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Estimation of the Parameter of an Exponential Distribution When Applying Maximum Likelihood and Probability Plot Methods Using Simulation

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Abstract

Exponential Distribution is probably the most important distribution in reliability work. In this paper, estimating the scale parameter of an exponential distribution was proposed through out employing maximum likelihood estimator and probability plot methods for different samples size. Mean square error was implemented as an indicator of performance for assumed several values of the parameter and computer simulation has been carried out to analysis the obtained results.

Key words: maximum likelihood estimators; probability plot methods; exponential distribution.

Introduction

Exponential distribution is one of the most important distributions which can be used in many places such as in the statistics, engineering, physics, chemistry and others [1]. It is good to use with reliability because value of failure is constant since it has one parameter[2]. Exponential gives distribution of time between independent events occurring at a constant rate-equivalently, and it is a special case of both weibull and gamma distributions [2]. An important property of this distribution is that its memory is less so it is used in any system in the life and to solve problems of survival theory and analysis live table and it is called life distribution [3].

Exponential distribution has the density function below:-

$$f(t) = \lambda \cdot e^{-\lambda t} \qquad t, \lambda>0 \qquad \text{---------------------------------(1)}$$

Where λ is scale parameter and a continuous random variable X is said to have an exponential distribution with rate parameter λ as shown in figure(1).

Noted that

This distribution is valuable and has the following advantages see; [4]
(1) A single and easily estimated parameter
(2) Is mathematically tractable
(3) Has fairly wide applicability

Another properties of exponential distribution are listed in table (1)

From table (1) note that

$$F(t;\lambda) = 1 - e^{-\lambda t} \qquad \text{------------------------------------ (2)}$$

If T~exp(λ) and U represent a uniform random varible from[0,1] then:-

$$f(u)=\begin{cases}1 & \text{If } u\in[0,1]\\ 0 & \text{Otherwise}\end{cases}$$

$$t=-\frac{1}{\lambda}\ln(1-u)$$

$$1-y=e^{-\lambda t} \qquad t,\ \lambda>o$$

$$\because |J|=\frac{dy}{dx}=\lambda\cdot e^{-\lambda t}$$

$$g(t)=f\left[u^{-1}(t)\right]\cdot|J|$$

$$g(t)=\lambda e^{-\lambda t}$$

$$\therefore t=-\frac{1}{\lambda}\ln(1-y) \qquad \text{-------------------------------------- (3)}$$

Maximum Likelihood Estimator

In estimating unknown parameters the most popular method is the Maximum-likelihood estimator (MLE). One important reason is that the MLE is asymptotically optimal in that it approximates the minimum variance unbiased (MVU) estimator for large data records [4].Maximum Likelihood Estimator (MLE) represents a very general method of point estimation which is applicable whether the regularity condition are or are not satisfied [5]. Consider estimation of λ when :-

Let $T_1,T_2,\ldots,T_n$ be a random sample of size n≥2 from exp(λ) then the loglikehood function L(λ) is J.P.F of $t_1,t_2,\ldots,t_n$.

Hence

$$f(t_1,t_2,\ldots,t_n,\lambda)=f(t_1,\lambda)\cdot f(t_2,\lambda)\cdot\ldots,f(t_n,\lambda) \qquad \text{---- ------------------------------- (4)}$$

$$=\lambda\cdot e^{-\lambda t_1}\cdot\lambda\cdot e^{-\lambda t_2}\ldots\lambda\cdot e^{-\lambda t_n}$$

$$=\lambda^n\cdot e^{-\lambda\sum_{i=1}^{n}t_i}$$

By taking $L(\lambda)=f(t_1,t_2,\ldots,t_n,\lambda)$ --------------------------------- (5)

$$\ln L(\lambda)=\ln\lambda^n-\lambda\sum_{i=1}^{n}t_i$$

And

$$\frac{\partial\ln L(\lambda)}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}t_i$$

By equating $\frac{\partial\ln L(\lambda)}{\partial\lambda}=0$

$$0=\frac{n}{\lambda}-\sum_{i=1}^{n}t_i$$

We get an estimator of λ which is denoted by $\hat{\lambda}$

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i} \qquad \text{---------------------------------- (6)}$$

Probability Plotting Method (P.P.E)

To estimate the parameter, one can also use the graphical method called probability plotting method using the data obtained [6].

