Strongly (Completely) Hollow Sub-modules II

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Abstract

Let M be an R-module, where R is commutative ring with unity. In this paper we study the behavior of strongly hollow and quasi hollow submodule in the class of strongly comultiplication modules. Beside this we give the relationships between strongly hollow and quasi hollow submodules with V-coprime, coprime, bi-hollow submodules.

Key Words : Strongly hollow and quasi hollow submodules, strongly comultiplication modules, V-coprime module, V-coprime submodule, coprime submodule, bi-hollow module, bi-hollow submodule.

Introduction

Throughout this paper, all rings are commutative rings with identity and all modules are unital module. In this research we investigate some properties of strongly hollow and quasi hollow submodules in the class of strongly comultiplication modules, see proposition 2.7, proposition 2.8.
Next, we introduce the relationships between strongly hollow and quasi hollow modules with coprime modules, V-coprime modules and bi-hollow modules.
Also, we study the relationships between strongly hollow and quasi hollow submodule with V-coprime (bi-hollow)-submodules.

1- Some Basic Definitions

1.1 Definition: [1,4.2]

Let $0 \neq L \leq M$, then L is called a strongly-hollow submodule (briefly, SH-submodule) if for every $L_1, L_2 \leq M$ with $L \leq L_1 + L_2$ implies $L \leq L_1$ or $L \leq L_2$, we say that an R-module M is a strongly-hollow module if M is a strongly hollow submodule of itself.

1.2 Remark:

Let $0 \neq L \leq M$, L is a SH-submodule if for each $L_1, \ldots, L_n \leq M$ with $L \leq L_1 + L_2 + \ldots + L_n$, implies $L \leq L_1$ or $L \leq L_2$ or … or $L \leq L_n$.

1.3 Definition: [1, 4.2]

Let $0 \neq L \leq M$, then L is called a completely hollow submodule (briefly, CH-submodule) if for any collection $\{L_\lambda\}_{\lambda\in\Lambda}$ of R-submodules of M with $L = \sum_{\lambda\in\Lambda} L_\lambda$, implies $L = L_\lambda$ for some $\lambda\in\Lambda$.

We say that an R-module M is completely hollow (briefly, CH-module) if M is completely hollow submodule of itself.

1.4 Definition: [2, Definition 1.13]

Let $0 \neq L \leq M$, then L is called a quasi-hollow submodule (briefly, qH-submodule) if for each $L_1, L_2 \leq M$ with $L \leq L_1 + L_2$, then either $L = L_1$ or $L = L_2$.
An R-module M is called a quasi-hollow module if M is a quasi-hollow submodule of itself.

1.5 Remark: [2, Remark 1.14]

Let $0 \neq L \leq M$, L is a quasi-hollow submodule if for each $L_1, \ldots, L_n \leq M$ with $L \leq L_1 + L_2 + \ldots + L_n$, then either $L = L_1$ or $L = L_2$ or … or $L = L_n$.

1.6 Definition: [3]

Let M be an R-module M is called distributive if for each $N, K, L \leq M$, $N \cap (K + L) = (N \cap K) + (N \cap L)$.

1.7 Definition: [4]

Let M be an R-module and let $N \leq M$. N is called a strongly irreducible submodule (briefly, SI-submodule) if for each $K, L \leq M$, $N \supseteq K \cap L$ implies either $N \supseteq K$ or $N \supseteq L$.

1.8 Definition: [5]

Let M be an R-module and let $N \leq M$. N is called an irreducible submodule if for each $K, L \leq M$, $N = K \cap L$ implies either $N = K$ or $N = L$.

2- SH (qH,CH) Submodules and Strongly Comultiplication Modules

We start this section by the following definition.

2.1 Definition: [6, Definition 2.1]

An R-module M is called strongly comultiplication if $I = \underset{R}{\text{ann}}\,\underset{M}{\text{ann}}\,I$ for every ideal I of R, and M is comultiplication, where M is comultiplication if for any $L \le M$, there exists an ideal $I \le R$ such that $L = \underset{M}{\text{ann}}\,\underset{R}{\text{ann}}\,L$.

Equivalently, M is strongly comultiplication if for every $L \le M$ and for every ideal $I \le R$, $I = \underset{R}{\text{ann}}\,\underset{M}{\text{ann}}\,I$ and $L = \underset{M}{\text{ann}}\,\underset{R}{\text{ann}}\,L$.

