Fixed Point Theorem for Uncommuting Mappings

Salwa S. Abd
Alaa Abd-ullah
Dept. of Mathematics/College of Education for Pure Science(Ibn Al- Haitham)
University of Baghdad

Received in:19 June 2012 Accepted in:15 October 2012

Abstract

In this paper we prove a theorem about the existence and uniqueness common fixed point for two uncommenting self-mappings which defined on orbitally complete G-metric space. Where we use a general contraction condition.

Key words : G-metric space, orbitaliy complete, commuting mappings, common fixed point.

Introduction and preliminaries

A number of authors have defined contractive type mappings on a usual complete metric space X which are generalizations of the well known Banach's contraction principle [1:pp. 175-206], and which have the property that each such mapping has a unique fixed point . The fixed point can always be found by using Picard iteration (i.e. iterative sequence[Zidler: pp.15-30]), beginning with some initial choice $x_0 \in X$. And then many authors have extended, generalized and improved Banach's contraction principle in different ways some these ways are depending on commuting mappings, compatible mappings, weakly commuting mappings, … ets (such as, see [3,4,5,6]).

Recently, Branciari [7] introduced a generalization of metric space and proved a general version of Banach's contraction principle. And then ,P.Das[8], P.Das and L.Dey [9] , S.Mordi [10] and Akram, Zafar and Siddiqui[11] prove other results about the existence of fixed points and common fixed points for mappings defined on complete G-metric space.

Throughout this paper R^+ is denoted by non-negative real numbers and N is positive integer numbers. Now we begin with the following definition.

Definition 1. 1[11]: Let X be a nonempty set. Suppose that the mapping $\rho: X \times X \to R^+$ such that for all x , y ∈ X and for all distinct points z ,v ∈ X\ {x, y}, satisfies:

1. $\rho (x, y) = 0$ if and only if $x = y$,

2. $\rho (x, y) = \rho (y, x)$,

3. $\rho (x, y) \le \rho (x, z) + \rho (z,v) + \rho (v, y)$, (rectangular property),

Then the ordered pair (X, ρ) is called a generalized metric space (or shortly G-metric space.).

Note that, any metric space is G-metric space but the converse is not true, for examples,

Example1.2: Let X={a,b,c,d,}. Define $\rho : X \times X \to R$ by

$\rho (a,b)= \rho (b,a)= 3$, $\rho (b,c)= \rho (c,b)= \rho (a,c)= \rho (c,a)=1$,

$\rho (a,d)= \rho (d,a)= \rho (b,d)= \rho (d,b)= \rho (c,d)= \rho (d,c)=4$.

It is easily to show that (X, ρ) is G-metric space and it is not metric space ,since

$$\rho (a,b) > \rho (a,c)+ \rho (c,b)$$
$$3 > 1 + 1$$

Example1.3: Consider X=R , $\mu : X \times X \to R$ and $\mu (x,y)= (x-y)^2$,

Clearly μ is not G-metric space and so is not metric space since, for x=2, y=0, z=1 and w=1/2.We have

$$\mu (2,0) > \mu (2,1)+ \mu (1,1/2) + \mu (1/2 ,0)$$

Example1.4: Let $\rho: R^2 \to R^+$ be a mapping such that

$$\rho(x,y)=\max\{\mu (x,z), \mu (z,w), \mu (w,y)\},$$

whereas in example above, then ρ is G-metric space. Therefore, G-metric space is a proper extension of a metric space.

Also, one can generate many G-metric spaces by usual sense, such as:

Example 1.5: If ρ(x,y) G-metric space

$$\rho_1 (x,y)= \rho(x,y) / (1+ \rho(x,y)) \quad \text{also G-metric space.}$$

Remark1.6 [7]: the G-metric space is continues function on X × X.

Remark1.7 [11]: As in the usual metric space settings, a G-metric space is a topological space with respect to the basis given by

$B=\{B(x,r): x \in X, r \in R^+\}$,where $B(x,r)=\{y \in X: \rho (x,y) < r\}$ is open ball centered by x and with radius r.

