

# Extend Differential Transform Methods for Solving Differential Equations with Multiple Delay

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## Abstract

In this paper, we present an approximate analytical and numerical solutions for the differential equations with multiple delay using the extend differential transform method (DTM). This method is used to solve many linear and non linear problems.

**Key words:** (differential transform method, solving differential equation, multiple delay).

## Introduction

In this paper, we extend the differential transform method (DTM) to solve the  $n$ th order differential equations with multiple delay of the form:

$$y^{(n)}(x) = f(x, y(x), y(x - r_1), y(x - r_2), \dots, y(x - r_m)) \quad , \quad m \in \mathbb{N} \quad \dots(1)$$

where  $y : I \longrightarrow \mathbb{R}$ ,  $f : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ ,  $I \subset \mathbb{R}$ ,  $r_i > 0$ ,  $i = 1, 2, \dots, m$

The differential transform method was first applied in the engineering domain in [1]. In general, the DTM is applied to find the solution of electric circuit problems [2]. The DTM is numerical method based on Taylor series expansion, which is constructed as an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the DTM obtains a polynomial series solution by means of an iterative procedure [3]. Recently the application of differential transform method is successfully extended to obtain approximate solutions to linear and nonlinear functional equations.

Delay differential equations are observed in many fields of science and technology, such as biology engineering and physics. Many dynamic population first order nonlinear scalar equation of the form

$$y'(x) = g(y(x)) - g(y(x - L)) \quad \dots(2)$$

may be used as a model for certain population growth if individuals have a constant life span  $L$ , where  $y(x)$  is the population size at time  $t$  and  $g(y)$  is the birth rate.

Recently, various methods such as, monotone iterative technique, a domain decomposition method and the spline functions method have been considered for approximate solutions of DDE [4].

Hence, due to practical reasons and the papers mentioned above, we have been motivated to deal with DDE and develop DTM for both linear and nonlinear delay differential equations. According to the best of our knowledge, DTM has not been studied for DDE till now. With this technique, it is possible to obtain highly accurate numerical solution, analytical solution and as well as exact solutions.

The aim this paper is to extend the method of differential transformation for solving differential equations with multiple delay of difference types as in equation (1).

## Differential Transform Method

The differential transform of a function  $y(x)$  is defined as, [2]:

$$Y(k) = \frac{1}{k!} \left[ \frac{d^k}{dx^k} y(x) \right]_{x=x_0}, \quad k \in \mathbb{N} \quad \dots(3)$$

where  $Y(k)$  refers to the differential transform of a given function  $y(x)$ , and  $x_0$  is the initial state.

Throughout this paper, we use the small and capital letters to represent the original and transformed functions, respectively.

The inverse of the differential transform  $Y(k)$  is defined by

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x - x_0)^k \quad \dots(4)$$

The automatic computation of DTM might be done. In this case, the following steps should be taken into consideration successively:

- i) The differential transform of each term in the DDE is computed ;
- ii) The recurrence equation is obtained ;
- iii)  $Y(0), Y(1), Y(2), Y(3), \dots$  are calculated by the recurrence equation and given initial condition ;
- iv) Finally, these values are substituted back into eq. (4).

## Preliminaries of the DTM

The following theorems give the properties of the DTM which can be easily derived from equations (3) and (4), for their proofs and more details (see [5], [6]).

**Theorem (1):-** If  $y(x) = f(x) \pm g(x)$ , then  $Y(k) = F(k) \pm G(k)$ .

**Theorem (2):-** If  $y(x) = c f(x)$ , then  $Y(k) = c F(k)$

**Theorem (3):-** If  $y(x) = \frac{d^n f(x)}{dx^n}$ , then  $Y(k) = \left[ \frac{(k+n)!}{k!} \right] F(k+n), k \in \mathbb{N}$ .

