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GENERALIZED STEFFENSEN INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS

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Abstract. Firstly we give a important integral inequality which is generalized Steffensen's inequality. Then, we establish weighted version of generalized Steffensen's inequality for local fractional integrals. Finally, we obtain several inequalities related these inequalities using the local fractional integral.

1. Introduction

In [17], J. S. Steffensen established the following result which is known as Steffensen's inequality in the literature.

Theorem 1.1. *Let a and b be real numbers such that $a < b$, $f, g : [a, b] \to \mathbb{R}$ be integrable functions such that f is nonincreasing and for every $x \in [a, b]$, $0 \le g(x) \le 1$. Then*

$$\int_{b-\lambda}^{b} f(x)dx \le \int_{a}^{b} f(x)g(x)dx \le \int_{a}^{a+\lambda} f(x)dx \tag{1.1}$$

where

$$\lambda = \int_{a}^{b} g(x)dx.$$

The most basic inequality which deals with the comparison between integrals over a whole interval $[a, b]$ and integrals over a subset of $[a, b]$ is the following inequality. The inequality (1.1) has attracted considerable attention and interest from mathematicans and researchers. Due to this, over the years, the interested reader is also refered to ([1], [2], [4]- [7], [10], [11] and [18]) for integral inequalities.

In [19], Wu and Srivastava proved the following inequality which is weighted version of the inequality (1.1).

Theorem 1.2. *Let f, g and h be integrable functions defined on $[a, b]$ with f nonincreasin. Also let $0 \le g(x) \le h(x)$ for all $x \in [a, b]$. Then, the following inequalities hold:*

$$\begin{aligned}
&\int_{b-\lambda}^{b} f(x)h(x)dx \\
\le\ & \int_{b-\lambda}^{b} \left(f(x)h(x) - [f(x) - f(b-\lambda)]\,[h(x) - g(x)]\right) dx \le \int_{a}^{b} f(x)g(x)dx \\
\le\ & \int_{a}^{a+\lambda} \left(f(x)h(x) - [f(x) - f(a+\lambda)]\,[h(x) - g(x)]\right) dx \le \int_{a}^{a+\lambda} f(x)h(x)dx
\end{aligned}$$

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where λ *is given by*

$$\int_a^{a+\lambda} h(x)dx = \int_a^b g(x)dx = \int_{b-\lambda}^b h(x)dx.$$

2. Preliminaries

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [20, 21] and so on.

Recently, the theory of Yang's fractional sets [20] was introduced as follows.

For $0 < \alpha \le 1$, we have the following α-type set of element sets:

Z^α : The α-type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, ..., \pm n^\alpha, ...\}$.

Q^α : The α-type set of the rational numbers is defined as the set $\{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p, q \in Z,\ q \neq 0\}$.

J^α : The α-type set of the irrational numbers is defined as the set $\{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p, q \in Z,\ q \neq 0\}$.

R^α : The α-type set of the real line numbers is defined as the set $R^\alpha = Q^\alpha \cup J^\alpha$.

If a^α, b^α and c^α belongs the set R^α of real line numbers, then

(1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belongs the set R^α;

(2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;

(3) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$;

(4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;

(5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;

(6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;

(7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2.1. *[20] A non-differentiable function* $f : R \to R^\alpha$, $x \to f(x)$ *is called to be local fractional continuous at* x_0, *if for any* $\varepsilon > 0$, *there exists* $\delta > 0$, *such that*

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, *where* $\varepsilon, \delta \in R$. *If* $f(x)$ *is local continuous on the interval* (a, b), *we denote* $f(x) \in C_\alpha(a, b)$.

Definition 2.2. *[20] The local fractional derivative of* $f(x)$ *of order* α *at* $x = x_0$ *is defined by*

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(\alpha + 1)(f(x) - f(x_0))$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha ... D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, ...$

Definition 2.3. *[20] Let* $f(x) \in C_\alpha[a, b]$. *Then the local fractional integral is defined by,*

$${}_aI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha+1)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha+1)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t_j = t_{j+1} - t_j$ *and* $\Delta t = \max\{\Delta t_1, \Delta t_2, ..., \Delta t_{N-1}\}$, *where* $[t_j, t_{j+1}]$, $j = 0, ..., N-1$ *and* $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$ *is partition of interval* $[a, b]$.

