FREE VIBRATION OF VARIABLE THICKNESS PLATES USING CHARACTERISTIC ORTHOGONAL POLYNOMIAL STRIP FUNCTIONS SUBJECTED TO DIFFERENT COMBINATIONS OF BOUNDARY CONDITIONS

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ABSTRACT: A set of characteristic orthogonal polynomial strip functions was used to study the flexural vibration of variable thickness plates in one direction. The starting function in the set, which is satisfying the geometry and natural conditions, is developed from the solution of a static beam subjected to an arbitrary load that expanded into a Taylor series. The Gram-Schmidt orthogonalization process is used to generate the reminder functions of the set. The natural frequencies of the plat are calculated by using the finite strip method together with the transition matrix. Numerical results are obtained for different combination of boundary conditions, tapered ratio and aspect ratio. It is demonstrated that the method as well as the orthogonal strip functions yield good accuracy and fast convergence by comparisons with available results in the literature. This technique can also be used to solve some other complicated problems in the engineering applications.

1. INTRODUCTION:

Plates of variables thickness are commonly used as structural elements in many engineering applications such as aerospace, civil, and ocean engineering systems. The natural frequencies and mode shapes of the plate are useful in their operation environments. In the literature, there are many publications concerning the flexural vibration of uniform and non-uniform plates, see for examples Lissisa [1-5]. In last decades, many of the researchers are used different admissible functions to obtain the natural frequencies of rectangular and skew plates of variable thickness. Ashour [6] used the eigen functions derived from the solution of a classical beam vibration as basis functions. Cheung and Zhou [7] used a set of admissible functions that are developed from static solution of beam under arbitrary Taylor series load in Rayleigh-Ritz method. Zhou [8] used a combination of sine series and polynomials as the basis functions in Rayleigh-Ritz method. Grossi and Bhat [9] used boundary characteristic orthogonal polynomials in Rayleigh-Ritz method and applied Rayleigh-Schmidt method in their analysis. Singh and Saxena [10-12] used boundary characteristic orthogonal polynomials in two variables in Rayleigh-Ritz method. Also Lewi and lam [13], and Bhat [14] used two-dimensional orthogonal polynomials. Malhotra [15] used the conventional beam functions as the basis functions. Bhat [16] used a set of orthogonal polynomial functions in Rayleigh-Ritz method.

In this paper, a set of orthogonal polynomial shape functions in x-direction is developed to study the free vibration of variable thickness plate in y-direction. The first strip function in the set that satisfies the geometry and natural boundary conditions is the first term in the series solution obtained by Cheung and Zhou [7] as a new set of admissible functions that are the static solution of the beam under an arbitrary static load expanded into Taylor series. The rest of the orthogonal polynomial strip functions are generated through Gram-Schmidt orthogonalization process, which are also satisfy the geometry and natural boundary conditions. It is shown that, this set is the same as developed by Bhat [16]. The finite strip method in conjunction with transition matrix [17-18] is used to calculate the natural frequencies of the plate for different boundary conditions, tapered ratio and aspect ratio. Good accuracy and fast convergence for all cases presented in this paper are demonstrated when compared with available results in the literature.

2. THE CHARACTERISTIC ORTHOGONAL POLYNOMIAL STRIP FUNCTIONS:

One of the admissible functions that are generated by Cheung and Zhou [7] is adopted in the present work. Consider a beam of length equal l and unit width under an arbitrary static load $p(x)$. If the width and the depth of the beam are small compared to its length, the classical beam theory is valid. The non-dimensional

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differential equation that is governing the static deflection of the beam may be put in the form

$$\frac{d^4 y}{d\xi^4} = P(\xi) \tag{1}$$

where $\xi = \frac{x}{l}$, $P(\xi) = \frac{l^4}{EI} p(\xi\, l)$ and EI is the flexural rigidity of the beam. The arbitrary static load $P(\xi)$ can be expanded into Taylor series as

$$P(\xi) = \sum_{i=0}^{\infty} P_i^{'} \xi^i \tag{2}$$

where $P_i^{'}$ are the constants that can be determined if $P(\xi)$ is a given function. Using equation (2) into equation (1), the general solution of the differential equation can be put in the form

$$y(x) = \sum_{i=0}^{\infty} P_i [b_0^i + b_1^i x + b_2^i x^2 + b_3^i x^3 + x^{i+4}] \tag{3}$$

