leq \mathbb{B}(2\theta_1+1, 2\theta_2+1)M(a,b)+\mathbb{B}(\theta_1+\theta_2+1, \theta_1+\theta_2+1)N(a,b),
\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (2.3) and (2.4) respectively.

If $p = -1$ in Theorem 2.2, then we have following new result for harmonic beta-convex functions.

Corollary 2.6. *Let $f, g : \mathcal{I} \setminus \{0\} \subset \mathbb{R} \to \mathbb{R}$ be two harmonic beta-convex functions. If $fg \in \mathscr{L}[a,b]$, then, we have*

$$\begin{aligned}
&8f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) \\
&\quad - \mathbb{B}(\theta_1+\theta_2+1, \theta_1+\theta_2+1)M(a,b)+\mathbb{B}(2\theta_1+1, 2\theta_2+1)N(a,b) \\
&\leq \frac{ab}{b-a}\int\limits_0^1 \frac{f(x)g(x)}{x^2}\mathrm{d}x \\
&\leq \mathbb{B}(2\theta_1+1, 2\theta_2+1)M(a,b)+\mathbb{B}(\theta_1+\theta_2+1, \theta_1+\theta_2+1)N(a,b),
\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given in (2.3) and (2.4) respectively.

Theorem 2.3. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, b]$. If f is generalized beta-convex function, then*

$$\int_a^b (b^p - x^p)^\alpha (x^p - a^p)^\beta \left(\frac{f(x)}{x^{1-p}}\right) \mathrm{d}x \le (b^p - a^p)^{\alpha+\beta+1} \Big[k_1(\theta) f(a) + k_2(\theta) f(b)\Big],$$

where

$$k_1(\theta) := \mathbb{B}(\alpha + \theta_1 + 1, \beta + \theta_2 + 1), \tag{2.7}$$

and

$$k_2(\theta) := \mathbb{B}(\alpha + \theta_2 + 1, \beta + \theta_1 + 1), \tag{2.8}$$

respectively.

Proof. Using Lemma 1.1 and the fact that f is generalized beta-convex function, we have

$$\begin{aligned}
&\int_a^b (b^p - x^p)^\alpha (x^p - a^p)^\beta \left(\frac{f(x)}{x^{1-p}}\right) \mathrm{d}x \\
&= (b^p - a^p)^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta f(M_p(x, y; t)) \mathrm{d}t \\
&\le (b^p - a^p)^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta [t^{\theta_1} (1-t)^{\theta_2} f(a) + (1-t)^{\theta_1} t^{\theta_2} f(b)] \mathrm{d}t \\
&= (b^p - a^p)^{\alpha+\beta+1} \Big[k_1(\theta) f(a) + k_2(\theta) f(b)\Big].
\end{aligned}$$

This completes the proof. □

If $\theta_1 = \theta = \theta_2$ in Theorem 2.3, then we have

Corollary 2.7. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, b]$. If f is Breckner generalized tgs-convex function, then*

$$\int_a^b (b^p - x^p)^\alpha (x^p - a^p)^\beta \left(\frac{f(x)}{x^{1-p}}\right) \mathrm{d}x \le (b^p - a^p)^{\alpha+\beta+1} k(\theta) \Big[f(a) + f(b)\Big],$$

where

$$k(\theta) := \mathbb{B}(\alpha + \theta + 1, \beta + \theta + 1). \tag{2.9}$$

If $\theta_1 = -\theta = \theta_2$ in Theorem 2.3, then we have

Corollary 2.8. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, b]$. If f is Godunova-Levin-Dragomir generalized tgs-convex function, then*

$$\int_a^b (b^p - x^p)^\alpha (x^p - a^p)^\beta \left(\frac{f(x)}{x^{1-p}}\right) \mathrm{d}x \le (b^p - a^p)^{\alpha+\beta+1} h(\theta) \Big[f(a) + f(b)\Big],$$

where

$$h(\theta) := \mathbb{B}(\alpha - \theta + 1, \beta - \theta + 1). \tag{2.10}$$

If $p = -1$ in Theorem 2.3, then we have

Corollary 2.9. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, b]$. If f is harmonic beta-convex function, then*

$$\int_a^b \left(\frac{1}{b} - \frac{1}{x}\right)^\alpha \left(\frac{1}{x} - \frac{1}{a}\right)^\beta \left(\frac{f(x)}{x^{1-p}}\right) \mathrm{d}x \le \left(\frac{1}{b} - \frac{1}{a}\right)^{\alpha+\beta+1} \Big[k_1(\theta) f(a) + k_2(\theta) f(b)\Big],$$

where

$$k_1(\theta) := \mathbb{B}(\alpha+\theta_1+1, \beta+\theta_2+1), \tag{2.11}$$

and

$$k_2(\theta) := \mathbb{B}(\alpha+\theta_2+1, \beta+\theta_1+1), \tag{2.12}$$

respectively.