Definition1.9[11]: Let (X, ρ) be a G-metric space. A sequence $\{x_n\}$ in X is said to to be a Cauchy sequence if for any $\varepsilon > 0$ there exists n_ε in N such that for all m, n ∈ N and $m, n > n_\varepsilon$, one has $\rho(x_n,x_{n+m})<\varepsilon$.

The space (X,ρ) is called complete if every Cauchy sequence in X is convergent.
Definition1.10: Let T be a self mapping on X. Let $x_0 \in X$. A sequence $\{T^n x\}$ in X is said to be an orbit of x by T and denoted by $O(x, n)= \{x,Tx,T^2x,\dots,T^nx\}$, for all $n \in N$. Also, $O(x, \infty) =\{x,Tx,T^2x,\dots\}$.
Definition 1.11[9]: Let T be mapping on a G-metric space (X, ρ) into itself. (X, ρ) is said to be T-orbitally complete if and only if every Cauchy sequence in $O(x, \infty)$ converges in X, for some $x \in X$.
Now we introduced the following concept
Definition 1.12: Let T_1S be two self mappings on a G-metric space X. X is called ST-orbitally complete if for $x_0 \in X$ the sequence $\{x_0,Tx_0,STx_0,TSTx_0,\dots\}$ converges to a point in X.

or the sequence $\{x_n\}$ converges to a point in X where

$$x_0 \in X,\ x_{2n+1}=Tx_{2n},\ x_{2n+2}=Sx_{2n+1} \qquad \dots (1)$$

For all $n \in N \cup \{0\}$.
Definition1.13: A point x in X is a common fixed point of two self-mappings on G-metric space X if $Tx = Sx = x$.
Definition1.14: Let T and S be self mappings on G-metric space X. T and S are commuting mappings if there exists a point x in X such that $T x = S x$ and $T S x = S T x$..

Main results

Let Φ be a family of functions such that $\varphi \in \Phi$ mean that $\varphi : R^+ \to R^+$ is continuous from the right, non-decreasing and satisfy the condition

$$\varphi(t) < t \text{ for } t > 0 \text{ and } \varphi(0)=0.$$

It is easy to have the following lemma

Lemma2.1: If $\varphi_1, \varphi_2 \in \Phi$, then there is some $\varphi_3 \in \Phi$ such that $\max\{\varphi_1(t), \varphi_2(t)\} \le \varphi_3(t)$ for all $t > 0$.
Proof: we can see φ_3 as $\varphi_1 + \varphi_2$.
Lemma 2.1[12]: Let $\varphi \in \Phi$, then $\varphi^n(t) \to 0$ as $n \to +\infty$ for every $t > 0$.
Proposition2.3: Let (X,ρ) be a G-metric space. Let $S,T : X \to X$ be mappings. If for each x,y in X and T and S satisfy the condition:

$$\rho(STx,TSy) \le \max\{\varphi_1(1/2[\rho(x,Sy)+ \rho(y,Tx)]),\ \varphi_2(\rho(x,Tx)),\ \varphi_3(\rho(y,Sy)),\ \varphi_4(\rho(x,y))\} \text{ for all } x,y \in X, \text{ where } \varphi_i \in \Phi\ (i = 1,2,3,4) \qquad .. (2)$$