The first term in the series solution will be taken as the first strip function in the set of orthogonal polynomial strip functions, i.e.

$$y_o(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3 + x^4 \tag{4}$$

Where $a_i (i = 0, 1, 2, 3)$ are arbitrary constants to be determined from the boundary conditions. For clamped ends

$$y_o(0) = y_o(1) = \frac{dy_o(0)}{dx} = \frac{dy_o(1)}{dx} = 0.0 \tag{5}$$

Then the strip function (4) becomes

$$y_o(x) = x^2 - 2x^3 + x^4 \tag{6}$$

Equation (6) is the same as the strip function developed by Bhat [16]. The subsequent orthogonal polynomial functions are generated from Gram-Schmidt process [19]. The first orthogonal function $y_1(x)$ is obtained from

$y_1(x) = (x - B_1) y_o(x)$

where $B_1 = \int_0^1 x\, y_o^2\, dx \Big/ \int_0^1 y_o^2\, dx$. The higher terms in the set can be obtained by the recurrence relation between three successive members of the orthogonal polynomials set, i.e. $y_k(x) = (x - B_k) y_{k-1} - C_k y_{k-2}$, $k \geq 2$ where

$$B_k = \int_0^1 x\, y_{k-1}^2\, dx \Big/ \int_0^1 y_{k-1}^2\, dx,$$

$$C_k = \int_0^1 x\, y_{k-1} y_{k-2}\, dx \Big/ \int_0^1 y_{k-2}^2\, dx.$$

All the orthogonal polynomials that generated are satisfied the geometry and natural boundary conditions of the beam.

3. PROBLEM FORMULATION:

Assume a homogenous, isotropic, and linearly elastic rectangular plate in (x-y) plane as depicted in Fig. 1. The plate is of uniform in x-direction and varying thickness $h(y)$ in y-direction. Under the assumption of the classical deformation theory, the partial differential equation governing the vibration of such plate is given by

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}\left[D(y)\left(\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2}\right)\right] \\ &+ \frac{\partial^2}{\partial y^2}\left[D(y)\left(\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2}\right)\right] \\ &+ 2\frac{\partial^2}{\partial x \partial y}\left[D(y)(1-\nu)\frac{\partial^2 W}{\partial x \partial y}\right] = -\rho h(y) \frac{\partial^2 W}{\partial t^2}\end{aligned} \tag{7}$$

where $D(y) = \frac{E h^3(y)}{12(1-\nu^2)}$ is the flexure rigidity of the plate, ν is the Poisson's ratio, E the Young's modulus, ρ is the material density, and W is the deflection of the plate. By normalizing the coordinate system with respect to the plate dimensions as follows: $\xi = \frac{x}{a}$ and $\eta = \frac{y}{b}$ and the variable thickness $h(y)$ with respect to the thickness of the plate h_o at $\eta = 0$ as $f(\eta) = \frac{h(\eta)}{h_0}$, equation (7) can be written as

$$\begin{aligned}&W_{\xi\xi\xi\xi} + 2\beta^2 W_{\xi\xi\eta\eta} + \beta^4 W_{\eta\eta\eta\eta} + 2\beta^2 f_1(\eta) + \left\{ \beta^2 W_{\eta\eta\eta} + W_{\xi\xi\eta} \right\} \\ &+ \beta^2 f_2(\eta) \left\{ \beta^2 W_{\eta\eta} + \nu W_{\xi\xi} \right\} - \lambda^2 f_3(\eta) W = 0,\end{aligned} \tag{8}$$

where $\lambda^2 = \frac{\rho h_o a^4 \omega^2}{D_0}$ is the square of the natural frequency of the plate , D_o is the flexural rigidity of the plate at $\eta = 0$, $\beta = \frac{a}{b}$ is the aspect ratio of the plate and

$$\left.\begin{aligned} f_1(\eta) &= \frac{1}{f^3(\eta)} \frac{df^3(\eta)}{d\eta} \\ f_2(\eta) &= \frac{1}{f^3(\eta)} \frac{d^2 f^3(\eta)}{d\eta^2} \\ f_3(\eta) &= \frac{1}{f^2(\eta)} \end{aligned}\right\} \tag{9}$$