It is clear that every strongly comultiplication is comultiplication, but the converse is not true as the following examples show:

1. Z_4 as Z-module is comultiplication, see [2, Example 1.5(3)]. But it is not strongly comultiplication, since for the ideal I = <6>, $I \ne \underset{Z}{\text{ann}}\,\underset{Z_4}{\text{ann}} < 6 >$, because $\underset{Z_4}{\text{ann}} < 6 > = < \bar{2} >$ and $\underset{Z}{\text{ann}} < \bar{2} > = < 2 > \ne I$.

2. Consider Z-module Z_{3^∞} is comultiplication, see [2, Example 1.5(1)]. But it is not strongly comultiplication, since if we take I = <2>, then $\underset{Z_{3^\infty}}{\text{ann}} < 2 > = 0_{Z_{3^\infty}}$ and $\underset{Z}{\text{ann}}\,0_{Z_{3^\infty}} = Z$. So $I \ne \underset{Z}{\text{ann}}\,\underset{Z_{3^\infty}}{\text{ann}}\,I$.

3. Z_2 as Z_4-module.
Z_4 is comultiplication ring, but Z_2 is not strongly comultiplication. Since $< \bar{0} > \ne \underset{Z_4}{\text{ann}}\,\underset{Z_2}{\text{ann}} < \bar{0} >$ because $\underset{Z_2}{\text{ann}} < \bar{0} > = Z_2$ and $\underset{Z_4}{\text{ann}}\,Z_2 = < \bar{2} > \ne < \bar{0} >$.

2.2 Remark:

Let R be a ring. Then R is comultiplication if and only if R is strongly comultiplication.

Proof: It is clear.

Now we can give the following examples:

1. Z_n as Z_n-module is strongly comultiplication ring (comultiplication).
2. $R=Z_2[x,y]/<x^2,y^2>$ is strongly comultiplication ring. All ideals of R are $< \bar{x} >, < \bar{y} >, < \overline{x,y} >, < \overline{xy} >$. $\underset{R}{\text{ann}} < \bar{x} > = < \bar{x} >$, $\underset{R}{\text{ann}} < \bar{y} > = < \bar{y} >$, $\underset{R}{\text{ann}} < \overline{x,y} > = < \bar{x} \cdot \bar{y} >$ and $\underset{R}{\text{ann}} < \bar{x} \cdot \bar{y} > = < \bar{x}, \bar{y} >$.

Recall that a ring R is QF if R self-injective ring and noetherian. Equivalently, if R is noertherian and every ideal is an annihilator ($I = \underset{R}{\text{ann}}\,\underset{R}{\text{ann}}\,I$) [7], hence we have:

2.3 Remark:

Every QF-ring is strongly comultiplication (comultiplication) ring.

The following lemmas are needed in our work.

2.4 Lemma:

Let M be an R-module. Let $I_1, I_2 \le R$. Then $\underset{M}{\text{ann}}(I_1 + I_2) = \underset{M}{\text{ann}}\,I_1 \cap \underset{M}{\text{ann}}\,I_2$.

Proof: It is clear, so is omotted.

2.5 Lemma:

Let M be a strongly comultiplication R-module, and let $L_1, L_2 \leq M$. Then $\underset{R}{\text{ann}}(L_1 \cap L_2) = \underset{R}{\text{ann}} L_1 + \underset{R}{\text{ann}} L_2$.

Proof:

Let $L_1, L_2 \leq M$. Since M is comultiplication, then $L_1 = \underset{M}{\text{ann}} I_1$, $L_2 = \underset{M}{\text{ann}} I_2$ for some $I_1, I_2 \leq R$, hence $\underset{R}{\text{ann}}(L_1 \cap L_2) = \underset{R}{\text{ann}}(\underset{M}{\text{ann}}(I_1 + I_2))$, by lemma 2.4

$= I_1 + I_2$, since M is strongly comultiplication

$= \underset{R}{\text{ann}}(\underset{M}{\text{ann}} I_1) + \underset{R}{\text{ann}}(\underset{M}{\text{ann}} I_2)$

$= \underset{R}{\text{ann}} L_1 + \underset{R}{\text{ann}} L_2$

2.6 Lemma:

Let M be a strongly comultiplication R-module. Let $I_1, I_2 \leq R$. Then $\underset{M}{\text{ann}}(I_1 \cap I_2) = \underset{M}{\text{ann}} I_1 + \underset{M}{\text{ann}} I_2$.