Here, it follows that ${}_aI_b^\alpha f(x) = 0$ *if* $a = b$ *and* ${}_aI_b^\alpha f(x) = -{}_bI_a^\alpha f(x)$ *if* $a < b$. *If for any* $x \in [a, b]$, *there exists* ${}_aI_x^\alpha f(x)$, *then we denoted by* $f(x) \in I_x^\alpha[a, b]$.

Lemma 2.1. *[20] We have*

i) $\dfrac{d^\alpha x^{k\alpha}}{dx^\alpha} = \dfrac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k-1)\alpha)} x^{(k-1)\alpha}$;

ii) $\dfrac{1}{\Gamma(\alpha+1)} \int_a^b x^{k\alpha}(dx)^\alpha = \dfrac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right)$, $k \in R$.

The interested reader is able to look over the references [3], [8], [9], [12]- [16], [20]- [24] for local freactional theory.

In this study, generalized Steffensen's inequality is established. Then, some inequalities related generalized this inequality are given by using local fractional integrals.

3. Main Results

We start the following important inequality for local fractional integrals:

Theorem 3.1 (Generalized Steffensen's Inequality). *Let $f(x), g(x) \in I_x^{\alpha}[a,b]$ such that f never increases and $0 \le g(x) \le 1$ on $[a,b]$ with $a<b$. Then*

$$ {}_{b-\lambda}I_b^{\alpha} f(x) \le {}_{a}I_b^{\alpha} f(x)g(x) \le {}_{a}I_{a+\lambda}^{\alpha} f(x) \tag{3.1}$$

where

$$\lambda^{\alpha} = \Gamma(\alpha+1)\ {}_{a}I_b^{\alpha} g(x). \tag{3.2}$$

Proof. For the proof of theorem, we give two different methods:

First method: By direct computation, we get

$$\frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} f(x)(dx)^{\alpha} - {}_{a}I_b^{\alpha} f(x)g(x) \tag{3.3}$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} \left[f(x) - f(a+\lambda)\right]\left[1-g(x)\right](dx)^{\alpha}$$

$$+\frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^{b} \left[f(a+\lambda) - f(x)\right] g(x)(dx)^{\alpha}.$$

Using the equality (3.3), because f is nonincreasing, we obtain the second inequality of (3.1).

Similarly, we have

$${}_{a}I_b^{\alpha} f(x)g(x) - \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^{b} f(x)(dx)^{\alpha} \tag{3.4}$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_a^{b-\lambda} \left[f(x) - f(b-\lambda)\right] g(x)(dx)^{\alpha}$$

$$+\frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^{b} \left[f(b-\lambda) - f(x)\right]\left[1-g(x)\right](dx)^{\alpha}.$$

Using the equality (3.4), because f is nonincreasing, we obtain the first inequality of (3.1). Thus, the proof is completed.

Second method: Now, we prove the same of above Theorem in a deffirent way.

Because f is nonincreasing, the second inequality of (3.1) may be derived as follows:

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_a^{a+\lambda} f(x)(dx)^\alpha - {}_aI_b^\alpha f(x)g(x)\\
&= \frac{1}{\Gamma(\alpha+1)}\int_a^{a+\lambda} f(x)\left[1-g(x)\right](dx)^\alpha - \frac{1}{\Gamma(\alpha+1)}\int_{a+\lambda}^{b} f(x)g(x)(dx)^\alpha\\
&\geq \frac{f(a+\lambda)}{\Gamma(\alpha+1)}\int_a^{a+\lambda} \left[1-g(x)\right](dx)^\alpha - \frac{1}{\Gamma(\alpha+1)}\int_{a+\lambda}^{b} f(x)g(x)(dx)^\alpha\\
&= \frac{f(a+\lambda)}{\Gamma(\alpha+1)}\left[\lambda^\alpha - \int_a^{a+\lambda} g(x)(dx)^\alpha\right] - \frac{1}{\Gamma(\alpha+1)}\int_{a+\lambda}^{b} f(x)g(x)(dx)^\alpha\\
&= \frac{1}{\Gamma(\alpha+1)}\int_{a+\lambda}^{b} \left[f(a+\lambda)-f(x)\right]g(x)(dx)^\alpha \geq 0.
\end{aligned}
$$