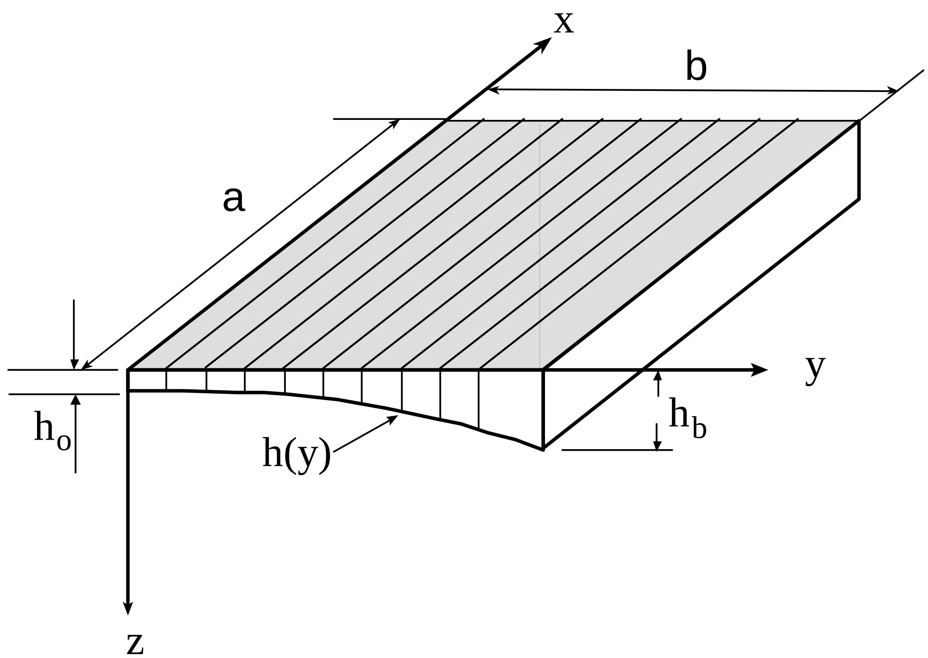

Fig. 1 Geometry of rectangular plate of variable thickness

For a plate striped in the x-direction as shown in Fig.1, the shape function $W(x,\eta)$ may be assumed in the form

$$W(x,\eta) = \sum_{i=0}^{N} X_i(x) Y_i(\eta) \tag{10}$$

where $Y_i(\eta)$ are the unknown functions in η-direction and $X_i(x)$ are the strip functions in x-direction that are developed in the previous section. Substituting equation (10), in equation (8), multiplying both sides by $X_j(x)$, and integrating from 0 to 1, we may obtain

$$\sum_{i=0}^{N} q_{ij}Y_i + 2b^2 n_{ij} Y_i'' + b^4 m_{ij} Y_i^{iv} + 2b^2 \phi_1 \left\{ b^2 m_{ij} Y_i''' + n_{ij} Y_i' \right\} + b^2 \phi_2 \left\{ b^2 m_{ij} Y_i'' + \nu n_{ij} Y_i \right\} - \lambda^2 \phi_3 m_{ij} Y_i = 0$$

$$j = 0, 1, 2,.., N \tag{11}$$

where

$$m_{ij} = \int_0^1 X_i(x) X_j(x)\,dx\,,$$

$$n_{ij} = \int_0^1 \frac{d^2 X_i(x)}{dx^2} X_j(x)\,dx\,, \tag{12}$$

$$q_{ij} = \int_0^1 \frac{d^4 X_i(x)}{dx^4} X_j(x)\,dx$$

From the orthogonality properties, $m_{ij} = q_{ij} = 0$ for $i \neq j$, this is true for clamped-clamped boundary conditions in the x-direction. Therefore, equation (11) becomes