Theorem 2.4. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, b]$. If $|f|^{\frac{r}{r-1}}$ is generalizedbeta-convex function, then*

$$\begin{aligned}&\int_a^b (b^p - x^p)^\alpha (x^p - a^p)^\beta \left(\frac{f(x)}{x^{1-p}}\right) \mathrm{d}x \\ &\leq (b^p - a^p)^{\alpha+\beta+1} \mathbb{B}(r\alpha+1, r\beta+1) \left[\left\{|f(a)|^{\frac{r}{r-1}} + |f(b)|^{\frac{r}{r-1}}\right\} \mathbb{B}(\theta_1+1, \theta_2+1)\right]^{\frac{r-1}{r}}.\end{aligned}$$

Proof. Using Lemma 1.1, Holder's inequality and the fact that $|f|^{\frac{r}{r-1}}$ is generalized beta-convex function, then

$$\begin{aligned}&\int_a^b (b^p - x^p)^\alpha (x^p - a^p)^\beta \left(\frac{f(x)}{x^{1-p}}\right) \mathrm{d}x \\ &= (b^p - a^p)^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta f(M_p(x, y; t)) \mathrm{d}t \\ &\leq (b^p - a^p)^{\alpha+\beta+1} \left[\int_0^1 t^{r\alpha}(1-t)^{r\beta} \mathrm{d}t\right]^{\frac{1}{r}} \left[\int_0^1 |f(M_p(x, y; t))|^{\frac{r}{r-1}} \mathrm{d}t\right]^{\frac{r-1}{r}} \\ &\leq (b^p - a^p)^{\alpha+\beta+1} \mathbb{B}(r\alpha+1, r\beta+1) \left[\int_0^1 \left\{t^{\theta_1}(1-t)^{\theta_2}|f(a)|^{\frac{r}{r-1}} + (1-t)^{\theta_2} t^{\theta_1} |f(b)|^{\frac{r}{r-1}}\right\} \mathrm{d}t\right]^{\frac{r-1}{r}} \\ &\leq (b^p - a^p)^{\alpha+\beta+1} \mathbb{B}(r\alpha+1, r\beta+1) \left[\left\{|f(a)|^{\frac{r}{r-1}} + |f(b)|^{\frac{r}{r-1}}\right\} \mathbb{B}(\theta_1+1, \theta_2+1)\right]^{\frac{r-1}{r}}.\end{aligned}$$

This completes the proof. □

If $\theta_1 = \theta = \theta_2$ in Theorem 2.4, then we have

Corollary 2.10. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, b]$. If $|f|^{\frac{r}{r-1}}$ is Breckner generalized tgs-convex function, then*

$$\begin{aligned}&\int_a^b (b^p - x^p)^\alpha (x^p - a^p)^\beta \left(\frac{f(x)}{x^{1-p}}\right) \mathrm{d}x \\ &\leq (b^p - a^p)^{\alpha+\beta+1} \mathbb{B}(r\alpha+1, r\beta+1) \left[\left\{|f(a)|^{\frac{r}{r-1}} + |f(b)|^{\frac{r}{r-1}}\right\} \mathbb{B}(\theta+1, \theta+1)\right]^{\frac{r-1}{r}}.\end{aligned}$$

If $\theta_1 = -\theta = \theta_2$ in Theorem 2.4, then we have

Corollary 2.11. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a, b]$. If $|f|^{\frac{r}{r-1}}$ is Godunova-Levin-Dragomir type of tgs-convex function, then, we have*

$$\begin{aligned}&\int_a^b (b^p - x^p)^\alpha (x^p - a^p)^\beta \left(\frac{f(x)}{x^{1-p}}\right) \mathrm{d}x \\ &\leq (b^p - a^p)^{\alpha+\beta+1} \mathbb{B}(r\alpha+1, r\beta+1) \left[\left\{|f(a)|^{\frac{r}{r-1}} + |f(b)|^{\frac{r}{r-1}}\right\} \mathbb{B}(1-\theta, 1-\theta)\right]^{\frac{r-1}{r}}.\end{aligned}$$