The first inequality of (3.1) can be proved in a similar way. However, the second inequality implies the first.

Indeed, let

$$G(x) = 1 - g(x)$$

and

$$\Lambda^\alpha = \Gamma(\alpha+1)\ {}_aI_b^\alpha G(x).$$

Note that $0 \leq G(x) \leq 1$ if $0 \leq g(x) \leq 1$ in (a,b).

Suppose the second inequality of (3.1) holds. Then, we obtain

$$
\begin{aligned}
\frac{1}{\Gamma(\alpha+1)}\int_a^b f(x)G(x)(dx)^\alpha &\leq \frac{1}{\Gamma(\alpha+1)}\int_a^{a+\Lambda} f(x)(dx)^\alpha\\
\frac{1}{\Gamma(\alpha+1)}\int_a^b f(x)(dx)^\alpha - \frac{1}{\Gamma(\alpha+1)}\int_a^{a+\Lambda} f(x)(dx)^\alpha &\leq \frac{1}{\Gamma(\alpha+1)}\int_a^b f(x)g(x)(dx)^\alpha\\
\frac{1}{\Gamma(\alpha+1)}\int_{a+\Lambda}^b f(x)(dx)^\alpha &\leq \frac{1}{\Gamma(\alpha+1)}\int_a^b f(x)g(x)(dx)^\alpha.
\end{aligned}
$$

Because of

$$\Lambda^\alpha = \Gamma(\alpha+1)\ {}_aI_b^\alpha G(x) = b^\alpha - a^\alpha - \lambda^\alpha$$

we have the identity

$$\Lambda + a = b - \lambda. \tag{3.5}$$

From (3.5), we get the inequality

$$\frac{1}{\Gamma(\alpha+1)}\int_{b-\lambda}^{b} f(x)(dx)^\alpha \leq {}_aI_b^\alpha f(x)g(x)$$

which is the first inequality of (3.1). The proof is thus completed. □

In order to prove weighted version of generalized Steffensen's inequality we need the following lemma:

Lemma 3.1. *Let f, g and h belong to $I_x^\alpha[a,b]$. Suppose also that λ is a real number such that*

$${}_aI_{a+\lambda}^\alpha h(x) = {}_aI_b^\alpha g(x) = {}_{b-\lambda}I_b^\alpha h(x).$$

Then, we have

$$\begin{aligned} & {}_aI_b^\alpha f(x)g(x) \\ = & \frac{1}{\Gamma(\alpha+1)}\int_a^{a+\lambda}\left(f(x)h(x)-\left[f(x)-f(a+\lambda)\right]\left[h(x)-g(x)\right]\right)(dx)^\alpha \\ & +\frac{1}{\Gamma(\alpha+1)}\int_{a+\lambda}^{b}\left[f(x)-f(a+\lambda)\right]g(x)(dx)^\alpha \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} & {}_aI_b^\alpha f(x)g(x) \\ = & \frac{1}{\Gamma(\alpha+1)}\int_{b-\lambda}^{b}\left(f(x)h(x)-\left[f(x)-f(b-\lambda)\right]\left[h(x)-g(x)\right]\right)(dx)^\alpha \\ & +\frac{1}{\Gamma(\alpha+1)}\int_{a}^{b-\lambda}\left[f(x)-f(b-\lambda)\right]g(x)(dx)^\alpha. \end{aligned} \tag{3.7}$$