**Theorem (4):-** If  $y(x) = f(x).g(x)$ , then  $Y(k) = \sum_{k_1=0}^k F(k_1)G(k - k_1), k \in \mathbb{N}$ .

Moreover, we need the following theorem (see [4], [5]) :

**Theorem (5):-** If  $y(x) = f_1(x) \cdot f_2(x) \dots f_{n-1}(x) \cdot f_n(x)$ , then

$$Y(k) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} F_1(k) F_2(k_2 - k_1) \dots F_{n-1}(k_{n-1} - k_{n-2}) F_n(k_2 - k_1)$$

**Main Result**

The following theorems are proved to give the differential transform of given functions with constant delay.

Theorem (6):- The differential transform of  $y(x) = f(x - r)$ ,  $r \geq 1$ , is

$$Y(k) = \sum_{h_1}^N (-1)^{h_1-k} \binom{h_1}{k} r^{h_1-k} F(h_1), N \rightarrow \infty$$

**Proof:** - Using the definition of the inverse DTM where the definition given by eq. (4), we get:

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} F(k)(x - r - x_0)^k \\ &= F(0)(x - r - x_0)^0 + F(1)(x - r - x_0)^1 + F(2)(x - r - x_0)^2 + F(3)(x - r - x_0)^3 + \dots \\ &= [F(0) - rF(1) + r^2F(2) - r^3F(3) + \dots](x - x_0)^0 + [F(1) - 2F(2) + 3r^2F(3) - 4r^3F(4) + \dots] \\ &\quad (x - x_0)^1 + [F(2) - 3F(3) + 6r^2F(4) - 10r^3F(5) + \dots](x - x_0)^2 + \dots \\ &= \sum_{h_1=0}^{\infty} (-1)^{h_1} \binom{h_1}{0} r^{h_1-0} F(h_1)(x - x_0) + \sum_{h_1=1}^{\infty} (-1)^{h_1-1} \binom{h_1}{1} r^{h_1-1} F(h_1)(x - x_0)^1 + \\ &\quad \sum_{h_1=2}^{\infty} (-1)^{h_1-2} \binom{h_1}{2} r^{h_1-2} F(h_1)(x - x_0)^2 + \sum_{h_1=3}^{\infty} (-1)^{h_1-3} \binom{h_1}{3} r^{h_1-3} F(h_1)(x - x_0)^3 + \dots \\ y(x) &= \sum_{k=0}^{\infty} \sum_{h_1=k}^{\infty} (-1)^{h_1-k} \binom{h_1}{k} r^{h_1-k} F(h_1)(x - x_0)^k \quad \dots(5) \end{aligned}$$

Taking in to account eq. (4) and eq. (5), we have  $Y(k)$  as:

$$Y(k) = \sum_{h_1}^N (-1)^{h_1-k} \binom{h_1}{k} r^{h_1-k} F(h_1), N \rightarrow \infty$$

**Theorem (7):-** The differential transform of  $y(x) = f_1(x - r_1) \cdot f_2(x - r_2)$ , provided that  $r_1 > 0$  and  $r_2 > 0$  is:

$$Y(x) = \sum_{k_1}^k \sum_{h_1}^N \sum_{h_2=k-k_1}^{\infty} (-1)^{h_1+h_2-k} \binom{h_1}{k} \binom{h_1}{k_1} \binom{h_2}{k-k_1} r^{h_1-k} r^{h_2-k+k_1} F_1(h_1) F_2(h_2), \text{ for } N \rightarrow \infty$$

**Proof:** let the differential transforms of  $f_1(x - r_1)$  and  $f_2(x - r_2)$  at  $x = x_0$  be  $G_1(k)$  and  $G_2(k)$ , respectively.