Then the sequence $\{x_n\}$ defined by (1) is a Cauchy sequence.
Proof: Let $x_0 \in X$ and $\{x_n\}$ be a sequence as in (1).The proof includes two steps:
Step 1: to show that $\lim_{n\to\infty} \rho(x_n,x_{n+1})=0$, let $x_1,x_2 \in \{x_n\}$ and $M = \max\{\rho(x_0,x_1), \rho(x_1,x_2)\}$.
Since all φ_i are non-decreasing functions by (2),

$$\rho(x_2,x_3)=\rho(STx_0,TSx_1)$$
$$\le \max\{\varphi_1(1/2[\rho(x_0,Sx_1)+\rho(x_1,Tx_0)]),\varphi_2(\rho(x_0,Tx_0)),\varphi_3(\rho(x_1,Sx_1)),\varphi_4(\rho(x_0,x_1))\}$$
$$\le \max\{\varphi_1(M), \varphi_2(M),\varphi_3(M),\varphi_4(M)\}$$
$$\le \varphi(M) \qquad \dots (3)$$

where $\varphi \in \Phi$. Therefore, we have

$$\rho(x_3,x_4)= \rho(STx_1,TSx_2)$$
$$\le \max\{\varphi_1(1/2[\rho(x_1,Sx_2)+\rho(x_2,Tx_1)]),\varphi_2(\rho(x_1,Tx_1)),\varphi_3(\rho(x_2,Sx_2)),\varphi_4(\rho(x_1,x_2))\}$$
$$\le \max\{\varphi_1(M), \varphi_2(\varphi(M)), \varphi_3(M),\varphi_4(M)\} \le \varphi(M), \qquad \dots (4)$$

Using (2),(3) and (4),we get

$$\begin{aligned}\rho(x_4,x_5)&=\rho(STx_2,TSx_3)\\ &\le \max\{\varphi_1(1/2[\rho(x_3,Sx_2)+\rho(x_2,Tx_3)]),\varphi_2(\rho(x_2,Tx_2),\varphi_3(\rho(x_3,Sx_3),\varphi_4(\rho(x_2,x_3)\}\\ &\le \max\{\ \varphi_1(\varphi\ (M)),\ \varphi_2(\varphi\ (M)),\ \varphi_3(\varphi\ (M)),\ \varphi_4(\varphi\ (M))\}\\ &\le \varphi^2(M) \qquad \ldots(5)\end{aligned}$$

Again from (2),(4) and (5), we get

$$\begin{aligned}\rho(x_5,x_6)&=\rho(STx_3,TSx_4)\\ &\le \max\{\varphi_1(1/2[\rho(x_3,Sx_4)+\ \rho(x_4,Tx_3)]),\ \varphi_2(\ \rho(x_3,Tx_3)),\ \varphi_3(\ \rho(x_4,Sx_4)),\\ &\quad \varphi_4(\rho(x_3,x_4))\}\\ &\le \max\{\ \varphi_1(\varphi\ (M)),\ \varphi_2(\varphi\ ^2(M)),\ \varphi_3(\varphi\ (M)),\ \varphi_4(\varphi\ (M))\}\le \varphi\ ^2(M), \qquad ...(6)\end{aligned}$$

In general, by induction, we get

$$\rho(x_n,x_{n+1})\le \varphi^{[n/2]}(M)$$

for $n \ge 2$, where $[n/2]$ stands for the greatest integer not exceeding $n/2$. Since $\varphi \in \Phi$, by lemma2.2 it follows that $\varphi\ ^n(M) \to 0$ as $n \to +\infty$ for every $M > 0$. Thus, we obtain

$$\rho(x_n,x_{n+1})\to 0 \text{ as } n \to\infty. \qquad \ldots (7)$$

Step2: Suppose that proposition is not true. Then there exists an $\varepsilon > 0$ such that for each $i\in N$, there exist positive integers n_i, m_i, with $i \le n_i<m_i$, satisfying

$$\begin{aligned}&\varepsilon \le \rho(x_{ni},x_{mi})\\ &\quad \le \rho(x_{ni},x_{mi+1})\\ &\rho(x_{ni},x_{mi-1})< \varepsilon \text{ for } i = 1,2,.. \qquad ... (8)\end{aligned}$$

Set,

$$\begin{aligned}&\varepsilon_i= \rho(x_{ni},x_{mi+1})\\ &\rho_i = \rho(x_i,x_{i+1})\text{for } i = 1,2,... \qquad ,,. (9)\end{aligned}$$