Proof. The essumptions of the Lemma imply that

$$a \le a+\lambda \le b \text{ and } a \le b-\lambda \le b.$$

Firstly, we prove the validity of the equality (3.6). Indeed, by direct computation, we find that

$$\begin{aligned} & {}_aI_b^\alpha f(x)g(x) \\ = & \frac{1}{\Gamma(\alpha+1)}\int_a^{a+\lambda} f(x)g(x)(dx)^\alpha+\frac{1}{\Gamma(\alpha+1)}\int_{a+\lambda}^{b} f(x)g(x)(dx)^\alpha \\ & +\frac{f(a+\lambda)}{\Gamma(\alpha+1)}\left(\int_a^b g(x)(dx)^\alpha-\int_a^{a+\lambda} g(x)(dx)^\alpha-\int_{a+\lambda}^{b} g(x)(dx)^\alpha\right) \end{aligned} \tag{3.8}$$

Now, if we apply the following assumption of the Lemma:

$${}_aI_{a+\lambda}^\alpha h(x) = {}_aI_b^\alpha g(x)$$

to (3.8), we obtain

$$\begin{aligned} & {}_aI_b^\alpha f(x)g(x) \\ = & \frac{1}{\Gamma(\alpha+1)}\int_a^{a+\lambda}\left(f(x)g(x)+f(a+\lambda)\left[h(x)-g(x)\right]\right)(dx)^\alpha \\ & +\frac{1}{\Gamma(\alpha+1)}\int_{a+\lambda}^{b}\left[f(x)-f(a+\lambda)\right]g(x)(dx)^\alpha. \end{aligned} \tag{3.9}$$

If we add

$$\frac{1}{\Gamma(\alpha+1)}\int_a^{a+\lambda} f(x)h(x)(dx)^\alpha-\frac{1}{\Gamma(\alpha+1)}\int_a^{a+\lambda} f(x)h(x)(dx)^\alpha$$

to right side of (3.9) and also we use elementary analysis, then we easily get the equality (3.6).

Secondly, if we apply above the operations for the following assumption of the Lemma

$${}_aI_b^\alpha g(x) = {}_{b-\lambda}I_b^\alpha h(x)$$

and also we consider the case $a \le b-\lambda \le b$, then we obtain the equality (3.7). Thus, the proof is completed. $\square$

Now, we prove weighted version generalized Steffensen's inequality using local fractional integrals.

Theorem 3.2. *Let f, g and h belong to $I_x^{\alpha}[a,b]$ with f nonincreasing. Suppose also that $0 \le g(x) \le h(x)$ for all $x \in [a,b]$. Then, we have the following inequalities*

$$\begin{aligned}
& {}_{b-\lambda}I_b^{\alpha} f(x)h(x) \\
\le\ & \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^{b} \left(f(x)h(x) - [f(x)-f(b-\lambda)]\,[h(x)-g(x)] \right) (dx)^{\alpha} \\
\le\ & {}_aI_b^{\alpha} f(x)g(x) \\
\le\ & \frac{1}{\Gamma(\alpha+1)} \int_{a}^{a+\lambda} \left(f(x)h(x) - [f(x)-f(a+\lambda)]\,[h(x)-g(x)] \right) (dx)^{\alpha} \\
\le\ & {}_aI_{a+\lambda}^{\alpha} f(x)h(x)
\end{aligned} \tag{3.10}$$

where λ is given by

$${}_aI_{a+\lambda}^{\alpha} h(x) = {}_aI_b^{\alpha} g(x) = {}_{b-\lambda}I_b^{\alpha} h(x).$$

Proof. In view of the assumptions that the function f is nonincreasing on $[a,b]$ and that $0 \le g(x) \le h(x)$ for all $x \in [a,b]$, we find that

$$\frac{1}{\Gamma(\alpha+1)} \int_{a}^{b-\lambda} [f(x)-f(b-\lambda)]\, g(x)(dx)^{\alpha} \ge 0, \tag{3.11}$$

$$\frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^{b} [f(b-\lambda)-f(x)]\,[h(x)-g(x)]\,(dx)^{\alpha} \ge 0, \tag{3.12}$$

$$\frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^{b} [f(x)-f(a+\lambda)]\, g(x)(dx)^{\alpha} \le 0, \tag{3.13}$$

and

$$\frac{1}{\Gamma(\alpha+1)} \int_{a}^{a+\lambda} [f(a+\lambda)-f(x)]\,[h(x)-g(x)]\,(dx)^{\alpha} \le 0. \tag{3.14}$$