If $p = -1$ in Theorem 2.4, then we have

Corollary 2.12. *Let $f : \mathcal{I} \setminus \{0\} \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a,b]$. If $|f|^{\frac{r}{r-1}}$ is harmonic beta-convex function, then, we have*

$$\int_a^b \left(\frac{1}{b}-\frac{1}{x}\right)^{\alpha}\left(\frac{1}{x}-\frac{1}{a}\right)^{\beta}\left(\frac{f(x)}{x^{1-p}}\right)\mathrm{d}x$$
$$\leq \left(\frac{1}{b}-\frac{1}{a}\right)^{\alpha+\beta+1}\mathbb{B}(r\alpha+1,r\beta+1)\left[\left\{|f(a)|^{\frac{r}{r-1}}+|f(b)|^{\frac{r}{r-1}}\right\}\mathbb{B}(\theta_1+1,\theta_2+1)\right]^{\frac{r-1}{r}}.$$

Theorem 2.5. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a,b]$. If $|f|^r$ is beta-convex function, then, we have*

$$\int_a^b (b^p-x^p)^{\alpha}(x^p-a^p)^{\beta}\left(\frac{f(x)}{x^{1-p}}\right)\mathrm{d}x$$
$$\leq (b^p-a^p)^{\alpha+\beta+1}\left[\mathbb{B}(\alpha+1,\beta+1)\right]^{\frac{r-1}{r}}\left[k_1(\theta)|f(a)|^r+k_2(\theta)|f(b)|^r\right]^{\frac{1}{r}},$$

where $k_1(\theta)$ and $k_2(\theta)$ are given by (2.11) and (2.12) respectively.

Proof. Using Lemma 1.1, Holder's inequality and the fact that $|f|^r$ is beta-convex function, then

$$\int_a^b (b^p-x^p)^{\alpha}(x^p-a^p)^{\beta}\left(\frac{f(x)}{x^{1-p}}\right)\mathrm{d}x$$
$$= (b^p-a^p)^{\alpha+\beta+1}\int_0^1 t^{\alpha}(1-t)^{\beta}f(M_p(x,y;t))\mathrm{d}t$$
$$\leq (b^p-a^p)^{\alpha+\beta+1}\left[\int_0^1 (1-t)^{\alpha}t^{\beta}\mathrm{d}t\right]^{\frac{r-1}{r}}\left[\int_0^1 t^{\alpha}(1-t)^{\beta}\left|f(M_p(x,y;t))\right|^r\mathrm{d}t\right]^{\frac{1}{r}}$$
$$\leq (b^p-a^p)^{\alpha+\beta+1}\left[\mathbb{B}(\alpha+1,\beta+1)\right]^{\frac{r-1}{r}}$$
$$\times\left[\int_0^1 t^{\alpha}(1-t)^{\beta}\left[t^{\theta_1}(1-t)^{\theta_2}|f(a)|^r+(1-t)^{\theta_1}t^{\theta_2}|f(b)|^r\right]\mathrm{d}t\right]^{\frac{1}{r}}$$
$$= (b^p-a^p)^{\alpha+\beta+1}\left[\mathbb{B}(\alpha+1,\beta+1)\right]^{\frac{r-1}{r}}\left[k_1(\theta)|f(a)|^r+k_2(\theta)|f(b)|^r\right]^{\frac{1}{r}}.$$

This completes the proof. □

If $\theta_1=\theta=\theta_2$ in Theorem 2.5, then we have

Corollary 2.13. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a,b]$. If $|f|^r$ is Breckner type of tgs-convex function, then we have*

$$\int_a^b (b^p-x^p)^{\alpha}(x^p-a^p)^{\beta}\left(\frac{f(x)}{x^{1-p}}\right)\mathrm{d}x$$
$$\leq (b^p-a^p)^{\alpha+\beta+1}\left[\mathbb{B}(\alpha+1,\beta+1)\right]^{\frac{r-1}{r}}k^{\frac{1}{r}}(t)\left[|f(a)|^r+|f(b)|^r\right]^{\frac{1}{r}},$$

where $k(\theta)$ is given by (2.9).

