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On Almost Bounded Submodules

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Abstract

Let R be a commutative ring with identity, and let M be a unitary R-module. We introduce a concept of almost bounded submodules as follows: A submodule N of an R-module M is called an almost bounded submodule if there exists $x\in M$, $x\notin N$ such that $ann_R(N)=ann_R(x)$.

In this paper, some properties of almost bounded submodules are given. Also, various basic results about almost bounded submodules are considered.

Moreover, some relations between almost bounded submodules and other types of modules are considered.

Introduction

Every ring considered in this paper will be assumed to be commutative with identity and every module is unitary. We introduce the following: A submodule N of an R-module M is called an almost bounded submodule, if there exists $x\in M$, $x\notin N$ such that $ann_R(N)=ann_R(x)$, where $ann_RN=\{r:r\in R \text{ and } rN=0\}$.

Our concern in this paper is to study almost bounded submodules and to look for any relation between almost bounded submodules and certain types of well-known modules especially with prime modules.

This paper consists of two sections. Our main concern in section one, is to define and study almost bounded submodules. Also, we give some basic results for this concept.

In section two, we study the relation between almost bounded submodules and bounded modules. We show that the proper submodule of bounded module is not necessary to be almost bounded submodule and we give some conditions under which a proper submodule of bounded module is an almost bounded submodule. Next we investigate the relationships between almost bounded submodules, prime and fully stable module.

1- Basic Properties of Almost Bounded Submodules

In this section, we introduce the concept of almost bounded submodule. We establishe some basic properties of this concept.

First, we introduce the following definition.

1.1 Definition:

A proper submodule N of an R-module M is called almost bounded submodule if there exists $x\in M$, $x\notin N$ such that $ann_R(N)=ann_R(x)$.

An ideal I of a ring R is an almost bounded ideal if I is an almost bounded R-submodule.

1.2 Remarks and Examples:

1. Let $M=Z\oplus Z$ as a Z-module and $N=2Z\oplus 0$ be a submodule of M. Then N is an almost bounded submodule.
2. Every submodule of the Z-module Z is an almost bounded submodule.

Key words: almost bounded submodule, bounded module, prime module, quasi-prime module, fully stable module.

3. Consider the Z-module $M=Z\oplus Z_p$, where p is a prime number and the Z-suubmodule $N=qZ\oplus Z_p$, where q is any prime number. Then N an almost bounded sumodule.
4. For each positive integer n and n is not prime number, every proper submodule of a Z_n-module Z_n is not almost bounded submodule.
5. $<\bar{2}>$ as a Z-submodule of Z_{12} is not almost bounded. In general, let n be a positive integer, then the Z-module Z_n has no proper almost bounded submodule.
6. Let p be a prime number. The Z-module Z_p dose not contain any proper almost bounded submodule.

The following remark ensures that the almost boundedness property is not hereditary.

1.3 Remark:

A submodule of an almost bounded submodule need not be almost bounded in general. For example:

$M=Z\oplus Z_p$ as a Z-module, where p any prime number, $N=qZ\oplus Z_p$ be a submodule of M, where q is any prime number. Then N is an almost bounded submodule of M, but $K=0\oplus Z_p$ as a submodule of N which is not almost bounded submodule of N.

We state and prove the following proposition.

1.4 Proposition:

Let M_1 and M_2 be two R-modules, $M=M_1\oplus M_2$. If N_1 and N_2 are almost bounded R-submodules of M_1 and M_2 respectively, then $N_1\oplus N_2$ is an almost bounded R-submodule of M.

Proof: We have N_1 and N_2 are almost bounded R-submodules of M_1 and M_2 respectively. Then there exists $x\in M_1$, $x\notin N_1$ such that $ann_R N_1=ann_R(x)$ and also there exists $y\in M_2$, $y\notin N_2$ such that $ann_R N_2=ann_R(y)$. Therefore $(x,y)\in M_1\oplus M_2$, $(x,y)\notin N_1\oplus N_2$. Now, $ann_R(x,y) = ann_R(x)\cap ann_R(y) = ann_R N_1 \cap ann_R N_2 = ann_R(N_1\oplus N_2)$. Hence $N_1\oplus N_2$ is an almost bounded R-submodule of M.