Proof:

Since M is a stronglycomultiplication, so

$I_1 \cap I_2 = \underset{R}{\text{ann}}\ \underset{M}{\text{ann}}(I_1 \cap I_2)$ and $I_1 \cap I_2 = \underset{R}{\text{ann}}\ \underset{M}{\text{ann}} I_1 \cap \underset{R}{\text{ann}}\ \underset{M}{\text{ann}} I_2$

$= \underset{R}{\text{ann}}(\underset{M}{\text{ann}} I_1 + \underset{M}{\text{ann}} I_2)$

Thus $\underset{R}{\text{ann}}\ \underset{M}{\text{ann}}(I_1 \cap I_2) = \underset{R}{\text{ann}}(\underset{M}{\text{ann}} I_1 + \underset{M}{\text{ann}} I_2)$. Hence $\underset{M}{\text{ann}}(I_1 \cap I_2) = \underset{M}{\text{ann}} I_1 + \underset{M}{\text{ann}} I_2$.

2.7 Proposition:

Let M be a strongly comultiplication R-module. Then:

(1) Every non-zero proper ideal of R is SH-ideal if and only if every non-zero proper submodule of M is SI-submodule.

(2) Every non-zero proper ideal of R is qH-ideal if and only if every non-zero proper submodule of M is irreducible.

Proof:

(1) $\Rightarrow$ Let $<0> \neq N \lneqq M$, $N \supseteq L_1 \cap L_2$ where $L_1, L_2 \leq M$. Then

$\underset{R}{\text{ann}} N \subseteq \underset{R}{\text{ann}}(L_1 \cap L_2)$

$= \underset{R}{\text{ann}} L_1 + \underset{R}{\text{ann}} L_2$, by lemma 2.5.

But $\underset{R}{\text{ann}} N$ is a non-zero proper ideal. So by hypothesis it is SH. Hence $\underset{R}{\text{ann}} N \subseteq \underset{R}{\text{ann}} L_1$ or $\underset{R}{\text{ann}} N \subseteq \underset{R}{\text{ann}} L_2$. Then $\underset{M}{\text{ann}}\ \underset{R}{\text{ann}} N \supseteq \underset{M}{\text{ann}}\ \underset{R}{\text{ann}} L_1$ or $\underset{M}{\text{ann}}\ \underset{R}{\text{ann}} N \supseteq \underset{M}{\text{ann}}\ \underset{R}{\text{ann}} L_2$. It follows that $N \supseteq L_1$ or $N \supseteq L_2$.

$\Leftarrow$ Let $<0> \neq I \leq R$, $I \subseteq I_1 + I_2$ where $I_1, I_2 \leq R$. Then

$\underset{M}{\text{ann}} I \supseteq \underset{M}{\text{ann}}(I_1 + I_2)$

$= \underset{M}{\text{ann}} I_1 \cap \underset{M}{\text{ann}} I_2$, by lemma 2.4.

But $\underset{M}{\text{ann}} I \neq (0)$ and $\underset{M}{\text{ann}} I \lneqq M$, so by hypothesis, $\underset{M}{\text{ann}} I$ is SI. Thus $\underset{M}{\text{ann}} I \supseteq \underset{M}{\text{ann}} I_1$ or $\underset{M}{\text{ann}} I \supseteq \underset{M}{\text{ann}} I_2$. Then $\underset{R}{\text{ann}}\ \underset{M}{\text{ann}} I \subseteq \underset{R}{\text{ann}}\ \underset{M}{\text{ann}} I_1$ or $\underset{R}{\text{ann}}\ \underset{M}{\text{ann}} I \subseteq \underset{R}{\text{ann}}\ \underset{M}{\text{ann}} I_2$. It follows $I \subseteq I_1$ or $I \subseteq I_2$.

(2) a similar proof of part (1), so is omitted.

2.8 Proposition:

Let M be a strongly comultiplication R-module and let $<0> \neq N \subsetneqq M$. Then:

(a) $\underset{R}{\text{ann}}\,N$ is SI-ideal if and only if N is SH-submodule.

(b) $\underset{R}{\text{ann}}\,N$ is irreducible if and only if N is qH-submodule.

Proof:

(a) $\Rightarrow$ Let $N \subseteq L_1 + L_2$ where $L_1, L_2 \leq M$. So

$$\underset{R}{\text{ann}}\,N \supseteq \underset{R}{\text{ann}}\,(L_1 + L_2) = \underset{R}{\text{ann}}\,L_1 \cap \underset{R}{\text{ann}}\,L_2$$

Since $\underset{R}{\text{ann}}\,N$ is SI-ideal, $\underset{R}{\text{ann}}\,N \supseteq \underset{R}{\text{ann}}\,L_1$ or $\underset{R}{\text{ann}}\,N \supseteq \underset{R}{\text{ann}}\,L_2$. Then $\underset{M}{\text{ann}}\,\underset{R}{\text{ann}}\,N \subseteq \underset{M}{\text{ann}}\,\underset{R}{\text{ann}}\,L_1$ or $\underset{M}{\text{ann}}\,\underset{R}{\text{ann}}\,N \subseteq \underset{M}{\text{ann}}\,\underset{R}{\text{ann}}\,L_2$. Thus $N \subseteq L_1$ or $N \subseteq L_2$.