The following transformations have been employed:

Probability Plotting Model: $F = A + B$

C.D.F: F (t)

From table (1)

$$F(t) = 1 - e^{-\lambda t}$$

$$1 - F(t) = e^{-\lambda t}$$

$$\ln[1 - F(t)] = -\lambda t \qquad \text{------------------------------------(7)}$$

$$\ln[1 - F(t)] = \ln f(t)$$

$$\ln F(t) = -\lambda t \qquad \text{------------------------------------(8)}$$

Taking the natural logarithim of the sided for equation(8):

$$\ln[\ln F(t)] = \ln(-\lambda t) \qquad \text{----------------------------------(9)}$$

Hence

$\ln[\ln F(t)] = y$ & $-\ln \lambda = A_{\varphi}$ and $\ln(t) = \beta x$

Hence equation[9] will produce linear from:

$$\hat{y}_i = \hat{A}_i + Bx_i \qquad \text{--------------------------------(10)}$$

$$\therefore \hat{y}_i = -\ln \lambda - \ln(t) \qquad \text{--------------------------------(11)}$$

Now, in added error term to equation (10), the

$y_i = \hat{y}_i + e_i$; e~E(t)

Using estimater,we obtan estimater the Exponential parameter λ frome equation (11):

$$\therefore \hat{\lambda} = \exp(-y_i - \ln(t)) \qquad \text{-----------------------------(12)}$$

Now, parameter estimaterfor the other distribution, probability plotting model can be written as in equation (12) above.

Simulation & Empirical Work

One of the most important application of computer science is computer simulation [7]. Simulation can be used to show the eventual real effects of alternative conditions and courses of action. Simulation is also used when the real system cannot be engaged, because it may not be accessible, or it may be dangerous or unacceptable to engage, or it is being designed but not yet built, or it may simply not exist [8].simulation approaches offer great opportunities for working out probabilities, cofidence intervals and similar concepts [9].This analysis may be done, sometimes, through analytical or numerical methods, but the model may be too complex to be dealt with. Essentially, simulation process consists of building a computer model that describes the behavior of a system and experimenting with this computer model to reach conclusions that support decisions [10]. Sometimes, it is not feasible or possible, to build a prototype, yet we may obtain a mathematical model describing, through equations and constraints, the essential behavior of the system. In such extreme cases, we may use simulation to replicate real world studies that cannot be done, simulation exercises may encounter statistical pitfalls that degrade their performance, or fail to take advantage of the opportunities statistics can provide for controlling simulation error and producing statistically reliable results [10].

In order to make the best estimation of parameter of it can exponentail distribution for (MLE)and (P.P.E). We make a simulation prototype provide assumption of many cases whh icbe existed in real word and use the basic step process in any simulation experiment once we have estimated the corresponding simulation model.

Algorithms steps :-

(I) Frist step:-

Specified the assumed values by choosing different sample sizes of exponential distrbution, such as sample size (n=20) and sample size (n=50) and samle size (n=100)

Then choosing the values of assumption parameter λ in each several contrasts and choosing for the initial values of the parameter (scale) it as shown in Table (2):-

(II) Second step :-

Generation of data which include:

- Generated the random data which was taken from the uniform distribution in the interval [0,1] using Excel,and SPSS,software computer package.
- The generation of errors for all data and in method the random errors have been generated using the standard exponential distribution.

(III) Third step :-

This step contains the following:-

- Using the same value of $\hat{t}$ & $\hat{y}_i$ for amethods (MLE),(P.P.E)and applying the equation $\hat{t} = -\frac{1}{\lambda}\ln(1-y)$ & $\hat{y}_i = -\ln\lambda - \ln(t)$ as mentioned in (3),(11).
- Finding the time(t) by using the equation $t_i = \hat{t}_i + e$ where i=1,2,……….,n
- The value of $\hat{\lambda}$ of exponential distribution can be determined according to the estimators of (MLE)and(P.P.E) in equation (6),(12).

(IV) Fourth step: smoothing the obtained values

- In this step the iteration of data will be repeated 500 times to generate a new different error, so we obtain100 value of $\hat{t}$ for each contrast. Then the mean of each case will be calculated to find the estimated t .

(V) Fifth step :-

In this step the following comparison indicator will be employed to make a comporison between different values of λ and different methods for (MLE) and Probability Plotting estimation (P.P.E).

Conclusions & Future Work

As a consequence for practical work and taking the mean square error as the indicator of preference between the different estimator methods, the following results are obtained:-

(1) Sample size (n=20)
For the assumed contrast parameters (λ= 0.5) the MLHE and P.P.E estimators was given the best results.

(2) Sample size (n=50)
For the assumed contrast parameters (λ= 0.5) the MLHE and P.P.E estimators was given the best results.

(3) Sample size (n=100)
For the assumed contrast parameters (λ=0.5) the MLHE and P.P.E estimators was given the best results.

(4) The best results from different sample sizes (20, 50, 100) is sample size n=100 for the assumed contrast parameters (λ=0.5, 1, 2) the MLHE estimator method was given the best results.

(5) The best results from different sample sizes (20, 50, 100) is sample size n=50 for the assumed contrast parameters (λ=0.5, 1, 2) the P.P.E method was given the best results.

