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Approximation Properties of the Strong Difference Operators

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Abstract

In this paper, we study some approximation properties of the strong difference and study the relation between the strong difference and the weighted modulus of continuity .

Key words: Positive linear operator, weighted space, weighted modulus of continuity.

Introduction

In 2004 Rempulska ,L. and Skorupka ,M. [2] We introduce the strong differences of functions and their operators and we gave the Jackson type theorems for them. In this paper we generalized to results Rempulska to $L_{p,\alpha}(X)$-spaces .

Let $X=[a,b]$, a ,b ∈R we define:

$$L_{p,\alpha}(X)=\left\{f:X\to R: f \text{ is bounded measurable with norm } \|f\|_{p,\alpha}=\left(\int_a^b\left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx\right)^{\frac{1}{p}}<\infty\right\} \quad \ldots(1.1)$$

Where $\omega_\alpha(x)$ is positive continuous function ,$\lim_{x\to\infty}\omega_\alpha(x)=\infty$

$$C_{p,\alpha}(R^+)=\{f\in L_{p,\alpha}(R^+): f \text{ is continuous}\} \quad \ldots(1.2)$$

Let C be the set of all infinite matrices A=$[a_{nk}]$,n ∈ N,K∈ N_0 ,(where N is the set of all natural numbers, N_0={0} ∪ N) of functions in $C_{p,\alpha}$ having the following properties: [2]

1.) $a_{nk}(x)\geq 0$ for x ∈ X,n ∈ N, K ∈ N_0 ...(1.3)

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2.) $\sum_{k=0}^{\infty} a_{nk}(x) = 1$ for $x \in X$, $n \in N$, $K \in N_0$ ...(1.4)

3.) for every n , r $\in N$ the series $\sum_{k=0}^{\infty} k^r a_{nk}(x)$ is uniformly convergent on X and

$\sum_{k=0}^{\infty} k^r a_{nk}(x) \in C_{p,\alpha}$ ...(1.5)

4.) for every r $\in N$ there exists M > 0 independent on x $\in X$ and n $\in N$ such that for the

functions $T_{n,r}(x,A) = \sum_{k=0}^{\infty} a_{nk}(x)\left(\frac{k}{n} - x\right)^r$, $x \in X$

Such that $\|T_{n,r}(x,A)\|_{p,\alpha} \leq M$, $n \in N$ ...(1.6)

We define the global norm :

$$\|f\|_{\delta,p,\alpha} = \left(\int_a^b \sup_{y\in\left[x-\frac{\delta}{2},x+\frac{\delta}{2}\right]} \left|\frac{f(y)}{\omega_\alpha(y)}\right|^p dx\right)^{\frac{1}{p}}$$ a , b $\in$ R , $x \in X$, $\delta > 0$ .where R is the set

of all real numbers .

1.Basis concepts and Lemmas

Definition 1.1 [2]

For every $A \in C$, $f \in C_{p,\alpha}$ define:

$L_n(f,A,x) = \sum_{k=0}^{\infty} a_{nk}(x)\, f\left(\frac{k}{n}\right)$ k , n $\in N$, $x \in X$ ...(1.7)

Definition 1.2 [2]

For every $A \in C$, $f \in C_{p,\alpha}$ define the strong difference by:

$H_n(f,A,x) = \sum_{k=0}^{\infty} a_{nk}(x) \left|f\left(\frac{k}{n}\right) - f(x)\right|$ k , n $\in N$, $x \in X$ ...(1.8)

Properties of L_n(f,A,x) [2]

1.)From the definition of $L_n(f,A,x)$ we get $L_n(1,A,x) = 1$ ...(1.9)

2.) by def. (1.1) and (1.9):

$$L_n(f,A,x) - f(x) = \sum_{k=0}^{\infty} a_{nk}(x)\, f\left(\frac{k}{n}\right) - f(x) \qquad \ldots(1.10)$$

properties of the strong difference [2]

by def (1.1) , def(1.2), (1.10) :

1.): $H_n(f,A,x) - L_n\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|, A, x\right)$ ...(1.11)

2.) $\left|L_n(f,A,x) - f(x)\right| < H_n(f,A,x)$ ...(1.12)

New weighted modulus of continuity

We need to define a new weighted modulus :