$\Leftarrow$ Let $\underset{R}{\text{ann}}\,N \supseteq I_1 \cap I_2$ where $I_1, I_2 \leq R$. So

$$\underset{M}{\text{ann}}\,\underset{R}{\text{ann}}\,N \subseteq \underset{M}{\text{ann}}\,(I_1 \cap I_2) = \underset{M}{\text{ann}}\,I_1 + \underset{M}{\text{ann}}\,I_2$$

Then $N \subseteq \underset{M}{\text{ann}}\,I_1 + \underset{M}{\text{ann}}\,I_2$. Thus $N \subseteq \underset{M}{\text{ann}}\,I_1$ or $N \subseteq \underset{M}{\text{ann}}\,I_2$, since N is a SH-submodule, so $\underset{R}{\text{ann}}\,N \supseteq \underset{R}{\text{ann}}\,\underset{M}{\text{ann}}\,I_1$ or $\underset{R}{\text{ann}}\,N \supseteq \underset{R}{\text{ann}}\,\underset{M}{\text{ann}}\,I_2$. Hence $\underset{R}{\text{ann}}\,N \supseteq I_1$ or $\underset{R}{\text{ann}}\,N \supseteq I_2$. Then $\underset{R}{\text{ann}}\,N$ is SI-ideal.

(b) similar proof of part (a).

Now we have several consequences of the previous proposition.

2.9 Corollary:

Let M be a strongly comultiplication R-module and let $<0> \neq I \subsetneqq R$, $N \leq M$. Then:

(1) I is a SI-ideal if and only if $\underset{M}{\text{ann}}\,I$ is a SH-submodule.

(2) I is an irreducible if and only if $\underset{M}{\text{ann}}\,I$ is qH-submodule.

(3) Every non-zero submodule of M is SH-if and only if every non-zero proper ideal of R is SI.

(4) Every non-zero submodule of M is qH if and only if every non-zero proper ideal of R is irreducible.

Proof:

(1) Since M is strongly comultiplication, $I = \underset{R}{\text{ann}}\,\underset{M}{\text{ann}}\,I$. Hence $\underset{M}{\text{ann}}\,I \neq <0>$. Put $N = \underset{M}{\text{ann}}\,I$, so $I = \underset{R}{\text{ann}}\,N$. Hence $I = \underset{R}{\text{ann}}\,N$ is a SI-ideal if and only if $N = \underset{R}{\text{ann}}\,N$ is SH, see Proposition 2.7 part (a).

(2) It follow by Proposition 2.7 part (b) and a similar proof of part (1).

(3) Let $<0> \neq N \subsetneqq M$. Then $N = \underset{M}{\text{ann}}\,\underset{R}{\text{ann}}\,N$. Put $\underset{R}{\text{ann}}\,N = I$, then $N = \underset{M}{\text{ann}}\,I$, hence $I = \underset{R}{\text{ann}}\,N$ is SI-ideal if and only if $\underset{M}{\text{ann}}\,I = N$ is SH-submodule by part (1). Thus we get the result.

(4) It follows similarly from part (2).

2.10 Corollary:

Let R be a comultiplication ring, <0> ≠ I ≤ R. Then:

(1) I is SH-ideal if and only if $\underset{R}{\text{ann}}\, I$ is SI-ideal.

(2) I is qH-ideal if and only if $\underset{R}{\text{ann}}\, I$ is irreducible.

(3) I is SI-ideal if and only if $\underset{R}{\text{ann}}\, I$ is SH-ideal.

Proof: It follows directly from Proposition 2.8 and Corollary 2.9.

2.11 Corollary:

Let M be a strongly comultiplication and distributive R-module, let <0> ≠ N ≤ M. Then the following statements are equivalent:

(1) N is SH-submodule.

(2) N is qH-submodule.

(3) $\underset{R}{\text{ann}}\, N$ is irreducible ideal.

(4) $\underset{R}{\text{ann}}\, N$ is SI-ideal.

Proof:

(1) ⇔ (2) by [2, Proposition 1.16].

(1) ⇔ (3) and **(1) ⇔ (4)** by Proposition 2.8.