Using theorem (4), we have the differential transform of  $y(x)$  as:

$$Y(x) = \sum_{k_1=0}^k G_1(k)G_2(k-k_1) \dots(6)$$

From theorem (6), we get:

$$G_1(k) = \sum_{h_1=k_1}^N (-1)^{h_1-k} \binom{h_1}{k_1} r^{h_1-k} F(h_1), \text{ for } N \rightarrow \infty$$

and

$$G_2(k-k_1) = \sum_{h_1=k-k_1}^N (-1)^{h_2-k-k_1} \binom{h_2}{k-k_1} r^{h_2-k+k_1} F(h_2), \text{ for } N \rightarrow \infty$$

Substituting these values in to equation (6), we obtain :

$$Y(x) = \sum_{k_1}^k \sum_{h_1}^N \sum_{h_2=k-k_1}^{\infty} (-1)^{h_1+h_2-k} \binom{h_1}{k} \binom{h_1}{k_1} \binom{h_2}{k-k_1} r^{h_1-k} r^{h_2-k+k_1} F_1(h_1)F_2(h_2), N \rightarrow \infty$$

The next remark shows that the DTM in DDES is generalization to the DTM in ODES.

**Remark:-** If  $r_1 = 0$  and  $r_2 = 0$ , then theorem (7) reduces to theorem (4).

Similarly, as in theorem (7) we may generalize the result for the n-delays as in the next theorem:

**Theorem (8):-** The differential transform of  $y(x) = f_1(x - r_1) \cdot f_2(x - r_2) \cdot \dots \cdot f_n(x - r_n)$ , provided that  $r_i \geq 0, i = 1, 2, \dots, n$  is:

$$Y(k) = \sum_{k_{n-1}=0}^k \sum_{h_{n-2}=0}^{k_{n-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \sum_{h_1=k_1}^N \sum_{h_2=k_2-k_1}^N \dots \sum_{h_{n-1}=k_{n-1}-k_{n-2}}^N \sum_{h_n=k-k_{n-1}}^N (-1)^{h_1+h_2+\dots+h_n-k} \binom{h_1}{k_1} \binom{h_2}{k_2-k_1} \dots \binom{h_{n-1}}{k_{n-1}-k_{n-2}}$$

$$\binom{h_n}{k-k_{n-1}} r_1^{h_1-k} r_2^{h_2-k+k_1} \dots r_{n-1}^{h_{n-1}-k_{n-1}+k_{n-2}} r_n^{h_n-k+k_{n-1}} F_1(h_1)F_2(h_2) \dots F_{n-1}(h_{n-1})F_n(h_n), \text{ for } N \rightarrow \infty$$

... (7)

### Illustrative Examples

In this section, some liner and nonlinear differential equations with multiple delay are considered .By using DTM , we obtain an approximate and exact solutions when the original problem has an exact solution in polynomial form.

**Example 1:-** Let us consider the following initial value problem:

$$\frac{dy}{dx} + y(x-1) + y(x-2) = 2x - 2, 0 \leq x \leq 1 \dots(8)$$

with the initial condition

$$y(0) = 0 \quad \dots(9)$$

By applying the DTM, we can get the exact solution for eq. (8) and eq. (9).  
Indeed, using theorems (2), (3) and (6) the differential transform for equation (8) is found as

$$(k+1)Y(k+1) + \sum_{h_1=k_1}^N (-1)^{h_1-k} \binom{h_1}{k_1} h_1^{-k} Y(h_1) + \sum_{h_2=k}^N (-1)^{h_2-k} \binom{h_2}{k_1} 2^{h_2-k} Y(h_2) = 2\delta(K-1) - 2\delta(K) \dots(10)$$

where  $\delta(k - n)$  is the differential transform of  $x^n$  at  $x_0 = 0$  and it is easily show that:

$$\delta(k-n) \begin{cases} 1, k=n \\ 0, k \neq n, n=0,2 \end{cases} \quad \dots(11)$$

Considering the differential transform of  $y(x)$  at  $x_0 = 0$ , the initial conditions in equation (9) are transformed into  $Y(0) = 0$ , respectively.

Form equation(10), we obtain:

$$Y(1) = 1, Y(k) = 0, \text{ for } k \geq 2$$

Then, by using the inverse transform defined by equation (4), we obtain the exact solution  $y(x) = x$ .