$$\frac{d^4 Y_j}{d\eta^4} + \frac{1}{b^2} \sum_{i=0}^{N} \left\{ \left[Y_i'' + 2\phi_1 Y_i' + \nu \phi_2 Y_i \right] \frac{n_{ij}}{m_{jj}} \right\} + 2\phi_1 Y_j''' + \phi_2 Y_j'' + \frac{1}{b^4} \left[\frac{q_{jj}}{m_{jj}} - \lambda^2 \phi_3 \right] Y_j = 0.0$$

$$j = 0, 1, 2, ..N \tag{13}$$

The above system of fourth order differential equations can be rewritten as 4-N first order differential equations at any strip k as:

$$\frac{d\mathbf{Y}_k}{d\eta} = \{A\}_k \mathbf{Y}_k \tag{14}$$

where the coefficients of the matrix $\{A\}_k$, in general, are functions of η and the eigenvalue parameter λ. The vector $\mathbf{Y}_k$ is given by

$$\mathbf{Y}_k = \left[\bar{Y}_1 \bar{Y}_2 .. \quad \bar{Y}_i .. \quad \bar{Y}_N \right] \tag{15}$$

where

$$\bar{Y}_i = \left[Y_i\ Y_i'\ Y_i''\ Y_i''' \right] \tag{16}$$

Solving the above system of first order differential equations using the transition matrix technique [20] at any strip k, with boundaries $k-1$ and k yield

$$\mathbf{Y}_k = [\mathbf{B}]_k \ \mathbf{Y}_{k-1} \tag{17}$$

where $[\mathbf{B}]_k$ is called the transition matrix of the strip k and $\mathbf{Y}_k$, $\mathbf{Y}_{k-1}$ are the nodal vectors of the boundaries k, $k-1$.

4. BOUNDARY CONDITIONS:

In this paper any combination of the classical boundary conditions such as simply supported, clamped, or free boundary conditions at either y = 0 or y = b is considered. After normalizing the boundary conditions and using equation (10) , we may have

$$\frac{d^2 Y_j}{d\eta^2} = 0 \ , Y_j = 0 \quad (\eta = 0,\ \eta = 1), \qquad (18)$$

For simply supported edges,

$$\frac{d Y_j}{d\eta} = 0 \ , \ Y_j = 0 \quad (\eta = 0,\ \eta = 1). \qquad (19)$$

For camped edges, and

$$\begin{aligned} m_{jj}\frac{d^2Y_j}{d\eta^2} + c_{f_1}\sum_{i=0}^{N} n_{ij}Y_i &= 0 \\ m_{jj}\frac{d^3Y_j}{d\eta 3} + c_{f_2}\sum_{i=0}^{N} n_{ij}\frac{dY_i}{d\eta} &= 0 \end{aligned} \qquad (\eta = 0,\ \eta = 1) \qquad (20)$$

where $c_{f_1} = \frac{\nu}{b^2}$ and $c_{f_2} = \frac{(2-\nu)}{b^2}$ for free edges.

5. NUMERICAL RESULTS AND CONCLUSION:

To illustrate the validity of the proposed set of orthogonal polynomial strip functions in finite strip method, the natural frequencies for the free vibration plate of variable thickness in y-direction are calculated and compared with the available results obtained by different numerical methods. Numerical results are obtained for different combinations of classical boundary conditions, aspect ratio and tapered ratio.

A linearly tapered plate is assumed with non-dimensional variable thickness function defined by

$$f(\eta) = (1 + \delta\eta) \qquad (21)$$

where δ is the taper ratio given by $\delta = \frac{h_b - h_0}{h_0}$, where h_b is the thickness at $\eta = 1$.