If $\theta_1=-\theta=\theta_2$ in Theorem 2.5, then we have

Corollary 2.14. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a,b]$. If $|f|^r$ is Godunova-Levin-Dragomir generalized tgs-convex function, then*

$$
\begin{aligned}
&\int_a^b (b^p - x^p)^{\alpha}(x^p - a^p)^{\beta}\left(\frac{f(x)}{x^{1-p}}\right)\mathrm{d}x \\
&\leq (b-a)^{\alpha+\beta+1}\left[\mathbb{B}(1-\alpha,1-\beta)\right]^{\frac{r-1}{r}} h^{\frac{1}{r}}(t)\left[|f(a)|^r + |f(b)|^r\right]^{\frac{1}{r}},
\end{aligned}
$$

where $h(t)$ is given by (2.10).

If $p=-1$ in Theorem 2.5, then we have

Corollary 2.15. *Let $f : \mathcal{I}\setminus\{0\} \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a,b]$. If $|f|^r$ is harmonic beta-convex function, then, we have*

$$
\begin{aligned}
&\int_a^b \left(\frac{1}{b}-\frac{1}{x}\right)^{\alpha}\left(\frac{1}{x}-\frac{1}{a}\right)^{\beta}\left(\frac{f(x)}{x^{1-p}}\right)\mathrm{d}x \\
&\leq \left(\frac{1}{b}-\frac{1}{a}\right)^{\alpha+\beta+1}\left[\mathbb{B}(\alpha+1,\beta+1)\right]^{\frac{r-1}{r}}\left[k_1(\theta)|f(a)|^r + k_2(\theta)|f(b)|^r\right]^{\frac{1}{r}},
\end{aligned}
$$

where $k_1(\theta)$ and $k_2(\theta)$ are given by (2.11) and (2.12) respectively.

Now using Lemma 1.2 we derive some Hermite-Hadamard type inequalities.

Theorem 2.6. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{I}^0$ (the interior of $\mathcal{I}$) with $a<b$ and $f' \in \mathscr{L}[a,b]$. If $|f'|$ is beta-convex function, then*

$$
|R_f(a,b;p)| \leq \frac{b^p - a^p}{2p}\left[h_1(\theta_1,\theta_2)|f'(a)| + h_2(\theta_1,\theta_2)|f'(b)|\right],
$$

where

$$
\begin{aligned}
h_1(\theta_1,\theta_2) &:= b^{p-1}\mathbb{B}(\theta_1+1,\theta_2+1)\,{}_2F_1\Big(\frac{1}{p}-1,\theta_1+1;\theta_1+\theta_2+2;1-\frac{a^p}{b^p}\Big) \\
&\quad - 2b^{p-1}\mathbb{B}(\theta_1+2,\theta_2+1)\,{}_2F_1\Big(\frac{1}{p}-1,\theta_1+2;\theta_1+\theta_2+3;1-\frac{a^p}{b^p}\Big),
\end{aligned} \tag{2.13}
$$

and

$$
\begin{aligned}
h_2(\theta_1,\theta_2) &:= b^{p-1}\mathbb{B}(\theta_2+1,\theta_1+1)\,{}_2F_1\Big(\frac{1}{p}-1,\theta_2+1;\theta_1+\theta_2+2;1-\frac{a^p}{b^p}\Big) \\
&\quad - 2b^{p-1}\mathbb{B}(\theta_2+2,\theta_1+1)\,{}_2F_1\Big(\frac{1}{p}-1,\theta_2+2;\theta_1+\theta_2+3;1-\frac{a^p}{b^p}\Big),
\end{aligned} \tag{2.14}
$$

respectively.

Proof. Using Lemma 1.2, property of the modulus and the fact that $|f'|$ is beta-convex function, we have