**Example 2:**In this example, we consider the second order linear differential equation with multiple delay:

$$\frac{d^2y}{dx^2} + y(x-1) + y(x-2) = 2x^2 - 6x + 7, 0 \leq x \leq 1 \quad \dots(12)$$

and the following initial condition:

$$y(0) = 0, y'(0) = 0 \quad \dots(13)$$

Using theorem (3) and (6), the differential transform for eq. (12) is found as:

$$(k+1)(k+2)Y(k+2) + \sum_{h_1=k}^N (-1)^{h_1-k} \binom{h_1}{k} h_1^{-k} Y(h_1) + \sum_{h_2=k}^N (-1)^{h_2-k} \binom{h_2}{k} 2^{h_2-k} Y(h_2) = 2\delta(K-2) - 6\delta(K-1) + 7\delta(K) \quad \dots(14)$$

where  $\delta(k - n)$  is defined in equation (11)

Considering the differential transform of  $y(x)$  at  $x_0 = 0$ , the initial conditions in equation (13) are transformed into:

$$Y(0) = 0, Y(1) = 0 \quad \dots(15)$$

Taking  $N = 3$ , we obtain the following system of linear algebraic equations by using **equation (14) and eq.(15)** for  $k = 0, 1$ .

$$7Y(2) - 9Y(3) = 7$$

$$-6Y(2) + 21Y(3) = -6$$

Solving this system, we obtain:

$$Y(2) = 1, Y(3) = 0$$

Similarly, we have  $Y(k) = 0$ , for  $k \geq 4$ .  
 Then, by using equation (4), we obtain the exact solution  $y(x) = x^2$ .

**Example 3:-** consider the linear differential equation of third order

$$\frac{d^3y}{dx^3} + y(x) + y(x-0.3) = e^{-x+0.3}, 0 \leq x \leq 1 \quad \dots(16)$$

and the following initial conditions  
 $y(0) = 1, y'(0) = -1, y''(0) = 1 \quad \dots(17)$

Using equation (3), the differential transform of  $e^{-x+0.3}$  at  $x_0 = 0$  are obtained to be  $(\frac{1}{k!})(-1)^k e^{0.3}$ . then because of theorems (3) and (6) the differential transform of equation (16) is:

$$(k+1)(k+2)(k+3)Y(k+3) + Y(k) + \sum_{h_1=k}^N (-1)^{h_1-k} \binom{h_1}{k} (0.3)^{h_1-k} Y(h_1) = \frac{1}{k!} (-1)^k e^{0.3} \quad \dots(18)$$

Considering the differential transform of  $y(x)$  at  $x_0 = 0$  the initial conditions in equation (17) are transformed into:

$$Y(0) = 1, Y(1) = -1, Y(2) = \frac{1}{2} \quad \dots(19)$$

Taking  $N = 6$ , we obtain the following linear algebraic equations system by using **equation (18) and equation (19)** for  $k = 0, 1, 2, 3$ .

$$5.973Y(3) + 0.0081Y(4) - 0.00243Y(5) + 0.000729Y(6) = -0.995141192$$

$$0.27Y(3) + 23.892Y(4) + 0.0405Y(5) - 0.01458Y(6) = 0.950141192$$

$$-0.9Y(3) + 0.54Y(4) - 59.73Y(5) + 0.1215Y(6) = -0.325070596$$

$$2Y(3) - 1.2Y(4) - 0.9Y(5) + 119.46Y(6) = -0.224976468$$

Solving this system, we obtain:

$$Y(3) = -0.1666666589, Y(4) = 0.04166662168$$

$$Y(5) = -0.008333153417, Y(6) = 0.001388386354$$

Substituting these value of  $Y(K)$  for  $k = 0, 1, 2, 3$  into equation (4) we obtain the following approximate solution :

$$y(x) = 1 - x + 0.5x^2 - 0.1666666589x^3 + 0.04166662168x^4 - 0.008333153417x^5 + 0.001388386354x^6 + \dots$$

For N=8, the approximate solution is given as:

$$y(x) = 1 - x + 0.5x^2 - 0.1666666666x^3 + 0.0416666657x^4 - 0.0083333329417x^5 + 0.001388887387x^6 - 1.98084138 \times 10^{-4}x^7 + 2.479261343 \times 10^{-5}x^8 - 2.742273669 \times 10^{-6}x^9 + 2.61716267 \times 10^{-7}x^{10} - 1.626927858 \times 10^{-8}x^{11} + \dots$$

Comparison between the numerical results for N = 6, N = 8 and the exact solution  $y(x) = e^{-x}$  (see[3]), are given in table (1).