For each $f \in C_{p,\alpha}(X)$ and for each $\delta > 0$, we define:

$$\omega^*(f,\delta)_{p,\alpha} - \sup_{\substack{x,y>0 \\ |x-y|<\delta}} \left(\int_a^b \left|\frac{f(x) \quad f(y)}{\omega_{\alpha_1}(x) - \omega_{\alpha_1}(y)}\right|^p dx\right)^{\frac{1}{p}} \qquad \ldots(1.13)$$

Lemma (1.1) [3]

If is a bounded measurable function on $[a,b]$ a , b ∈ R , then:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) \quad n \in N \qquad \ldots(1.14)$$

Where $x_i = a + \frac{(b-a)(2i-1)}{2n}$

Lemma 1.2 [1]

Let $f, g \in L_{p,\alpha}(R^+)$ then $\|f+g\|_{p,\alpha} \le C(\|f\|_{p,\alpha} + \|g\|_{p,\alpha})$

where C is constant.

2.Main Results

We prove some properties of $L_n(f,A,x)$ and $H_n(f,A,x)$:

Lemma 2.1

Let $f \in L_{p,\alpha}(X)$ then $\|f\|_{\delta,p,\alpha} \le C_2 \|f\|_{p,\alpha}$ …(1.15)

where C_2 is constant.

Proof

$$\|f\|_{\delta,p,\alpha} = \left(\int_a^b \sup_{y \in \left[x-\frac{\delta}{2},\, x+\frac{\delta}{2}\right]} \left| \frac{f(y)}{\omega_\alpha(y)} \right|^p dx \right)^{\frac{1}{p}}$$

By lemma (1.1) we have :

$$\|f\|_{\delta,p,\alpha} \le \left(\frac{b-a}{n} \sum_{i=1}^{n} \sup_{y_i \in \left[x-\frac{\delta}{2},\, x+\frac{\delta}{2}\right]} \left| \frac{f(y_i)}{\omega_\alpha(y_i)} \right|^p \right)^{\frac{1}{p}}$$

$$\le \left(\frac{c}{n} \sum_{i=1}^{n} \left| \frac{f(y_i)}{\omega_\alpha(y_i)} \right|^p \right)^{\frac{1}{p}} \text{ where } c = b-a$$

$$\le \left(\frac{c}{n} \sum_{i=1}^{n} \left| \frac{f(y_i)}{\omega_\alpha(y_i)} \right|^p \right)^{\frac{1}{p}}$$

By lemma (1.1) we have :

$$\|f\|_{\delta,p,\alpha} \le C_2 \left(\int_a^b \left| \frac{f(x)}{\omega_\alpha(x)} \right|^p dx \right)^{\frac{1}{p}}$$

$$\le C_2 \|f\|_{p,\alpha}$$

Lemma 2.2

Let $A \in C$, $(1 \le p < \infty)$, $\alpha > 0$, $f \in C_{p,\alpha}$ then there exists T>0 such that

If the following satisfies:

1.) $\omega_\alpha^*(t) \le \left(\frac{k}{n}\right)$ $t \in X$, $k , n \in N$

2.) $\|x\|_{p,\alpha} \leq L \quad x \in X$, L is constant.

Then $\|L_n(\omega_\alpha^*(t), A, .)\|_{p,\alpha} \leq T$, $n \in N$ ...(1.16)

Where $\omega_\alpha^*(t)$ is a weighted function different from $\omega_\alpha(x)$.

Proof

$$\|L_n(\omega_\alpha^*(t), A, .)\|_{p,\alpha} = \left(\int_a^b \left|\frac{L_n(\omega_\alpha^*(t), A, .)}{\omega_\alpha(x)}\right|^p dx\right)^{1/p}$$

$$\|L_n(\omega_\alpha^*(t), A, .)\|_{p,\alpha} = \left(\int_a^b \left|\frac{\sum_{k=0}^{\infty} a_{nk}(x)\, \omega_\alpha^*(t)}{\omega_\alpha(x)}\right|^p dx\right)^{1/p}$$

$$\leq \left(\int_a^b \left|\frac{\sum_{k=0}^{\infty} a_{nk}(x) \left(\frac{k}{n}\right)}{\omega_\alpha(x)}\right|^p dx\right)^{1/p}$$

$$\leq \left(\int_a^b \left|\frac{\sum_{k=0}^{\infty} a_{nk}(x)\left(\frac{k}{n} - x + x\right)}{\omega_\alpha(x)}\right|^p dx\right)^{1/p}$$