Then we have

$$\begin{aligned}\varepsilon &\le \varepsilon_i\\ &= \rho(x_{ni},x_{mi+1})\\ &\le \rho(x_{ni},x_{mi-1})+ \rho(x_{mi-1},x_{mi}) + \rho(x_{mi},x_{mi-1})\\ &< \varepsilon+\rho_{mi-1}+\rho_{mi}\ ,\ i= 1,2,... \qquad ... (10)\end{aligned}$$

Taking the limit as $i\to+\infty$, we get $\lim \varepsilon_i = \varepsilon$. On the other hand, by (2) ,

$$\begin{aligned}\varepsilon_i &= \rho(x_{ni}\ ,x_{mi})\\ &\le \rho(x_{ni},x_{ni+1})+ \rho(x_{ni+1},x_{ni+2}) + \rho(x_{ni+2}\ ,x_{mi+2})+ \rho(x_{mi+2},x_{mi+1}) + \rho(x_{mi+1},x_{mi})\\ &=\rho_{ni}+\rho_{ni+1}+ \rho(x_{ni+2}\ ,x_{mi+2})+\rho_{mi+1}+\rho_{mi}\ ,\ i= 1,2,... \qquad ... (11)\end{aligned}$$

We will now analyze the term $\rho(x_{ni+2}\ ,x_{mi+2})$

$$\begin{aligned}\rho(x_{ni+2}\ ,x_{mi+2})&= \rho(STx_{ni},TSx_{mi})\\ &\le \max\{\varphi_1(1/2[\rho(x_{ni},Sx_{mi})+\ \rho(x_{mi},Tx_{ni})]),\ \varphi_2(\ \rho(x_{ni},Tx_{ni})),\ \varphi_3(\ \rho(x_{mi},Sx_{mi})),\\ &\quad \varphi_4(\ \rho(x_{ni},x_{mi}))\}\end{aligned}$$

$\leq \max\{\varphi_1(1/2[\rho(x_{ni},x_{mi+1})+\rho(x_{mi},x_{ni+1})]),\varphi_2(\rho(x_{ni},x_{ni+1})),\varphi_3(\rho(x_{mi},x_{mi+1})),$
$\varphi_4(\rho(x_{ni},x_{mi}))\}$
$\leq \max\{\varphi_1(1/2[\varepsilon_i+(\varepsilon_i+\rho_{ni-1}+\rho_{ni})]),\varphi_2(\rho_{ni}),\varphi_3(\rho_{mi}),\varphi_4(\varepsilon_i)\}$
$\leq \varphi(\varepsilon_i+\rho_{ni-1}+\rho_{mi}+\rho_{ni})=\varphi(k_i)$ … (12)

where $k_i=\varepsilon_i+\rho_{ni-1}+\rho_{mi}+\rho_{ni}$.

Substituting (11) into (10), taking the limit as $i\to+\infty$, and using the right continuity of φ, we get

$$\varepsilon=\lim_{i\to\infty}\varepsilon_i\leq\lim_{k_i\to\varepsilon+}\varphi(k_i)=\varphi(\varepsilon)<\varepsilon, \quad \dots (13)$$

which is a contradiction. $\lim_{n\to\infty}\rho(x_n,x_m)=0$. Thus $\{x_n\}$ is a Cauchy sequence.

Now we prove our results:

Theorem 2.4:Let (X,ρ) be a G-metric space. Let S and T be self mappings on X satisfying (2) of proposition 2.3 If S or T is continuous and X is ST-orbitally complete, then S and T have a unique common fixed point.