The converse of proposition (1.4) is not true in general as the following example shows.

1.5 Example:

Consider $M=Z_6\oplus Z_{12}$ as a Z-module. Let $N= N_1\oplus N_2=<\bar{3}>\oplus<\bar{2}>$ be a Z-submodule of M. Then N is an almost bounded submodule of M. Since $ann_Z N=ann_Z(<\bar{3}>\oplus<\bar{2}>)= ann_Z<\bar{3}>\cap ann_Z<\bar{2}>=2Z\cap 6Z=6Z$ and there exists $(\bar{2},\bar{2})\in M$, $(\bar{2},\bar{2})\notin N$ such that $ann_Z N=ann_Z(\bar{2},\bar{2})=ann_Z(\bar{2})\cap ann_Z(\bar{2})=3Z\cap 6Z=6Z$. But $N_1=<\bar{3}>$ and $N_2=<\bar{2}>$ is not almost bounded submodules of M_1 and M_2 respectively. Since for each $x\in Z_6$, $x=\bar{1},\bar{2},\bar{4},\bar{5}\notin N_1$, $ann_Z(\bar{1})=6Z$, $ann_Z(\bar{2})=3Z$, $ann_Z(\bar{4})=3Z$, $ann_Z(\bar{5})=6Z$. Therefore for each $x\in Z_6$, $x\notin N_1$ $ann_Z(x)\neq ann_Z N_1 = ann_Z<\bar{3}> =2Z$. Thus N_1 is not almost bounded submodule of M_1.

In the same way, N_2 is not almost bounded.

Using the mathematical induction, we obtain the following corollary.

1.6 Corollary:

Let $M_1, M_2, \ldots, M_n$ be a finite collection of R-modules and $M= M_1\oplus M_2\oplus\ldots\oplus M_n$. If N_1, N_2, … and N_n are almost bounded R-submodules of M_1, M_2, … and M_n respectively, then $N= N_1\oplus N_2\oplus\ldots\oplus N_n$ is an almost bounded submodule of M.

So, we have the following applications of (1.4)

1.7 Corollary:

Let N_1 and N_2 be two almost bounded submodules of an R-module M. Then $N_1\oplus N_2$ is an almost bounded submodule of $M\oplus M$.

Proof: We have N_1 and N_2 are almost bounded submodules of M, means there exists $x\in M$, $x\notin N_1$ such that $ann_R N_1=ann_R(x)$ and there exists $y\in M$, $y\notin N_2$ such that $ann_R N_2=ann_R(y)$, implies $(x,y)\notin N_1\oplus N_2$. Now, we claim that $ann_R(N_1\oplus N_2)=ann_R(x,y)$. Let $r\in ann_R(x,y)$. Then $r(x,y)=(0,0)$, implies $rx=0$ and $ry=0$. Therefore $r\in ann_R(x)= ann_R N_1$ and

$r\in ann_R(y)=ann_R N_2$. Thus $r\in ann_R N_1\cap ann_R N_2$. But $ann_R(N_1\oplus N_2)= ann_R N_1\cap ann_R N_2$, so we get $r\in ann_R(N_1\oplus N_2)$.

Conversely, let $r\in ann_R(N_1\oplus N_2)$. Then $r(a,b)=(0,0)$ for all $(a,b)\in N_1\oplus N_2$ which implies $ra=0$ for all $a\in N_1$ and $rb=0$ for all $b\in N_2$ which implies $rN_1=0$ and $rN_2=0$. Thus $r\in ann_R N_1$ and $r\in ann_R N_2$. But $ann_R N_1=ann_R(x)$ and $ann_R(y)=ann_R N_2$. This implies that $r\in ann_R(x)$ and $r\in ann_R(y)$, that is, $rx=0$ and $ry=0$. Thus $(rx,ry)=(0,0)$, so that $r(x,y)=(0,0)$. Hence $r\in ann_R(x,y)$. This completes the proof.

The following corollary is a special case of proposition (1.4).