By lemma (1.2) we have :

$$\leq C\left(\int_a^b \left|\frac{\sum_{k=0}^{\infty} a_{nk}(x)\left(\frac{k}{n} - x\right)}{\omega_\alpha(x)}\right|^p dx\right)^{1/p} + C\left(\int_a^b \left|\frac{\sum_{k=0}^{\infty} a_{nk}(x)\, x}{\omega_\alpha(x)}\right|^p dx\right)^{1/p}$$

$$\leq C\left(\int_a^b \left|\frac{\sum_{k=0}^{\infty} a_{nk}(x)\left(\frac{k}{n} - x\right)}{\omega_\alpha(x)}\right|^p dx\right)^{1/p} + C\left(\int_a^b \left|\frac{|x|^\alpha \sum_{k=0}^{\infty} a_{nk}(x))}{\omega_\alpha(x)}\right|^p dx\right)^{1/p}$$

$$\leq C\|T_{n,1}(x, A)\|_{p,\alpha} + C\left(\int_a^b \left|\frac{x}{\omega_\alpha(x)}\right|^p dx\right)^{1/p}$$

$$\leq C\|T_{n,1}(x, A)\|_{p,\alpha} + C\|x\|_{p,\alpha}$$

By (1.6) and by assumption we have:

$$\|L_n(\omega_\alpha^*(t), A, .)\|_{p,\alpha} \leq C\,M + C\,L = T$$

Lemma 2.3

Let $A \in C$, $(1 \leq p < \infty)$, $\alpha > 0$, $f \in C_{p,\alpha}$ such that then there exist

$C_2 > 0$ such that $\|L_n(f,A,.)\|_{p,\alpha} \le C_2 \|f\|_{p,\alpha}$ ...(1.17)

Proof

$$\|L_n(f,A,.)\|_{p,\alpha} = \left(\int_a^b \left|\frac{L_n(f,A,.)}{\omega_\alpha(x)}\right|^p dx\right)^{1/p}$$

$$\|L_n(f,A,.)\|_{p,\alpha} = \left(\int_a^b \left|\frac{\sum_{k=0}^{\infty} a_{nk}(x)\, f(\frac{k}{n})}{\omega_\alpha(x)}\right|^p dx\right)^{1/p}$$

Since $\|f\|_{p,\alpha} \le \|f\|_{\delta,p,\alpha}$

$$\|L_n(f,A,.)\|_{p,\alpha} \le \left(\int_a^b \sup_{y\in\left|x-\frac{\delta}{2},x+\frac{\delta}{2}\right|} \left|\frac{\sum_{k=0}^{\infty} a_{nk}(y) f(y)}{\omega_\alpha(y)}\right|^p dx\right)^{1/p}$$

$$\le \left(\int_a^b \sup_{y\in\left|x-\frac{\delta}{2},x+\frac{\delta}{2}\right|} \left|\frac{f(y)}{\omega_\alpha(y)}\right|^p \left|\sum_{k=0}^{\infty} a_{nk}(y)\right|^p dx\right)^{1/p}$$

By (1.4) we have:

$$\|L_n(f,A,.)\|_{p,\alpha} \le \left(\int_a^b \sup_{y\in\left|x-\frac{\delta}{2},x+\frac{\delta}{2}\right|} \left|\frac{f(y)}{\omega_\alpha(y)}\right|^p dx\right)^{1/p}$$

$$\le \|f\|_{\delta,p,\alpha}$$

BY lemma (2.1) we have :

$$\|L_n(f,A,.)\|_{p,\alpha} \le C_2 \|f\|_{p,\alpha}$$

Lemma 2.4

Let A $\in C$, $(1 \le p < \infty)$, $\alpha > 0$, f, $f^{(1)} \in C_{p,\alpha}$ then there exist $W > 0$ such that $\|H_n(f,A,.)\|_{p,\alpha} \le W \|f^{(1)}\|_{p,\alpha}$ ...(1.18)

where $f^{(1)} = \frac{d}{dx}\left(\frac{f(x)}{\omega_\alpha(x)}\right)$.