$$\begin{aligned}
&|R_f(a,b;p)| \\
&= \left|\frac{b^p-a^p}{2p}\int_0^1 [ta^p+(1-t)b^p]^{1-\frac{1}{p}}(1-2t)f'([ta^p+(1-t)b^p]^{\frac{1}{p}})\mathrm{d}t\right| \\
&\le \frac{b^p-a^p}{2p}\int_0^1 [ta^p+(1-t)b^p]^{1-\frac{1}{p}}(1-2t)\left[t^{\theta_1}(1-t)^{\theta_2}|f'(a)|+(1-t)^{\theta_1}t^{\theta_2}|f'(b)|\right]\mathrm{d}t \\
&= \frac{b^p-a^p}{2p}\left[\int_0^1 t^{\theta_1}(1-t)^{\theta_2}(1-2t)[ta^p+(1-t)b^p]^{1-\frac{1}{p}}|f'(a)|\mathrm{d}t\right. \\
&\qquad\left.+\int_0^1 (1-t)^{\theta_1}t^{\theta_2}(1-2t)[ta^p+(1-t)b^p]^{1-\frac{1}{p}}|f'(b)|\mathrm{d}t\right] \\
&= \frac{b^p-a^p}{2p}\left[\left\{b^{p-1}\mathbb{B}(\theta_1+1,\theta_2+1)\,{}_2F_1\left(\frac{1}{p}-1,\theta_1+1;\theta_1+\theta_2+2;1-\frac{a^p}{b^p}\right)\right.\right. \\
&\qquad\left.-2b^{p-1}\mathbb{B}(\theta_1+2,\theta_2+1)\,{}_2F_1\left(\frac{1}{p}-1,\theta_1+2;\theta_1+\theta_2+3;1-\frac{a^p}{b^p}\right)\right\}|f'(a)| \\
&\qquad+\left\{b^{p-1}\mathbb{B}(\theta_2+1,\theta_1+1)\,{}_2F_1\left(\frac{1}{p}-1,\theta_2+1;\theta_1+\theta_2+2;1-\frac{a^p}{b^p}\right)\right. \\
&\qquad\left.\left.-2b^{p-1}\mathbb{B}(\theta_2+2,\theta_1+1)\,{}_2F_1\left(\frac{1}{p}-1,\theta_2+2;\theta_1+\theta_2+3;1-\frac{a^p}{b^p}\right)\right\}|f'(b)|\right] \\
&= \frac{b^p-a^p}{2p}\left[h_1(\theta_1,\theta_2)|f'(a)|+h_2(\theta_1,\theta_2)|f'(b)|\right].
\end{aligned}$$

This completes the proof. □

We now discuss some special cases of Theorem 2.6.

If $\theta_1=\theta=\theta_2$ in Theorem 2.6, then we have a following new result.

Corollary 2.16. *Let $f:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\mathcal{I}^0$ (the interior of $\mathcal{I}$) with $a<b$ and $f'\in\mathscr{L}[a,b]$. If $|f'|$ is Breckner type of tgs-convex function, then*

$$|R_f(a,b;p)|\le\frac{b^p-a^p}{2p}h(\theta)\left[|f'(a)|+|f'(b)|\right],$$

where

$$\begin{aligned}
h(\theta):=\;&b^{p-1}\mathbb{B}(\theta+1,\theta+1)\,{}_2F_1\left(\frac{1}{p}-1,\theta+1;2\theta+2;1-\frac{a^p}{b^p}\right) \\
&-2b^{p-1}\mathbb{B}(\theta+2,\theta+1)\,{}_2F_1\left(\frac{1}{p}-1,\theta+2;2\theta+3;1-\frac{a^p}{b^p}\right).
\end{aligned}\tag{2.15}$$

If $\theta_1=-\theta=\theta_2$ in Theorem 2.6, then we have following new result.

Corollary 2.17. *Let $f:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\mathcal{I}^0$ (the interior of $\mathcal{I}$) with $a<b$ and $f'\in\mathscr{L}[a,b]$. If $|f'|$ is Godunova-Levin-Dragomir generalized tgs-convex function, then*

$$|R_f(a,b;p)|\le\frac{b^p-a^p}{2p}l(\theta)\left[|f'(a)|+|f'(b)|\right],$$

where

$$\begin{aligned}
l(\theta):=\;&b^{p-1}\mathbb{B}(1-\theta_1,1-\theta_2)\,{}_2F_1\left(\frac{1}{p}-1,1-\theta;2-2\theta;1-\frac{a^p}{b^p}\right) \\
&-2b^{p-1}\mathbb{B}(2-\theta_1,1-\theta_2)\,{}_2F_1\left(\frac{1}{p}-1,2-\theta;3-2\theta;1-\frac{a^p}{b^p}\right).
\end{aligned}\tag{2.16}$$

If $p=1$ in Theorem 2.6, then we have following new result.

Corollary 2.18. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{I}^0$ (the interior of $\mathcal{I}$) with $a < b$ and $f' \in \mathscr{L}[a,b]$. If $|f'|$ is beta-convex function, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_0^1 f(x)\mathrm{d}x\right| \le \frac{b-a}{2}\left[h_1(\theta_1,\theta_2)|f'(a)| + h_2(\theta_1,\theta_2)|f'(b)|\right],$$

where $h_1(\theta_1,\theta_2)$ and $h_2(\theta_1,\theta_2)$ are given by (2.13) and (2.14) respectively.