The results of simulation for different sample sizes (n=20, 50, and100) are listed in the table (3)

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Table (1):Some Properties of an Exponential Distribution

Mean	$\frac{1}{\lambda}$
Variance	$\frac{1}{\lambda^2}$
Median	$\frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}$
First quartile	$\frac{\ln 4 - \ln 3}{\lambda}$
Third quartile	$\frac{\ln 4}{\lambda}$
Survival function	$R(t) = 1 - (1 - e^{-\lambda t})$
Hazard function	$h(t) = \frac{f(t)}{1 - F(t)} = \frac{\lambda \cdot e^{-\lambda t}}{1 - (1 - e^{-\lambda t})} = \lambda$
$F^{-1}(y)$	$-\frac{\ln(1-y)}{\lambda}, 0 < y < 1$
(Scdf) **F(t)**	$F(t;\lambda) = 1 - e^{-\lambda t}$

Where (Scdf) is standard cumulative density function.

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مجلة إبن الهيثم للعلوم الصرفة و التطبيقية
العدد 1 المجلد 25 السنة 2012

Table (2):Assumed contrast parameter

λ
0.5
1 Table (3)
2

Table (3):Estimation of Scale Parameter of Exponential Distribution For (MLH) and (P.P.E)

Sample	Assumed Parameter	Estimator	Indicator	
	λ	$\hat{\lambda}$	MSE MLH	MSE P.P.E
20	0.5	0.327733	0.162901	. 02128338
	1	0.550218	1.08878	. 08668342
	2	0.616011	10.11944	. 34332615
50	0.5	0.285936	0.093745	. 02059832
	1	0.602633	0.327592	. 08300531
	2	0.717962	3.364929	. 32840239
100	0.5	0.344761	0.0245	. 02084406
	1	0.537857	0.21746	. 08338210
	2	0.65473	1.829821	.33505150

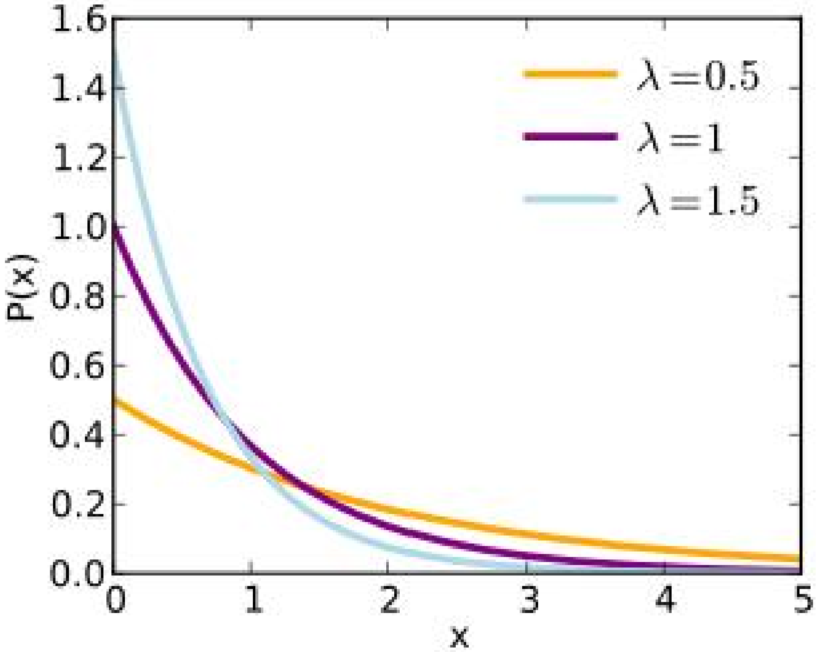

Fig.(1): The Probaability Density Function for Exponential Distribution

Ibn Al-Haitham Journal for Pure and Applied Science
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العدد 1 المجلد 25 السنة 2012

تخمين معلمة التوزيع الأسي بتطبيق طريقتي دالة الامكان الاعظم ودالة الرسم البياني الاحتمالي باستخدام المحاكاة

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الخلاصة

في هذا البحث تم تخمين معلمة القياس للتوزيع الاسي من خلال تطبيق طريقتي دالة الامكان الاعظم، والرسم البياني الاحتمالي ولأحجام وعينات مختلفة مع توليفات افتراضية عديدة للمعلمة. استخدم مؤشرا" معدل مربعات الخطأ كمؤشر لأفضل اداء باستخدام تقنية المحاكاة الحاسوبية و تحليل القيم والنتائج المستحصلة.

الكلمات المفتاحية : تقدير الجوار الاعظم ،طريقة الرسم الاحتمالي، التوزيع الاسي .