Proof

$$\|H_n(f,A,.)\|_{p,\alpha}=(\int_a^b\left|\frac{H_n(f,A,.)}{\omega_\alpha(x)}\right|^p dx)^{1/p}$$

By (1.11) we have:

$$\|H_n(f,A,.)\|_{p,\alpha}=(\int_a^b\left|\frac{L_n(\left|f(\frac{k}{n})-f(x)\right|,A,x)}{\omega_\alpha(x)}\right|^p dx)^{1/p}$$

$$=\left\|L_n(\left|f\left(\frac{k}{n}\right)-f(x)\right|,A,x)\right\|_{p,\alpha}$$

By lemma (2.3) we have:

$$\|H_n(f,A,.)\|_{p,\alpha}\le C_2\left\|(f\left(\frac{k}{n}\right)-f(.)\right\|_{p,\alpha}$$

$$\le C_2\left\|\int_{\frac{k}{n}}^{x}\left(\frac{f}{\omega_\alpha}\right)^{(1)}(u).du\right\|_p$$

$$\le C_2\left\|sup\left(\frac{f}{\omega_\alpha}\right)^{(1)}\right\|_p$$

$$\le C_2\left(\int_a^b sup\left|\left(\frac{f}{\omega_\alpha}\right)^{(1)}(u)\right|^p du\right)^{\frac{1}{p}}$$

$$\le C_2\left\|f^{(1)}\right\|_{k,p,\alpha}$$

BY the same line of proof lemma (2.1) we have:

$$\|H_n(f,A,.)\|_{p,\alpha}\le W\left\|f^{(1)}\right\|_{p,\alpha}$$ Where $W=C_2^2$

Theorem 2.5

Let $A\in C$, $(1\le p<\infty)$, $\alpha>0$, $f\in C_{p,\alpha}$ is increasing then there exists

$K>0$ then $\|H_n(f,A,.)\|_{p,\alpha}<K\omega^*(f,h)_{p,\alpha}$ …(1.19)

Proof

We consider the stieklov function f_h for $f\in C_{p,\alpha}$:

$$f_h(x) = \frac{1}{h}\int_0^h f(x+u).du \qquad x\in R^+,\ h>0$$

$$\left|f\left(\tfrac{k}{n}\right)-f(x)\right| = \left|f\left(\tfrac{k}{n}\right)-hf_h\left(\tfrac{k}{n}\right)+hf_h\left(\tfrac{k}{n}\right)-hf_h(x)+hf_h(x)-f(x)\right|$$

$$\leq \left|f\left(\tfrac{k}{n}\right)-hf_h\left(\tfrac{k}{n}\right)\right|+\left|hf_h\left(\tfrac{k}{n}\right)-hf_h(x)\right|+\left|hf_h(x)-f(x)\right|$$

… …(1.20)

$$\|H_n(f,A,.)\|_{p,\alpha} = \int_a^b \left|\frac{H_n(f,A_n)}{\omega_\alpha(x)}\right|^p dx)^{1/p}$$

By (1.11) we have:

$$\|H_n(f,A,.)\|_{p,\alpha} = \int_a^b \left|\frac{L_n(\left|f\left(\frac{k}{n}\right)-f(x)\right|,A,x)}{\omega_\alpha(x)}\right|^p dx)^{1/p}$$

By (1.20) we have:

$$\leq \int_a^b \left|\frac{L_n(\left|f\left(\frac{k}{n}\right)-hf_h\left(\frac{k}{n}\right)\right|+\left|hf_h\left(\frac{k}{n}\right)-hf_h(x)\right|+\left|hf_h(x)-f(x)\right|,A,x)}{\omega_\alpha(x)}\right|^p dx)^{1/p}$$

By lemma (1.2) we have:

$$\|H_n(f,A,.)\|_{p,\alpha} \leq C(\int_a^b \left|\frac{L_n(\left|f\left(\frac{k}{n}\right)-hf_h\left(\frac{k}{n}\right)\right|,A,x)}{\omega_\alpha(x)}\right|^p dx)^{\frac{1}{p}} + C\int_a^b \left|\frac{L_n(hf_h(\frac{k}{n})-hf_h(x),A,x)}{\omega_\alpha(x)}\right|$$

$$+C(\int_a^b \left|\frac{L_n(\left|hf_h(x)-f(x)\right|,A,x)}{\omega_\alpha(x)}\right|^p dx)^{1/p}$$

$$\leq C\left\|L_n(\left|f\left(\tfrac{k}{n}\right)-hf_h\left(\tfrac{k}{n}\right)\right|,A,.)\right\|_{p,\alpha} + C\ \|H_n(hf,A,.)\|_{p,\alpha} +$$

$$C\ \|L_n|hf_h(.)-f(.)|\|_{p,\alpha}$$

By lemma (2.3) and lemma (2.4) we have :

$$\|H_n(f,A,.)\|_{p,\alpha} \leq C_3\left\|f\left(\tfrac{k}{n}\right)-hf_h\left(\tfrac{k}{n}\right)\right\|_{p,\alpha}+C_4h\|f_h^{(1)}\|_{p,\alpha}+C_5\ \|hf_h(.)-f(.)\|_{p,\alpha}$$