1.8 Corollary:

Let M be an R-module, N be an almost bounded submodule of M. Then $N^2=N\oplus N$ is an almost bounded submodule of $M^2=M\oplus M$.

Proof: From hypothesis N is an almost bounded submodule of M. Then there exists $x\in M$, $x\notin N$ such that $ann_R N=ann_R(x)$. Thus $(x,x)\in M^2=M\oplus M$ and $(x,x)\notin N^2=N\oplus N$ since $ann_R(x,x)=ann_R(x)\cap ann_R(x)=ann_R N=ann_R(N\oplus N)$. Hence $ann_R(x,x)=ann_R(N\oplus N)$ which is what we wanted.

Now, we have the following proposition:

1.9 Proposition:

Let $M= M_1\oplus M_2$ be a direct sum of two R-modules M_1 and M_2. If L_1 is an almost bounded submodule of M_1 and $ann_R(y)=ann_R M_2$ for some $y\in M_2$, $y\neq 0$, then $L_1\oplus M_2$ is an almost bounded submodule of M.

Proof: We have L_1which is an almost bounded submodule of M_1, then there exists $x\in M_1$, $x\notin L_1$ such that $ann_R L_1=ann_R(x)$, $y\in M_2$. Then $(x,y)\in M_1\oplus M_2$ and $(x,y)\notin L_1\oplus M_2$. We claim that $ann_R(L_1\oplus M_2)=ann_R(x,y)$. Now to prove our assumption. Let $r\in ann_R(L_1\oplus M_2)= ann_R L_1\cap ann_R M_2$. Then $r\in ann_R L_1\cap ann_R M_2$, so $r\in ann_R L_1$ and $r\in ann_R M_2=ann_R(y)$. Therefore $r\in ann_R(x)$ and $r\in ann_R(y)$. Thus $rx=0$ and $ry=0$ means $(rx,ry)=(0,0)$, which implies $r(x,y)=(0,0)$ and hence $r\in ann_R(x,y)$.

Conversely, let $r\in ann_R(x,y)$. Then $r(x,y)=(0,0)$, which implies $(rx,ry)=(0,0)$. Therefore $rx=0$ and $ry=0$. Thus $r\in ann_R(x)=ann_R L_1$ and $r\in ann_R(y)=ann_R M_2$. Hence $r\in ann_R L_1\cap ann_R M_2$, which implies $r\in ann_R(L_1\oplus M_1)$. Therefore $r\in ann_R(L_1\oplus M_2)= ann_R(x,y)$.

Next, we have the following remark.

1.10 Remark:

A direct summand of almost bounded need not be an almost bounded.

For example:

It is known that $N=qZ\oplus Z_p$ is an almost bounded submodule of a Z-module M, where p,q is any prime numbers and $M=Z\oplus Z_p$. But Z_p is not almost bounded because Z_p has no proper almost bounded submodule.

We have seen by the following proposition that the class of almost bounded submodule is closed under homomorphic image and inverse image.

1.11 Proposition:

Let M and M' be two R-modules and let θ: $M \longrightarrow M'$ be an isomorphism. Then:

1. If N' is an almost bounded submodule of M', then $\theta^{-1}(N')$ is also almost bounded submodule of M.
2. If N is an almost bounded submodule of M, then $\theta(N)$ is an almost bounded submodule of M'.

Proof: 1. Assume that N' is an almost bounded submodule of M', then there exists $y\notin N'$ such that $ann_R(y)=ann_R N'$. Since θ is an epimorphisim, then there exists $x\in M$ such that $\theta(x)=y$. It is clear that $x\notin\theta^{-1}(N')$. We claim that $ann_R(\theta^{-1}(N'))=ann_R(x)$, let $r\in ann_R(x)$. Then $rx=0$, which implies $\theta(rx)=0$. Thus $r\theta(x)=0$. This means $r\in ann_R(\theta(x))=ann_R(y)=ann_R N'$. Thus $rN'=0$, which implies $\theta^{-1}(rN')=0$. Then $r\theta^{-1}(N')=0$ and implies $r\in ann_R(\theta^{-1}(N'))$.