Using the equality (3.7) together with the inequalities (3.11) and (3.12), we obtain that

$$\begin{aligned}
& {}_{b-\lambda}I_b^{\alpha} f(x)h(x) \\
\le\ & \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^{b} \left(f(x)h(x) - [f(x)-f(b-\lambda)]\,[h(x)-g(x)] \right) (dx)^{\alpha} \\
\le\ & {}_aI_b^{\alpha} f(x)g(x).
\end{aligned} \tag{3.15}$$

Using the equality (3.6) together with the inequalities (3.13) and (3.14) either, we get that

$$\begin{aligned}
& {}_aI_b^{\alpha} f(x)g(x) \\
\le\ & \frac{1}{\Gamma(\alpha+1)} \int_{a}^{a+\lambda} \left(f(x)h(x) - [f(x)-f(a+\lambda)]\,[h(x)-g(x)] \right) (dx)^{\alpha} \\
\le\ & {}_aI_{a+\lambda}^{\alpha} f(x)h(x).
\end{aligned} \tag{3.16}$$

Combining the inequalities (3.15) and (3.16), we easily deduce required inequalities. □

In particular, if we chose $h(t) = 1$ in (3.10), we obtain the following refinement of generalized Steffensen's inequality.

Corollary 3.1. *Let $f(x), g(x) \in I_x^\alpha [a,b]$ such that f never increases and $0 \le g(x) \le 1$ on $[a,b]$ with $a < b$. Then*

$$\begin{aligned}
& {}_{b-\lambda}I_b^\alpha f(x) \\
\le\ & \frac{1}{\Gamma(\alpha+1)} \int\limits_{b-\lambda}^{b} \left(f(x) - [f(x) - f(b-\lambda)]\,[1-g(x)]\right)(dx)^\alpha \\
\le\ & {}_aI_b^\alpha f(x)g(x) \\
\le\ & \frac{1}{\Gamma(\alpha+1)} \int\limits_{a}^{a+\lambda} \left(f(x) - [f(x) - f(a+\lambda)]\,[1-g(x)]\right)(dx)^\alpha \\
\le\ & {}_aI_{a+\lambda}^\alpha f(x)
\end{aligned}$$

where

$$\lambda^\alpha = \Gamma(\alpha+1)\ {}_aI_b^\alpha g(x).$$

Theorem 3.3. *Let f, g and h belong to $I_x^\alpha [a,b]$ with f nonincreasing. Also let*

$$0 \le \psi(x) \le g(x) \le h(x) - \psi(x)$$

for all $x \in [a,b]$. Then we have the inequalities

$$\begin{aligned}
& {}_{b-\lambda}I_b^\alpha f(x)h(x) + \frac{1}{\Gamma(\alpha+1)} \int\limits_{a}^{b} \left|[f(x) - f(b-\lambda)]\,\psi(x)\right|(dx)^\alpha \\
\le\ & {}_aI_b^\alpha f(x)g(x) \\
\le\ & {}_aI_{a+\lambda}^\alpha f(x)h(x) - \frac{1}{\Gamma(\alpha+1)} \int\limits_{a}^{b} \left|[f(x) - f(a+\lambda)]\,\psi(x)\right|(dx)^\alpha
\end{aligned}$$

where λ is given by

$${}_aI_{a+\lambda}^\alpha h(x) = {}_aI_b^\alpha g(x) = {}_{b-\lambda}I_b^\alpha h(x).$$