If $p = -1$ in Theorem 2.6, then we have following new result.

Corollary 2.19. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{I}^0$ (the interior of $\mathcal{I}$) with $a < b$ and $f' \in \mathscr{L}[a,b]$. If $|f'|$ is harmonic beta-convex function, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_0^1 \frac{f(x)}{x^2}\mathrm{d}x\right| \le \frac{ab(b-a)}{2}\left[h_1(\theta_1,\theta_2)|f'(a)| + h_2(\theta_1,\theta_2)|f'(b)|\right],$$

where $h_1(\theta_1,\theta_2)$ and $h_2(\theta_1,\theta_2)$ are given by (2.13) and (2.14) respectively.

Theorem 2.7. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{I}^0$ (the interior of $\mathcal{I}$) with $a < b$ and $f' \in \mathscr{L}[a,b]$. If $|f'|^r$ is beta-convex function, then*

$$\begin{aligned}
&|R_f(a,b;p)| \\
&\le \frac{b^p-a^p}{2p}\left(b^{1-p}\,{}_2F_1\Big(\frac{1}{p}-1,1;2;1-\frac{a^p}{b^p}\Big) - 4b^{1-p}\,{}_2F_1\Big(\frac{1}{p}-1,2;3;1-\frac{a^p}{b^p}\Big)\right)^{1-\frac{1}{r}} \\
&\quad\times\left[h_1(\theta_1,\theta_2)|f'(a)|^r\mathrm{d}t + h_2(\theta_1,\theta_2)|f'(b)|^r\mathrm{d}t\right]^{\frac{1}{r}},
\end{aligned}$$

where $h_1(\theta_1,\theta_2)$ and $h_2(\theta_1,\theta_2)$ are given by (2.13) and (2.14) respectively.

Proof. Using Lemma 1.2, property of the modulus, power mean's inequality and the fact that $|f'|^r$ is beta-convex function, we have

$$\begin{aligned}
&|R_f(a,b;p)| \\
&= \left|\frac{b^p-a^p}{2p}\int_0^1 [ta^p+(1-t)b^p]^{1-\frac{1}{p}}(1-2t)f'([ta^p+(1-t)b^p]^{\frac{1}{p}})\mathrm{d}t\right| \\
&\le \frac{b^p-a^p}{2p}\left(\int_0^1 (1-2t)[ta^p+(1-t)b^p]^{1-\frac{1}{p}}\mathrm{d}t\right)^{1-\frac{1}{r}} \\
&\quad\times\left(\int_0^1 (1-2t)[ta^p+(1-t)b^p]^{1-\frac{1}{p}}\left[t^{\theta_1}(1-t)^{\theta_2}|f'(a)|^r + (1-t)^{\theta_1}t^{\theta_2}|f'(b)|^r\right]\mathrm{d}t\right)^{\frac{1}{r}} \\
&= \frac{b^p-a^p}{2p}\left(\int_0^1 (1-2t)[ta^p+(1-t)b^p]^{1-\frac{1}{p}}\mathrm{d}t\right)^{1-\frac{1}{r}} \\
&\quad\times\left[\int_0^1 t^{\theta_1}(1-t)^{\theta_2}(1-2t)[ta^p+(1-t)b^p]^{1-\frac{1}{p}}|f'(a)|^r\mathrm{d}t\right. \\
&\qquad\qquad\left.+\int_0^1 (1-t)^{\theta_1}t^{\theta_2}(1-2t)[ta^p+(1-t)b^p]^{1-\frac{1}{p}}|f'(b)|^r\mathrm{d}t\right]^{\frac{1}{r}} \\
&= \frac{b^p-a^p}{2p}\left(b^{1-p}\,{}_2F_1\Big(\frac{1}{p}-1,1;2;1-\frac{a^p}{b^p}\Big) - 4b^{1-p}\,{}_2F_1\Big(\frac{1}{p}-1,2;3;1-\frac{a^p}{b^p}\Big)\right)^{1-\frac{1}{r}} \\
&\quad\times\left[h_1(\theta_1,\theta_2)|f'(a)|^r\mathrm{d}t + h_2(\theta_1,\theta_2)|f'(b)|^r\mathrm{d}t\right]^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

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