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On the other hand, let $r \in ann_R(\theta^{-1}(N'))$. Then $r\theta^{-1}(N')=0$, which implies $\theta^{-1}(rN')=0$. This means $rN'=0$. Therefore $r \in ann_R N'=ann_R(y)=ann_R(\theta(x))$. Thus $r \in ann_R(\theta(x))$ and from this, we get $r\theta(x)=0$ which implies $\theta(rx)=0$. Then $rx=0$ and hence $r \in ann_R(x)$. Thus $ann_R(x)=ann_R(\theta^{-1}(N'))$ which completes the proof.

2. Suppose that N is an almost bounded submodule of M. Then $\exists x \in M$, $x \notin N$ such that $ann_R(x)=ann_R N$. Since $x \in M$, we get $\theta(x) \in M'$. We claim that $\theta(x) \notin \theta(N)$. Suppose that $\theta(x) \in \theta(N)$. Then $\theta(x)= \theta(n)$ for some $n \in N$, which implies that $\theta(x) - \theta(n)=0$, so that $\theta(x-n)=0$. Thus $x-n=\theta^{-1}(0)$ and hence $x-n=0$. Then $x=n \in N$. Therefore $x \in N$ which is a contradiction. Hence $\theta(x) \notin \theta(N)$. To show that $ann_R(\theta(N))=ann_R(\theta(x))$. Let $r \in ann_R(\theta(x))$.

Then $r\theta(x)=0$, which implies $\theta(rx)=0$. Thus $rx=0$, that is $r \in ann_R(x)=ann_R N$. Then $r \in ann_R N$, which implies that $rN=0$, so that $\theta(rN)=0$. Then $r\theta(N)=0$. Hence $r \in ann_R\theta(N))$. Therefore $ann_R(\theta(x)) \subseteq ann_R(\theta(N))$. By using the same way, we can prove the other inclusion. Hence $ann_R(\theta(N))= ann_R(\theta(x))$ which is what we wanted.

The condition (θ: M $\longrightarrow$ M' is an isomorphism) in proposition (1.11) can not be dropped as the following example shows.

1.12 Example:

1. Let $\theta: Z \oplus Z_4 \longrightarrow Z_4$ be a projection map such that $\theta(x,y)=y$ for all $(x,y) \in Z \oplus Z_4$. Let $N=<3> \oplus <\bar{2}>$ be a submodule of $Z \oplus Z_4$. It is easily to show that N is an almost bounded submodule of $Z \oplus Z_4$. But $\theta(N)$ is a submodule of Z_4 and it is not almost bounded submodule of Z_4 by (remarks and examples (1.2) (5)).

2. Let θ: $Z_4 \longrightarrow Z_4 \oplus Z$ be an injection map such that $\theta(x)=(x,0)$ for all $x \in Z_4$, let $N'=<\bar{2}> \oplus <3>$ be an almost bounded submodule of $Z_4 \oplus Z$. It is know that Z_4 has no proper almost bounded submodule. Since $(\theta^{-1}(N')$ is a submodule of Z_4, then $(\theta^{-1}(N')$ is not almost bounded submodule of Z_4 by (remarks and examples (1.2) (5)).

2- Modules Related to Almost Bounded Submodules

In this section, we study the relationships between almost bounded submoduls and bounded modules, prime and fully stable modules.

We start with the following definition which will be needed.

Recall that an R-module M is said to be bounded module, if there exists an element $x \in M$ such that $ann_R M=ann_R(x)$, [1].

By using this concept, we have the following.

2.1 Remark:

A submodule N of a bounded R-module M is not necessary be an almost bounded. For example Z_4 as a Z_4-module is bounded module, but $<\bar{2}>$ is not almost bounded submodule.

Recall that an R-module M is called a quasi-prime R-module if and only if $ann_R N$ is a prime ideal for each non-zero submodule N of M, [2].

Recall that a submodule N of an R-module M is called essential if $N \cap K \neq 0$ for every non-zero submodule K of M, [1].

The following proposition gives a sufficient condition under which every submodule of a bounded module is an almost bounded.