Proof. By the assumptions that the function f is nonincreasing on $[a,b]$ and that

$$0 \le \psi(x) \le g(x) \le h(x) - \psi(x)$$

for all $x \in [a,b]$, it follows that

$$\begin{aligned}
& \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} [f(x)-f(a+\lambda)]\,[h(x)-g(x)]\,(dx)^\alpha \qquad (3.17)\\
& +\frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^{b} [f(a+\lambda)-f(x)]\,g(x)(dx)^\alpha \\
= & \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} |f(x)-f(a+\lambda)|\,[h(x)-g(x)]\,(dx)^\alpha \\
& +\frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^{b} |f(a+\lambda)-f(x)|\,g(x)(dx)^\alpha \\
\geq & \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} |f(x)-f(a+\lambda)|\,\psi(x)(dx)^\alpha \\
& +\frac{1}{\Gamma(\alpha+1)} \int_{a+\lambda}^{b} |f(a+\lambda)-f(x)|\,\psi(x)(dx)^\alpha \\
= & \frac{1}{\Gamma(\alpha+1)} \int_a^{b} |[f(x)-f(a+\lambda)]\,\psi(x)|\,(dx)^\alpha.
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
& \frac{1}{\Gamma(\alpha+1)} \int_{b-\lambda}^{b} [f(b-\lambda)-f(x)]\,[h(x)-g(x)]\,(dx)^\alpha \qquad (3.18)\\
& +\frac{1}{\Gamma(\alpha+1)} \int_{a}^{b-\lambda} [f(x)-f(b-\lambda)]\,g(x)(dx)^\alpha \\
\geq & \frac{1}{\Gamma(\alpha+1)} \int_a^{b} |[f(x)-f(b-\lambda)]\,\psi(x)|\,(dx)^\alpha.
\end{aligned}$$

If we use the equalities (3.6) and (3.7) and the inequalities (3.17) and (3.18), we obtain required inequalities. □

Corollary 3.2. *Under the same assumptions of Theorem 3.3 with $h(x)=1$ and $\psi(x)=M^\alpha$, then the following inequalities hold:*

$$\begin{aligned}
& {}_{b-\lambda}I_b^\alpha f(x) + \frac{M^\alpha}{\Gamma(\alpha+1)} \int_a^b |[f(x)-f(b-\lambda)]|\,(dx)^\alpha \\
\leq & \ {}_aI_b^\alpha f(x)g(x) \\
\leq & \ {}_{a+\lambda}I_b^\alpha f(x) - \frac{M^\alpha}{\Gamma(\alpha+1)} \int_a^b |[f(x)-f(a+\lambda)]|\,(dx)^\alpha
\end{aligned}$$

where $M^\alpha \in \mathbb{R}_+^\alpha \cup \{0^\alpha\}$ and

$$\lambda^\alpha = \Gamma(\alpha+1)\ {}_aI_b^\alpha g(x).$$

Finally, we give a general result on a considerably improved version of generalized Steffensen's inequality by introducing the additional paramaters λ_1 and λ_2.

Theorem 3.4. *Let $f(x), g(x) \in I_x^\alpha [a,b]$ such that f never increases on $[a,b]$. Also let*

$$0^\alpha \leq \lambda_1^\alpha \leq \lambda^\alpha = \Gamma(\alpha+1)\ {}_aI_b^\alpha g(x) \leq \lambda_2^\alpha \leq (b-a)^\alpha$$

and

$$0 \leq M^\alpha \leq g(x) \leq (1-M)^\alpha$$

for all $x \in [a,b]$. Then, we have the inequalities

$$\begin{aligned} & {}_{b-\lambda_1}I_b^\alpha f(x) + \frac{f(b)}{\Gamma(\alpha+1)} (\lambda-\lambda_1)^\alpha + \frac{M^\alpha}{\Gamma(\alpha+1)} \int_a^b |[f(x)-f(b-\lambda)]| \, (dx)^\alpha \\ \leq & \quad {}_aI_b^\alpha f(x)g(x) \\ \leq & \quad {}_aI_{a+\lambda_2}^\alpha f(x) + \frac{f(b)}{\Gamma(\alpha+1)} (\lambda_2-\lambda)^\alpha - \frac{M^\alpha}{\Gamma(\alpha+1)} \int_a^b |[f(x)-f(a+\lambda)]| \, (dx)^\alpha . \end{aligned} \tag{3.19}$$