**Example 4:-** Consider the nonlinear DDE with multiple delays:

$$\frac{d^2y}{dx^2} + y(x - \frac{\pi}{4})y(x - \frac{\pi}{4}) = \sin x \cos x - \cos x + \frac{1}{2}, 0 \leq x \leq 1 \quad \dots(20)$$

and the initial conditions

$$y(0) = 1, y'(0) = 1, \quad \dots(21)$$

Using theorem (3) and (7) the differential transform of equation (20) is obtained as

$$(k + 1)(k + 2)Y(k + 2) + Y(k) + \sum_{k_1=0}^k \sum_{h_1=k_1}^N \sum_{h_2=k-k_1}^N (-1)^{h_1+h_2-k} \binom{h_1}{k} \binom{h_2}{k-k_1} \left(\frac{\pi}{4}\right)^{h_1-k} \left(\frac{\pi}{4}\right)^{h_1-k+k_1} Y(h_1)Y(h_2) =$$

$$\frac{-1}{k!} \left[ \frac{d^k}{dx^k} (\sin x \cos x) \right] - \frac{-1}{k!} \left[ \frac{d^k}{dx^k} (\cos x) \right] + \frac{1}{2} \delta(k) \quad \dots(22)$$

where  $\delta(k - n)$  is defined in eq. (11), with  $n = 0$ .

The initial conditions in eq. (21) are transformed in to:

$$Y(0) = 1, Y(1) = 0$$

From eq. (22), we obtain:

$$Y(2) = \frac{-1}{2!}, Y(3) = 0, Y(4) = \frac{1}{4!}, Y(5) = 0$$

$$Y(6) = \frac{-1}{6!}, Y(7) = 0, Y(8) = \frac{1}{8!}, Y(9) = 0$$

$$Y(10) = \frac{-1}{10!}, Y(11) = 0, Y(12) = \frac{1}{12!}, Y(13) = 0, \dots$$

Substituting these values in to eq. (4), we obtain the following analytical solution,

$$y(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12} + \dots$$

Which is formally the same as Maclaurin Series of  $\cos x$ . In fact  $y(x) = \cos x$  is the exact solution for eq. (20) and (21).

## Conclusions

In this paper the differential transform method is developed for solving differential equations with multiple constant delays. First, some new theorems are provided and then used to solve linear and nonlinear DDES. The obtained results are found to be very accurate in comparison with the exact solution.

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**Table (1):Comparison of numerical results**

x	N = 6	N = 8	Exact
0.0	1	1	1
0.1	0.90483734	0.90483739	0.90483741
0.2	0.81873067	0.81873073	0.81873075
0.3	0.74081810	0.74081814	0.74081822
0.4	0.67320002	0.67320001	0.67032004
0.5	0.60653066	0.60653065	0.60653065
0.6	0.54881137	0.54881161	0.54881163
0.7	0.49658521	0.49653034	0.49658530
0.8	0.49328974	0.44932890	0.44932896
0.9	0.40656969	0.40656966	0.40656965
1	0.36787959	0.36787946	0.36787944

## توسيع طريقة التحويلات التفاضلية لحل المعادلات التفاضلية ذي التباطؤ المتعدد

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### الخلاصة

في هذا البحث، قدمنا طرائق تقريبية وطرائق عديدة لحل المعادلات التفاضلية ذي التباطؤ المتعدد بأستعمال التوسيع لطريقة التحويلات التفاضلية. أستعملت هذه الطريقة لحل العديد من المسائل الخطية وغير الخطية .