2.2 Proposition:

Let M be a cyclic quasi-prime R-module and N be a proper essential submodule of M. Then N is an almost bounded submodule.

Proof: Assume that N is proper submodule of an R-module M, then there exists $y \in M$, $y \notin N$. Since N is essential submodule of M, thus there exists $r \in R$, $r \neq 0$. Thus $ann_R ry \supseteq ann_R N$. But M quasi-prime, so $ann_R ry = ann_R y$. Then $ann_R y \supseteq ann_R N \supseteq ann_R M$. Let $t \in ann_R y$. Then $ty=0$, but M is cyclic. Thus $y=cx$ for some $c \in R$. Therefore $tcx=0$ which implies that $tc \in ann_R(x)$. Thus either $c \in ann_R(x)$ or $t \in ann_R(x)$. If $c \in ann_R(x)$, then $cx=y=0$. This is a contradiction. Thus $t \in ann_R(x) = ann_R M \subseteq ann_R N$. Therefore $ann_R(y) = ann_R N$ and hence N is an almost bounded submodule of M.

An R-module M is said to be uniform module if every nonzero submodule of M is essential, [1].

Now, we deduce the following corollary.

2.3 Corollary:

Let M be a cyclic uniform R-module and $ann_R M$ is prime ideal of R. Let N be a proper submodule of M. Then N is an almost bounded submodule.

Proof: The result follows from the definition of a uniform module, [2,Corollary (1.2.8)] and proposition (2.2).

Recall that an R-module M is said to be a multiplication module if for every submodule N of M, there exists an ideal I of R such that N=IM, [3].

An R-module M is called fully stable in case each submodule N of M is stable, where a submodule N is said to be stable, if $f(N)\subseteq N$ for each R-homomorphism $f:N\longrightarrow M$, [4].

So, we have the following proposition.

2.4 Proposition:

Let N be a proper submodule of an R-module M such that,

1. M is fully stable and bounded R-module.

2. $[N \underset{R}{:} M] \not\subseteq ann_R M$.

3. $ann_R M$ is prime ideal of R.

Then N is an almost bounded submodule of M.

Proof: From [1,corollary (1.1.9)], we get M is multiplication R-module and by [4,corollary (2.7)], we obtain $[ann_R M : ann_R(x)] \subseteq [(x) \underset{R}{:} M]$ for each $x\in M$.

Now, we have M is bounded. Then there exists $x\in M$ such that $ann_R M = ann_R(x)$. Therefore $[ann_R(x) \underset{R}{:} ann_R(x)] \subseteq [(x) \underset{R}{:} M]$, implies $R\subseteq [(x) \underset{R}{:} M]$. Thus $RM=<x>$ is cyclic. To prove N is an almost bounded submodule of M, we must show that $ann_R N = ann_R(x)$.

In the first, we claim that $x\notin N$. If $x\in N$, then $[(x) \underset{R}{:} M] \subseteq [N \underset{R}{:} M]$, but $[(x) \underset{R}{:} M]=R$. Therefore $[N \underset{R}{:} M]=R$, implies that $RM=[N \underset{R}{:} M]M=N$. Thus N=M which is a contradiction. Hence $x\notin N$.

It is easily to show that $ann_R(x)\subseteq ann_R N$.

On the other hand, let $r\in ann_R N$. Then rN=0 but M is multiplication [1,corollary (1.1.9)], then $r[N \underset{R}{:} M]M=0$ implies $r[N:M]\subseteq ann_R M$. But $ann_R M$ is prime ideal and $[N \underset{R}{:} M] \not\subseteq ann_R M$ by (2). Then $r\in ann_R M = ann_R(x)$ because M is bounded module. Thus $ann_R N = ann_R(x)$ and hence N is an almost bounded submodule of M.

The conditions $[N \underset{R}{:} M] \not\subseteq ann_R M$ and $ann_R M$ is prime ideal can not be dropped from proposition (2.4) as in the following example.