Proof: Let $x_0\in X$ and define $\{x_n\}$ as in (1). Then, by proposition 2.3, it follows that $\{x_n\}$ is a Cauchy sequence. Since X is a ST-orbitally complete G-metric space, $\{x_n\}$ is convergent to a limit u in X. Suppose that S is continuous. Then

$$u=\lim_{n\to\infty}x_{2n+2}=\lim_{n\to\infty}Sx_{2n+1}=S\lim_{n\to\infty}x_{2n+1}=Su. \quad \dots (14)$$

This implies that u is a fixed point of S. From (2), we get $\rho(u,Su)=0$ and

$\rho(u,Tu)=\rho(u,TSu)$
$\leq \rho(u,x_{2n+1})+\rho(x_{2n+1},x_{2n+2})+\rho(STx_{2n},TSu)$
$\leq \rho(u,x_{2n+1})+\rho(x_{2n+1},x_{2n+2})+\max\{\varphi_1(1/2[\rho(x_{2n},Su)+\rho(u,Tx_{2n})]),$
$\varphi_2(\rho(x_{2n},Tx_{2n})),\varphi_3(\rho(u,Su)),\varphi_4(\rho(x_{2n},u))\}$ … (15)

when $n\to\infty$, we get $\rho(u,Tu)=0$. Thus, we have $u=Su=Tu$. Therefore, u is the common fixed point of S and T. The proof for T continuous is similar.

We will now show that u is unique. Suppose that v is also a common fixed point of S and T. Then, from (2),

$\rho(u,v)=\rho(STu,TSv)$
$\leq \max\{\varphi_1(1/2[\rho(u,Sv)+\rho(v,Tu)]),\varphi_2(\rho(u,Tu)),\varphi_3(\rho(v,Sv)),\varphi_4(\rho(u,v)\}$
$=\varphi_1(1/2[\rho(u,v)+\rho(v,u)]),\varphi_2(\rho(u,u)),\varphi_3(\rho(v,v)),\varphi_4(\rho(u,v))\}$
$\leq \varphi(\rho(u,v))$.
… (16)

We write $\rho(u,v)\leq\varphi(\rho(u,v))$, which implies that $\rho(u,v)=0$, that is, $u=v$.
Therefore, the common fixed point of S and T is unique▪

Corollary 2.5: Let (X,ρ) be a ST-orbitally complete G-metric space. Let S and T be self mappings on X satisfying for all $x,y\in X$.

$$\rho(STx,TSy)\leq\alpha\max\{1/2[\rho(x,Sy)+\rho(y,Tx)],\rho(x,Tx),\rho(y,Sy),\rho(x,y)\} \text{ for all } x,y\in X,$$

The sequence $\{x_n\}$ is defined by (1). If S or T is continuous, then S and T have a unique common fixed point.

Proof. The proof follows by taking $\phi i(t) = \alpha t$ with $0 < \alpha < 1$ $(i = 1,2,3,4)$ in Theorem2.4▪

As special case of corollary2.5 we have theorem3.1 in [9], theorem 2.1 in [7] and theorem2.1 in[10]. Now we will prove the following corollary using another condition instead of continuity in Theorem 2.4.

Corollary 2.6 : Let (X, ρ) be a ST-orbitally complete G-metric space. Let S and T be self mappings on X satisfying (2) of proposition2.3, and, for each $u \in X$ with $u \neq Su$ or $u \neq Tu$, let

$$\inf \{\rho(x,u)+ \rho(x, Sx)+ \rho(y,Ty): x,y \in X \} > 0.$$

Then S and T have a unique common fixed point.
Proof: Let $x_0 \in X$ and $\{x_n\}$ defined by (1). From proposition2.3, $\{x_n\}$ is a Cauchy sequence. Since X is a ST-orbitally complete G-metric space, there exists $u \in X$ such that $\{x_n\}$ converges to u. Then we have

$$\begin{aligned}\rho(x_{2n+1},x_{2m+2})&=\rho(TSx_{2n-1},STx_{2m})\\ &\leq \max\{\varphi_1(1/2[\rho(x_{2n-1},Sx_{2m})+\rho(x_{2m},T_{2n-1})]),\varphi_2(\rho(x_{2n-1},x_{2n})),\\ &\quad \varphi_3(\rho(x_{2m},x_{2m+1})),\varphi_4(\rho(x_{2n-1},x_{2m}))\}\\ &\leq \max\{\varphi_1(1/2[\rho(x_{2n-1},x_{2m+1})+ \rho(x_{2m}, x_{2n})]),\varphi_2(\rho(x_{2n-1},x_{2n})),\\ &\quad \varphi_3(\rho(x_{2m},x_{2m+1})),\varphi_4(\rho(x_{2n-1},x_{2m}))\}.\end{aligned}$$