2.12 Lemma:

Let M be a strongly comultiplication R-module and R is distributive. Then M is distributive.

Proof:

Since $\underset{M}{\text{ann}}\,(I_1 \cap I_2) = \underset{M}{\text{ann}}\, I_1 + \underset{M}{\text{ann}}\, I_2$ by Lemma 2.6. So by [8, Lemma 3.16], M is distributive.

2.13 Corollary:

Let M be a strongly comultiplication over distributive ring R, and let <0> ≠ N ≤ M. Then the following statements are equivalent:

(1) N is SH-submodule.

(2) N is qH-submodule.

(3) $\underset{R}{\text{ann}}\, N$ is irreducible.

(4) $\underset{R}{\text{ann}}\, N$ is SI-ideal.

Proof: It follows from Lemma 2.12 and Corollary 2.11.

2.14 Proposition:

Let M be a strongly comultiplication R-module. Then R satisfies dcc(acc) on SI-ideal if and only if M satisfies acc(dcc) on SH-submodule.

Proof:

⇒ Let $N_1 \subseteq N_2 \subseteq \ldots$ be an ascending chain of SH-submodules; $N_i \neq$ <0> for each i = 1,2,… . So $\underset{R}{\text{ann}}\, N_i$ is SI-ideal for each i = 1,2,… , see Proposition 2.5 part (a). But $\underset{R}{\text{ann}}\, N_1 \supseteq \underset{R}{\text{ann}}\, N_2 \supseteq \ldots$. So by dcc on SI-ideal of R, there exists $n \in Z_+$ such that $\underset{R}{\text{ann}}\, N_n = \underset{R}{\text{ann}}\, N_{n+1} = \ldots$. Then $\underset{M}{\text{ann}}\, \underset{R}{\text{ann}}\, N_n = \underset{M}{\text{ann}}\, \underset{R}{\text{ann}}\, N_{n+1} = \ldots$. Thus we get $N_n = N_{n+1} = \ldots$.

⇐ The proof is similarly.

By a similar proof R satisfies acc on SI-ideals if and only if M satisfies dcc on SH-submodules.

3- SH(qH) and V-Coprime (Bihollow) Submodules

Recall that an R-module M is called coprime module if $\operatorname{ann}_R M = \operatorname{ann}_R \frac{M}{N}$, for every proper submodule N of M, see [9].
Equivalently, M is coprime if for every non-zero ideal I of R, either IM = <0> or IM = M, see [10].

A proper submodule K of M is called coprime submodule in M if M/K is a coprime R-module, see [1, Proposition 3.10].

3.1 Remark:
The concept of coprime module and SH(qH)-module are independent as the following examples show:

(1) The Z-module Q is coprime module, since for every r ∈ Z, either rQ = <0> or rQ = Q. But Q is not SH(not qH), see [2, Remark 1.4(3)].
(2) Z_4 as Z-module is SH(qH). But Z_4 is not coprime Z-module, since $2Z_4 = <\bar{2}> \neq Z_4$ and $2Z_4 \neq <\bar{0}>$.

Recall that a proper submodule N of an R-module M is called invariant if for each $f \in \mathrm{End}_R M$, $f(N) \subseteq N$, see [11]. Invariant submodule is called fully invariant submodule by some authors such as see [10].
In 2005, [12] define coprime submodules and modules as follows:

3.2 Definition:
Let K ≤ M be a fully invariant submodule, K is called a coprime submodule of M if for any fully invariant submodules L, L' ≤ M with $K \subseteq (L \underset{M}{:} L')$ implies $K \subseteq L$ or $K \subseteq L'$ where $(L \underset{M}{:} L') = \cap\{f^{-1}(L'); f \in \mathrm{End}_R M, f(L) = <0>\}$. M is called coprime module if M coprime submodule of itself.

3.3 Lemma:
Let L, L' be fully invariant submodules of an R-module M. Then $L + L' \subseteq (L \underset{M}{:} L')$, see [12, 4.1(ii)].

3.4 Remark:
The concept of coprime submodules (in the sense of J.Abu.), and (in the sence of J.Rios) are independent as the following examples show:
(1) Consider the Z-module Z, let N = <3>. N is coprime submodule in Z (in the sense of J.Abu) since $Z/N \simeq Z_3$ which is a coprime Z-module. But N is not coprime submodule (in the sense of J.Rios), because N = <3> ⊆ <4> + <5> = Z. But by previous lemma <4> + <5> ⊆ $(<4> \underset{Z}{:} <5>)$. So $N \subseteq (<4> \underset{Z}{:} <5>)$ and N ⊈ <4>, N ