Explicit Finite Difference Approximation for the Two-Dimensional Fractional Dispersion Equation

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Received in Feb,7,2010
Accepted in May,13,2010

Abstract

In this paper, we introduce and discuss an algorithm for the numerical solution of two-dimensional fractional dispersion equation. The algorithm for the numerical solution of this equation is based on explicit finite difference approximation. Consistency, conditional stability, and convergence of this numerical method are described. Finally, numerical example is presented to show the dispersion behavior according to the order of the fractional derivative and we demonstrate that our explicit finite difference approximation is a computationally efficient method for solving two-dimensional fractional dispersion equation.
Key words: Fractional derivative, Two-dimensional probem, Explicit Euler method, fractional dispersion equation, Stability, Convergence

Introduction

The space fractional dispersion equation is obtained from the classical dispersion equation by replacing the second space derivative by a fractional derivative. Numerical methods associated with integer-order differential equations have been treated extensively in the literature. On other hand, studies of the numerical methods and error estimates of fractional order differential equations are quite limited to date [1, 2, 3].

Many works by researchers from various fields of science and engineering deal with dynamical systems described by fractional partial differential equations, which have been used to represent many natural processes in physics [4], finance [5,6], and hydrology [7,8].

In this paper, we find the numerical solution of the two-dimensional fractional dispersion equation of the form:

$$\frac{\partial u(x,y,t)}{\partial t} = a(x,y)\frac{\partial^{\gamma} u(x,y,t)}{\partial x^{\gamma}} + b(x,y)\frac{\partial^{\beta} u(x,y,t)}{\partial y^{\beta}} + q(x,y,t) \qquad \text{............(1)}$$

subject to the initial condition

$$u(x,y,0) = f(x,y), \text{ for } x_0 < x < x_R \text{ and } y_0 < y < y_R \qquad \text{............(2)}$$

and the boundary conditions

$$\begin{aligned} &u(x_0,y,t) = 0, \text{ for } y_0 < y < y_R \text{ and } 0 \le t \le T \\ &u(x,y_0,t) = 0, \text{ for } x_0 < x < x_R \text{ and } 0 \le t \le T \\ &u(x_R,y,t) = g(y,t), \text{ for } y_0 < y < y_R \text{ and } 0 \le t \le T \\ &u(x,y_R,t) = k(x,t), \text{ for } x_0 < x < x_R \text{ and } 0 \le t \le T \end{aligned} \qquad \text{............(3)}$$

where a, b and f are known functions of x and y, g is a known function of y and t, k is aknwon function of x and t. γ and β are given fractional number. q is a knwon function of x, y and t.

We use a variation on the classical explicit Euler method. We prove this method by using a novel shifted version of the usual Grunwaled finite difference an approximation for the non-local fractional derivative operator.

Explicit Finite Difference Approximation for Solving the Two-Dimensional Fractional Dispersion Equation

In this section, we propose explicit finite difference approximation for solving the initial and boundary value problem two-dimensional fractional dispersion equation (1)-(3).

The finite difference method starts by dividing the x-interval $[x_0, x_R]$ into n subintervals to get the grid points $x_i = x_0 + i\Delta x$, where $\Delta x = (x_R - x_0)/n$ and $i=0,1,\dots,n$. Also we divide the y-interval $[y_0, y_R]$ into m subintervals to get the grid points $y_j = y_0 + j\Delta y$, where $\Delta y = (y_R - y_0)/m$ and $j=0,1,\dots,m$.

Also, the t-interval $[0,T]$ is divided into M subintervals to get the grid points $t_s = s\Delta t$, $s = 0, \dots, M$, where $\Delta t = T/M$.

Now, we evaluate eq.(1) at (x_i, y_j, t_s) and we use the explicit finite difference approximation to get

$$\frac{u(x_i,y_j,t_{s+1})-u(x_i,y_j,t_s)}{\Delta t} = a(x_i,y_j)\frac{\partial^{\gamma}u(x_i,y_j,t_s)}{\partial x^{\gamma}} + b(x_i,y_j)\frac{\partial^{\beta}u(x_i,y_j,t_s)}{\partial y^{\beta}} + q(x_i,y_j,t_s) \quad \text{.....(4)}$$

Then use the shifted Grunwald estimate to the γ, β - the fractional derivative, [9]:

$$\frac{\partial^{\gamma}u(x,y,t)}{\partial x^{\gamma}} = \frac{1}{(\Delta x)^{\gamma}}\sum_{k=0}^{M} g_{\gamma,k}u(x-(k-1)\Delta x, y, t) + O(\Delta x)$$

$$\dots\dots\dots\dots\dots\text{(5)}$$

$$\frac{\partial^{\beta}u(x,y,t)}{\partial y^{\beta}} = \frac{1}{(\Delta y)^{\beta}}\sum_{k=0}^{M} g_{\beta k}u(x, y-(k-1)\Delta y, t) + O(\Delta y)$$

to reduce eq.(4) as in the following form

$$\frac{u_{i,j}^{s+1}-u_{i,j}^{s}}{\Delta t} = a(x_i,y_j)\left[\frac{1}{\Delta x^{\gamma}}\sum_{k=0}^{i+1} g_{\gamma k}u_{i-k+1,j}^{s}\right] + b(x_i,y_j)\left[\frac{1}{\Delta y^{\beta}}\sum_{k=0}^{j+1} g_{\beta,k}u_{i,j-k+1}^{s}\right] + q(x_i,y_j,t_s)$$

$$\frac{u_{i,j}^{s+1}-u_{i,j}^{s}}{\Delta t} = \frac{a_{i,j}}{\Delta x^{\gamma}}\sum_{k=0}^{i+1} g_{\gamma,k}u_{i-k+1,j}^{s} + \frac{b_{i,j}}{\Delta y^{\beta}}\sum_{k=0}^{j+1} g_{\beta,k}u_{i,j-k+1}^{s} + q_{i,j}^{s},$$

$$i = 1,\dots,n-1,\ j = 1,\dots,m-1,\ s = 0,\dots,M \qquad \dots\dots\dots\dots\dots\text{(6)}$$

where $u_{i,j}^{s} = u(x_i,y_j,t_s)$, $a_{i,j} = a(x_i,y_j)$, $b_{i,j} = b(x_i,y_j)$, $q_{i,j}^{s} = q(x_i,y_j,t_s)$, $g_{\gamma,k} = (-1)^k\dfrac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{k!}$, k=0,1,2,… and $g_{\beta k} = (-1)^k\dfrac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}$, k=0,1,2,…

The resulting equation can be explicitly solved for $u_{i,j}^{s+1}$ to give

$$u_{i,j}^{s+1} = a_{i,j}\frac{\Delta t}{\Delta x^{\gamma}}\sum_{k=0}^{i+1} g_{\gamma,k}u_{i-k+1,j}^{s} + b_{i,j}\frac{\Delta t}{\Delta y^{\beta}}\sum_{k=0}^{j+1} g_{\beta,k}u_{i,j-k+1}^{s} + \Delta t q_{i,j}^{s} + u_{i,j}^{s} \qquad \dots\dots\dots\dots\dots\text{(7)}$$

Also form the initial condition and boundary conditions one can get

$u_{i,j}^{0} = f_{i,j}$, $i=0,\dots,n$

$u_{0,j}^{s} = 0$, $j=0,\dots,m$ and $s=1,\dots,M$

$u_{i,0}^{s} = 0$, $i=0,\dots,n$ and $s=1,\dots,M$

$u_{R,j}^{s} = g_{j}^{s}$, $j=0,\dots,m$ and $s=1,\dots,M$

$u_{i,R}^{s} = k_{i}^{s}$, $i=0,\dots,n$ and $s=1,\dots,M$

where $f_{i,j} = f(x_i, y_j, t_s)$, $g_j^s = g(y_j, t_s)$ and $k_i^s = (x_i, t_s)$

Stability Analysis of the Explicit Finite Difference Approximation

Define the following fractional partial difference operators:

$$\omega_{\gamma,x} u_{i,j}^{s} = \frac{a_{i,j}}{\Delta x^{\gamma}} \sum_{k=0}^{i+1} g_{\gamma,k} u_{i-k+1,j}^{s}$$

and

$$\omega_{\beta,y} u_{i,j}^{s} = \frac{b_{i,j}}{\Delta y^{\beta}} \sum_{k=0}^{j+1} g_{\beta,k} u_{i,j-k+1}^{s}$$

which is an $O(\Delta x)$ approximation to the $\gamma\, th$ fractional derivative and $O(\Delta y)$ approximation to the $\beta\, th$ fractional derivative term. Then eq.(7) may be written in the operator form

$$u_{i,j}^{s+1} = (1 + \Delta t\omega_{\gamma,x} + \Delta t\omega_{\beta,y}) u_{i,j}^{s} + \Delta t q_{i,j}^{s} \quad \text{................(8)}$$

eq.(8) may be written in form

$$u_{i,j}^{s+1} = (1 + \Delta t\omega_{\gamma,x})(1 + \Delta t\omega_{\beta,y}) u_{i,j}^{s} + \Delta t q_{i,j}^{s} \quad \text{..................(9)}$$

where

$$\underline{U}^{s} = [u_{1,1}^{s}, u_{2,1}^{s}, \dots, u_{n-1,1}^{s}, u_{1,2}^{s}, u_{2,2}^{s}, \dots, u_{n-1,2}^{s}, \dots, u_{1,m-1}^{s}, u_{2,m-1}^{s}, \dots, u_{n-1,m-1}^{s}]^{T}$$

To solve the problem for each fixed y_j to obtain an intermediate solution $u_{i,j}^{*}$ from

$$(1 + \Delta t\omega_{\gamma,x}) u_{i,j}^{*} + \Delta t q_{i,j}^{s} = u_{i,j}^{s+1} \quad \text{......................(10)}$$

Then solve for each fixed x_i

$$(1 + \Delta t\omega_{\beta,y}) u_{i,j}^{s} = u_{i,j}^{*} \quad \text{.......................(11)}$$

Now, we must prove each one-dimensional explicit system defined by the linear difference eqs. (10) and (11) is conditionally stable for all $1 < \gamma < 2$, $1 < \beta < 2$.

Theorem: The explicit system defined by the linear difference eqs.(10) and (11) with $1 < \gamma < 2$, $1 < \beta < 2$ is conditionally stable if

$$\frac{\Delta t}{\Delta x^{\gamma}} \le \frac{1}{\gamma a_{\max}} \quad \text{and} \quad \frac{\Delta t}{\Delta y^{\beta}} \le \frac{1}{\beta b_{\max}}$$

Proof:

At each grid point y_k, for $k = 1,\dots,m-1$, the system of equation defined by eq.(10) can be written in the explicit matrix form $\underline{U_k^{s+1}} = \underline{C_k}\,\underline{U_k^{*}} + \Delta t \underline{Q_k^{s}}$ where

$$\underline{U_k^{s+1}} = [u_{1,k}^{s+1}, u_{2,k}^{s+1}, \ldots, u_{n-1,k}^{s+1}]^T,$$

$$\underline{U_k^*} = [u_{1,k}^*, u_{2,k}^*, \ldots, u_{n-1,k}^*]^T,$$

$$\underline{Q_k^s} = [q_{1,k}^s, q_{2,k}^s, \ldots, q_{n-1\,k}^s]^T$$

$\underline{C_k}$ is the matrix of coefficients, and is the sum of a lower triangular matrix and a super diagonal matrix at the grid point y_k, where the matrix entries along the ith row are defined from eq.(10). For example, for $i = 1$ the equation becomes

$$u_{1,k}^{s+1} = \eta_{1,k} g_{\gamma,2} u_{0,k}^* + (1 + \eta_{1,k} g_{\gamma,1}) u_{1,k}^* + \eta_{1,k} g_{\gamma,0} u_{2,k}^* + \Delta t q_{1,k}^s$$

for $i = 2$ we have

$$u_{2,k}^{s+1} = \eta_{2,k} g_{\gamma,3} u_{0,k}^* + \eta_{2,k} g_{\gamma,2} u_{1,k}^* + (1 + \eta_{2,k} g_{\gamma,1}) u_{2,k}^* + \eta_{2,k} g_{\gamma,0} u_{3,k}^* + \Delta t q_{2,k}^s$$

and for $i = n - 1$ we get

$$u_{n-1,k}^{s+1} = \eta_{n-1,k} g_{\gamma,n} u_{0,k}^* + \cdots + \eta_{n-1,k} g_{\gamma,2} u_{n-2,k}^* + (1 + \eta_{n-1,k} g_{\gamma,1}) u_{n-1,k}^* +$$
$$\eta_{n-1,k} g_{\gamma,0} u_{n,k}^* + \Delta t q_{n-1,k}^s$$

Where the coefficients

$$\eta_{i,k} = a_{i,j} \frac{\Delta t}{\Delta x^\gamma},$$

Therefore the resulting matrix entries $C_{i,j}$ for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n-1$ by are defined by

$$C_{i,j} = \begin{cases} 1 + \eta_{i,k} g_{\gamma,1} & for \quad j = i \\ \eta_{i,k} g_{\gamma,2} & for \quad j = i-1 \\ \eta_{i,k} g_{\gamma,0} & for \quad j = i+1 \\ \eta_{i,k} g_{\gamma,j+1} & for \quad j < i+1 \end{cases}$$

According to the Greshgorin theorem [9], the eigenvalues of the matrix $\underline{C}$ lie in the union of the circles centered at $c_{i,i}$ with radius $r_i = \sum_{\substack{l=0 \\ l \neq i}}^{n} c_{i,l}$.

Here we have $c_{i,i} = 1 + \eta_{i,k} g_{\gamma,1} = 1 - \eta_{i,k}\gamma$ and $r_i = \sum_{\substack{l=0 \\ l \neq i}}^{n} c_{i,l} = \eta_{i,k} \sum_{\substack{l=0 \\ l \neq i}}^{i+1} g_{\gamma, i-l+1} \leq \eta_{i,k}\gamma$

and therefore $c_{i,i} + r_i \leq 1$. We also have $c_{i,i} - r_i \geq 1 - \eta_{i,k}\gamma - \eta_{i,k}\gamma = 1 - 2\eta_{i,k}\gamma$

$$= 1 - 2\left[a_{i,k} \frac{\Delta t}{\Delta x^\gamma}\right]\gamma \geq 1 - 2\left[a_{\max} \frac{\Delta t}{\Delta x^\gamma}\right]\gamma$$

Therefore, for the spectral radius of the matrix $\underline{C}$ to be at most one, it suffices to have

$$1 - 2\left[a_{\max} \frac{\Delta t}{\Delta x^\gamma}\right]\gamma \geq -1 \rightarrow \left[a_{\max} \frac{\Delta t}{\Delta x^\gamma}\right]\gamma \leq 1 \rightarrow [a_{\max}\gamma] \frac{\Delta t}{\Delta x^\gamma} \leq 1 \rightarrow \quad \frac{\Delta t}{\Delta x^\gamma} \leq \frac{1}{\gamma a_{\max}}$$

Same method above, resulting the system of equation defined by eq.(11) is then defined by

$$\underline{S_k}\,\underline{U_k^s} = \underline{U_k^*},$$

where

$$\underline{U_k^s} = [u_{k,1}^s, u_{k,2}^s, \ldots, u_{k,m-1}^s]^T,$$

$$\underline{U_k^*} = [u_{k,1}^*, u_{k,2}^*, \ldots, u_{k,m-1}^*]^T,$$

$\underline{S_k}$ is the matrix of coefficients, and is the sum of a lower triangular matrix and a super diagonal matrix at the grid point x_k for $k = 1,\ldots,n-1$. Therefore the resulting matrix entries $\underline{S_k}$ for $i = 1,2,\ldots,m-1$ and $j = 1,\ldots,m-1$ by are defined by

$$S_{i,j} = \begin{cases} 1+\psi_{k,i} g_{\beta,1} & for \quad j = i \\ \psi_{k,i} g_{\beta,2} & for \quad j = i-1 \\ \psi_{k,i} g_{\beta,0} & for \quad j = i+1 \\ \psi_{k,i} g_{\beta,j+1} & for \quad j < i+1 \end{cases}$$

where the coefficients

$$\psi_{i,k} = b_{i,j} \frac{\Delta t}{\Delta y^{\beta}}$$

So, and in the same way, according to the Greshgorin theorem [9], to get

$$\frac{\Delta t}{\Delta y^{\beta}} \leq \frac{1}{\beta b_{\max}}$$

Consistency and Convergent Analysis of the Explicit Finite Difference Approximation

We note that the three difference operators used in eq.(6) each have a local truncation error with $O(\Delta t)$, $O(\Delta x)$, and $O(\Delta y)$ respectively. The $O(\Delta t)$, for the time derivative term, is obtained from the classical Taylor's expansion. The $O(\Delta x)$ and $O(\Delta y)$ for the local truncation error of the fractional derivative terms was proved in [10]. Therefore, the explicit finite difference approximation is consistency. Theorem above shows that $(\omega_{\gamma,x}\omega_{\beta,y})u_{i,j}^{s}$ converges to the mixed fractional derivative linearly, as $O(\Delta x)+O(\Delta y)$. Therefore, the local truncation error of the explicit Euler method eq.(8) is $O(\Delta t)+O(\Delta x)+O(\Delta y)$.