To investigate the accuracy and the convergence of the proposed admissible functions, the first four natural frequencies are calculated for different two cases, CSCS with taper ratio $\delta = 0.5$, and CCCC with taper ratio $\delta = 0.4$ as shown in table 1. Different terms of the series solution in equation (10)(N) are considered. Some of the results that are obtained by using classical beam functions are calculated and presented to compare the convergent between the two admissible functions. Also, the results compared with that obtained from Grossi and Bhat [9] and Singh and Saxena [11]. For the purpose of brevity, the symbolism CFCS for example, means a plate with varying thickness in the y-direction having clamped, free, clamped and simple supported edges at the boundaries, x= 0, y = b, x =a, and y = 0, respectively (see Fig. 1). In all calculations, a Poisson's ratio $\nu = 0.3$ was used.

Table 2 demonstrates the natural frequencies for several combinations of boundary conditions, for two taper ratio δ (0,0.5), and two aspect ratio β (1,2). A comparison is made with some results obtained from other numerical methods. Good accuracy has been seen for all the cases considered in this paper.

Table 1: Convergence of the first four frequencies of square plate of variable thickness and comparison with data from different techniques (CSCS and CCCC plates)

	N	β	δ	λ_1	λ_2	λ_3
Classic	1	1	.5	36.0213	68.3958	
CSCS	2			36.0213	68.3958	85.2728
	3			35.9716	68.0789	85.2728
	4			35.9716	68.0789	85.2411
Polynomial	1			35.9672	68.1654	86.2478
CSCS	3			35.9227	67.8702	86.2478
	5			35.9173	67.8270	86.2303
	8			35.9165	67.8188	86.2229
Singh and Saxena [11]				35.9640	68.033	85.2000
Polynomial	1	1	.4	42.9255	87.6976	
CCCC	3			42.8647	87.4200	88.2030
	5			42.8547	87.3755	88.1805
Singh and Saxena [11]				42.9130	87.2900	87.5300

Table 2: The First four frequencies λ_i (i = 1,2,3,4) of tapered plate in y-direction for different boundary conditions

N		β	δ	λ_1	λ_2	λ_3	λ_4
5	CCCS	1	0.0	31.7809	63.1909	71.8811	100.879
5			0.5	39.6479	89.7691	77.9645	
5		2	0.0	73.2638	108.272	168.863	210.392
5			0.5	88.0933	134.256	211.415	
	Singh and Saxena [11]			88.2750	134.190	206.890	
5	CSCC	1	0.5	39.2255	78.7977	87.3291	
	Singh and Saxena [11]			39.2820	78.9690	86.404	
5		2	0.5	93.6774	134.575	206.9541	93.6773
	Singh and Saxena [11]			93.8310	134.520	202.580	
5	CFCC	1	0.0	24.2741	40.2840	62.9298	
5			0.5	27.6150	49.8425	68.0844	95.9289
5		2	0.0	31.8625	63.3429	103.4920	127.067
5			0.5	39.5637	75.8020	133.582	144.134
5	CFCF	1	0.0	22.3639	26.7401	43.8835	62.9323
5			0.5	26.8121	34.3755	54.7061	68.0833
5		2	0.0	22.3197	36.5568	62.9065	62.9389
5	CFCS	1	0.5	27.1122	44.2948	68.0726	91.3042
5		2	0.0	26.9892	62.9285	80.0701	108.2721
5			0.5	32.0426	74.2904	101.6199	142.2618
5	CSCS	1	0.0	28.9125	54.5913	70.1724	94.7093
5			0.5	35.9173	67.8270	86.2303	126.5393
5		2	0.0	54.5901	94.7081	158.9818	170.1827
5			0.5	67.7910	117.7756	196.9454	211.1377

6. CONCLUSION REMARKS:

The finite strip transition matrix method (FSTM) by using new set of orthogonal polynomial strip functions presented in this paper is used to investigate the flexure vibration of plate of generally variable thickness with general classical boundary conditions. The method is easily implemented in a computer program and yield a fast convergence and reliable results. The results obtained by the proposed method are compared with known results from different numerical and approximate methods. Excellent agreement has been achieved for all plate configurations considered in this paper. The comparison results illustrate the high accuracy and efficiency of the present method. Other complexity effects such as orthotropic, edges restrained against translation and rotation, concentrated masses, etc., can be easily implemented in the present method.

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