… ...(1.21)

$$\|f-hf_h\|_{p,\alpha} = \int_a^b \left|\frac{f(x)-hf_h(x)}{\omega_\alpha(x)}\right|^p dx)^{1/p}$$

$$\int_a^b \left|\frac{f(x)-\int_0^h f(x+u).du}{\omega_\alpha(x)}\right|^p dx)^{1/p} \qquad =$$

Let y =x + u let $|u| \leq h$, $h>0$

Since f is increasing we get:

$$\|f-hf_h\|_{p,\alpha} \leq \int_a^b \left|\frac{f(x)\quad f(y)}{\omega_\alpha(x)}\right|^p dx)^{1/p}$$

$$\leq \int_a^b \left|\frac{f(x)}{\omega_\alpha(x)}\ \frac{f(y)}{\omega_\alpha(y)}\right|^p dx)^{1/p}$$

By definition of the new weighted modulus of continuity we have :

$$\|f-hf_h\|_{p,\alpha} \leq \omega^*(f,h) \qquad \text{...(1.22)}$$

$$h\left\|f_h^{(1)}\right\|_{p,\alpha} = h\int_a^b \left|\frac{f_h^{(1)}}{\omega_\alpha}(x)\right|^p dx)^{1/p}$$

$$= h\int_a^b \left|\frac{\frac{1}{h}\int_0^h f^{(1)}(x+u).du}{\omega_\alpha(x)}\right|^p dx)^{1/p}$$

$$= \int_a^b \left|\frac{f(x+h)\quad f(x)}{\omega_\alpha(x)}\right|^p dx)^{1/p}$$

Let y=x + h

$$= \int_a^b \left|\frac{f(x)\quad f(y)}{\omega_\alpha(x)}\right|^p dx)^{1/p}$$

$$\leq \int_a^b \left|\frac{f(x)}{\omega_\alpha(x)}\ \frac{f(y)}{\omega_\alpha(y)}\right|^p dx)^{1/p}$$

$$\leq \sup \int_a^b \left|\frac{f(x)}{\omega_\alpha(x)}\ \frac{f(y)}{\omega_\alpha(y)}\right|^p dx)^{1/p}$$

By definition of the new weighted modulus of continuity we have :

$$h\left\|f_h^{(1)}\right\|_{p,\alpha} \leq \omega^*(f,h)_{p,\alpha} \qquad \text{...(1.23)}$$

Subtitling (1.23) and (1.22) in (1.21) we have :

$$\|\Pi_n(f,A,.)\|_{p,\alpha} \le C_3\,\omega^*(f,h)_{p,\alpha} + C_4\,\omega^*(f,h)_{p,\alpha} + C_5\,\omega^*(f,h)_{p,\alpha}$$

$$\le (C_3 + C_4 + C_5)\,\omega^*(f,h)_{p,\alpha}$$

Let $K = C_3 + C_4 + C_5$ we have:

$$\|\Pi_n(f,A,.)\|_{p,\alpha} \le K\omega^*(f,h)_{p,\alpha}$$

Conclusion

we find the relation between the strong difference and the new weighted modulus of continuity.

References

1.Husain , L .A. (2010) Approximations with weighted Function ,M. SC .Thesis ,AL-mustansirya University, Department Of mathematics ,College Of science ,Iraq , Baghdad

2..Rempulska , L. and Mariola Skorupka (2004) On strong Approximation of Functions By Certain Linear Operators",Math. J. Okayama Univ. 46, 153-161

3. Sendov , B. and Popov, V. A (1983) The Averaged Moduli Of Smoothness , the publishing House of the Bulgarian Academy of Sciences , Sofia.

خواص تقريب مؤثر الاختلاف القوي

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الخلاصة

في بحثنا هذا درسنا بعض خواص تقارب (the strong difference) ووجدنا علاقته بـــــ (modulus) الجديد .

الكلمات المفتاحية : مؤثر خطي موجب ، فضاء الوزن ، مقاس الوزن .