Thus, we obtain $\lim_{n\to\infty} \rho(x_{2n+1},u) = 0$. Assume that $u \neq Su$ or $u \neq Tu$.
Then, by hypothesis, we have

$$\begin{aligned}0 &< \inf \{\rho(x,u)+ \rho(x, Sx)+ \rho(y,Ty): x,y \in X \}\\ &= \inf \{\rho(x_{2n+1},u)+ \rho(x_{2n+1},Sx_{2n+1})+\rho(x_{2n+2},Tx_{2n+2}): n \in N \}\\ &= \inf \{\rho(x_{2n+1},u)+ \rho(x_{2n+1},x_{2n+2})+ \rho(x_{2n+2}, x_{2n+3}): n \in N \}\\ &= 0.\end{aligned}$$

This is a contradiction. Therefore, we have $u = Su = Tu$.On the other hand, we can prove the existence of a unique common fixed point of S and T by a method similar to that of Theorem 2.4.

We can prove the following corollary taking T= I ,the identity mapping, in Theorem 2.4 .
Corollary 2.7:Let (X, ρ) be a ST-orbitally complete G-metric space. Let S and T be self mappings on X satisfying

$$\rho(Sx,Sy)\leq \max\{\varphi_1(1/2[\rho(x,Sy)+ \rho(y,x)]), \varphi_3(\rho(y,Sy)), \varphi_4(\rho(x,y))\} \quad \text{for all } x,y \in X,$$

where $\varphi_i \in \Phi$ $(i = 1,3,4)$.
If S is a continuous, then S has a unique fixed point.

We can prove the following corollary taking T= I ,the identity mapping, in Corollary 2.5 .
Corollary 2.8: Let (X, ρ) be a ST-orbitally complete G-metric space. Let S be self mapping on X satisfying

$$\rho(Sx,Sy)\leq \alpha \max\{1/2[\rho(x,Sy)+\rho(y,x)],\rho(y,Sy),\rho(x,y)\} \quad \text{for all } x,y\in X,$$

for all $x,y \in X$. The sequence $\{x_n\}$ is defined by $x_0 \in X$, $x_{n+1} = S x_n$. If S is continuous, then S has a unique fixed point.

We can prove the following corollary taking T= I ,the identity mapping, in Corollary 2.6.

Corollary 2.9 : Let (X, ρ) be a ST-orbitally complete G-metric space. Let S be self mapping on X satisfying (2) of proposition 2.3, and, for each u ∈ X with u ≠ Su , let

$$\inf \{\rho(x,u)+ \rho(x, Sx) : x,y \in X \} > 0.$$

Then S have a unique fixed point.

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مبرهنة حول النقطة الصامدة لتطبيقين غير متبادلين

سلوى سلمان عبد
آلاء عبد الله
قسم علوم الرياضيات / كلية التربية للعلوم الصرفة (ابن الهيثم) / جامعة بغداد

استلم البحث في: 19 حزيران 2012 ، قبل البحث في: 15 تشرين الاول 2012

الخلاصة

في هذا البحث برهنا نتيجة حول وجود ووحدانية نقطة صامدة مشتركة ةلتطبيقين غير متبادلين معرفين على فضاء G – متري كامل مساريا ، إذ استخدمنا تطبيق انكماشي معمم .

الكلمات المفتاحية : فضاء G – متري ، كامل مساريا ، تطبيقات غير متبادلة، نقطة صامدة مشتركة.