This consistency of the explicit finite difference approximation together with the above result on conditional stability implies that the explicit finite difference approximation is convergent and this convergence is $O(\Delta x + \Delta y + \Delta t)$.

Numerical Example

In this section, numerical example is presented which confirm our theoretical results.

Example : Consider the two-dimensional fractional dispersion equation:

$$\frac{\partial u(x,y,t)}{\partial t} = \Gamma(0.5)x \frac{\partial^{1.5} u(x,y,t)}{\partial x^{1.5}} + \frac{\Gamma(1.4)y^{1.6}}{2} \frac{\partial^{1.6} u(x,y,t)}{\partial y^{1.6}} - x^{0.5} y^2 e^{2t} + xy^2 e^{2t}$$

subject to the initial condition

$u(x,y,0) = xy^2$, $0 < x < 0.5$, $0 < y < 0.5$

and the boundary conditions

$u(0,y,t) = 0$, $0 < y < 0.5$, $0 \leq t \leq 0.025$

$u(x,0,t) = 0,\ 0 < x < 0.5,\ 0 \le t \le 0.025$

$u(,0.5,y,t) = 0.5\, e^{2t},\ 0 < y < 0.5,\ 0 \le t \le 0.025$

$u(x,, 0.5,t) = 0.25e^{2t}x,\ 0 < x < 0.5,\ 0 \le t \le 0.025$

This fractional dispersion equation together with the above initial and boundary condition is constructed such that the exact solution is $u(x,y,t) = e^{2t}xy^2$.

Table (1) and (2) show the numerical solution using the explicit finite difference approximation. From table (1) and (2), it can be seen that thereisa good agreement between the numerical solution and exact solution.

6. Conclusions

In this paper, a numerical method for solving the two-dimensional fractional dispersion equation has been described and demonstrated. The explicit difference approximation is proved to be conditionally stable and converges. Furthermore numerical example is presented to show that good agreement between the numerical solution and exact solution has been noted.

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Table (1) The numerical solution of example by using the explicit finite difference approximation for $\Delta x = 0.1, \Delta y = 0.1$ **and** $\Delta t = 0.0125$

Numerical Solution	Exact Solution	Error
9.730E-4	1.02532 E -3	6.72814 E -2
7.876E-3	8.20252 E -3	3.26521 E -4
0.027	2.76835 E -2	6.83508 E -4
0.064	6.56202 E -2	1.62017 E -3
9.795E-4	1.05127 E -3	7.17711 E -5
7.759E-3	8.41017 E -3	6.51169 E -4
0.027	2.83843 E -2	1.38432 E 3
0.036	6.72814 E -2	3.12814 E -2

Table (2) The numerical solution of example by using the explicit finite difference approximation for $\Delta x = 0.125, \Delta y = 0.125$ **and** $\Delta t = 0.0125$

Numerical Solution	Exact Solution	Error
1.908E-3	2.00257 E -3	9.45686 E -5
0.015	1.60205 E -2	1.02055 E -3
0.052	5.40693 E -2	2.06935 E -3
1.929E-3	2.05326 E -3	1.24264 E -4
0.015	1.64261 E -2	1.42611 E -3
0.036	5.54381 E -2	1.94381 E -2

تقريب الفروق المنتهية الصريحةّ لمعادلة التشتت الكسرية ذات البعدين

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استلم البحث في 7 شباط 2010
قبل البحث في 13 ايار 2010

الخلاصة

في هذا البحث قدمنا وناقشنا خوارزمية للحل العددي لمعادلة التشتت الكسرية ذي البعدين. وان خوارزمية الحل العددي لتلك المعادلة قائمة على اساس تقريب الفروق المنتهية الصريحة. كما ناقشنا الاتساق ، والاستقرار الشرطي، والتقارب للطريقة العددية.

اخيرا قدمنا مثالا عدديا ليظهر سلوك التشتت حسب مرتبة الاشتقاق الكسرية وبينا أن تقريب الفروق المنتهية الصريحة هو طريقة فعالة حسابيا لمعادلة التشتت الكسرية ذي البعدين.

الكلمات المفتاحية: – مشتقة كسرية، مسألة ذي بعدين، طريقة اويلر الصريحة، معادلة التشتت الكسرية، الاستقرار التقارب.