2.5 Example:

Let $M=Z_6$ as a Z-module. Since M is bounded Z-module, see [1] and M is fully stable Z-module, see [4,example and remarks (3.7),(c)], but $ann_Z M=6Z$ is not prime ideal of Z. Let $N_1=<\bar{2}>$ and $N_2=<\bar{3}>$. $[N_1 \underset{Z}{:} M]=[<\bar{2}> \underset{Z}{:} Z_6]= 2Z \not\subseteq ann_Z M=6Z$ and $[N_2 \underset{Z}{:} M]=[<\bar{3}> \underset{Z}{:} Z_6]=3Z \not\subseteq ann_Z M$. Therefore N_1, N_2 are not almost bounded submodules of M.

An R-module M is said to be I-multiplication if each submodule of M is of the form AM for some idempotent ideal A of R, [4].

As an immediate consequence of proposition (2.4).

2.6 Corollary:

Let N be a proper submodule N of an R-module M such that:

1. M is I-multiplication bounded module

2. $ann_R M$ is prime ideal of R.
3. $[N \underset{R}{:} M] \not\subseteq ann_R M$.

Then N is an almost bounded submodule of M.

Proof: The result follows according to [4,theorem (2.9)] and proposition (2.4).

Recall that an R-module M is called a prime module if $ann_R M = ann_R N$ for every non-zero submodule N of M, [5], [6].

2.7 Proposition:

Let M be a prime R-module and N, K be two submodules of M such that $N \subseteq K \subseteq M$, K is an almost bounded submodule of M. Then N is an almost bounded submodule of M.

Proof: Assume that K is almost bounded submodule of M, that is there exists $x \in M$, $x \notin K$ such that $ann_R K = ann_R(x)$. Since, $x \notin K$, $N \subseteq K$. Then we obtain $x \notin N$. To prove $ann_R N = ann_R(x)$. $ann_R K \subseteq ann_R N$ (since $N \subseteq K \subseteq M$), implies $ann_R(x) \subseteq ann_R N$. Hence $ann_R(x) \subseteq ann_R N$.

Now, let $r \in R$, $r \in ann_R N = ann_R M$ for each submodule N of M (since M is prime module), but $ann_R M \subseteq ann_R K = ann_R(x)$. Therefore $r \in ann_R(x)$. Thus $ann_R N \subseteq ann_R(x)$. Hence N is an almost bounded submodule of M.

So, we have the following application of (2.7).

2.8 Corollary:

Let M be a prime R-module and N, K be two submodules of M such that N is an almost bounded submodule of M. Then $N \cap K$ is also almost bounded submodule of M.

Proof: It is know that $N \cap K \subseteq N$. So according to proposition (2.7), $N \cap K$ is an almost bounded submodule of M.

As a generalization of corollary (2.8), we give the following corollary.

2.9 Corollary:

Let M be a prime R-module and $\{N_i\}_{i=1}^{n}$ be a finite collection of submodules of M such that N_i is an almost bounded submodule of M for some i, i=1,2,…,n. Then $\bigcap_{i=1}^{n} N_i$ is also almost bounded submodule f M.

Proof: The proof is by induction on n and corollary (2.8).

The following example shows that the intersection of an infinite collection of almost bounded submodules of M need not be almost bounded submodule of M.

2.10 Example:

Consider Z as a Z-module, Z is prime Z-module. Since pZ is an almost bounded of Z, for each p where p is a prime number. However $\bigcap_{p\,is\,prime} pZ = 0$ is not almost bounded submodule of Z.

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الخلاصة

لتكن R حلقة ابدالية ذي عنصر محايد، وليكن M مقاساً احادياً أيسراً على الحلقة R. في هذا البحث قدمنا مفهوم مقاس جزئي مقيد تقريباً كما ياتي: يطلق على المقاس الجزئي N من المقاس M مقيد تقريباً اذا وجد عنصر $x \in M$ و $x \notin N$ بحيث ان $ann_R(N)=ann_R(x)$. في هذا البحث، اعطيت بعض الخواص وكذلك دُرست العديد من النتائج الاساسية حول المقاسات الجزئية المقيدة تقريباً . فضلا عن الى هذا دُرست بعض العلاقات بينه وبين انواع اخرى من المقاسات.