Proof. By direct computation, we obtain

$$\begin{aligned} & {}_aI_b^\alpha f(x)g(x) - {}_a I_{a+\lambda_2}^\alpha f(x) + \frac{f(b)}{\Gamma(\alpha+1)} \left(\lambda_2^\alpha - \int_a^b g(x)(dx)^\alpha \right) \\ = & \quad \frac{1}{\Gamma(\alpha+1)} \left(\int_a^b f(x)g(x)(dx)^\alpha - \int_a^{a+\lambda_2} f(x)(dx)^\alpha \right) \\ & + \frac{1}{\Gamma(\alpha+1)} \left(\int_a^{a+\lambda_2} f(b)(dx)^\alpha - \int_a^b f(b)g(x)(dx)^\alpha \right) \\ \leq & \quad \frac{1}{\Gamma(\alpha+1)} \int_a^b [f(x)-f(b)] \, g(x)(dx)^\alpha - \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} [f(x)-f(b)] \, (dx)^\alpha . \end{aligned} \tag{3.20}$$

Because of the following assumption of the Theorem

$$0^\alpha \leq \lambda_1^\alpha \leq \lambda^\alpha \leq \lambda_2^\alpha \leq (b-a)^\alpha ,$$

we find that

$$a^\alpha \leq a^\alpha + \lambda^\alpha \leq a^\alpha + \lambda_2^\alpha \leq b^\alpha$$

that is

$$a \leq a+\lambda \leq a+\lambda_2 \leq b.$$

Also, since f is nonincreasing, we have

$$f(x) - f(b) \geq 0$$

for all $x \in [a.b]$.

Onthe other hand, since the hypothesis of the Theorem, we we conclude that the function $f(x)-f(b)$ belong to $I_x^\alpha [a,b]$ and nonincreasing on $[a,b]$. Thus, substituting $f(x)-f(b)$ instead of $f(x)$ in Corollary

3.2, we find that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_a^b [f(x)-f(b)]\,g(x)(dx)^\alpha - \frac{1}{\Gamma(\alpha+1)}\int_{a+\lambda}^b [f(x)-f(b)]\,(dx)^\alpha \qquad (3.21)\\
\leq\ & -\frac{M}{\Gamma(\alpha+1)}\int_a^b |[f(x)-f(a+\lambda)]|\,(dx)^\alpha.
\end{aligned}$$

Combining the inequalities (3.20) and (3.21), we obtain

$$\begin{aligned}
&{}_aI_b^\alpha f(x)g(x) -{}_a I_{a+\lambda_2}^\alpha f(x) + \frac{f(b)}{\Gamma(\alpha+1)}\left(\lambda_2^\alpha - \lambda^\alpha\right)\\
\leq\ & -\frac{M^\alpha}{\Gamma(\alpha+1)}\int_a^b |[f(x)-f(a+\lambda)]|\,(dx)^\alpha.
\end{aligned}$$

which is the second inequality of (3.19).

In a similar way, we can prove that

$$\begin{aligned}
&{}_aI_b^\alpha f(x)g(x) -{}_{b-\lambda_1} I_b^\alpha f(x) - \frac{f(b)}{\Gamma(\alpha+1)}\left(\int_a^b g(x)(dx)^\alpha - \lambda_1^\alpha\right)\\
\geq\ & \frac{1}{\Gamma(\alpha+1)}\int_a^b [f(x)-f(b)]\,g(x)(dx)^\alpha + \frac{1}{\Gamma(\alpha+1)}\int_{b-\lambda}^b [f(b)-f(x)]\,(dx)^\alpha\\
\geq\ & \frac{M^\alpha}{\Gamma(\alpha+1)}\int_a^b |[f(x)-f(b-\lambda)]|\,(dx)^\alpha
\end{aligned}$$

which is the first inequality of (3.19). The proof is thus completed. □

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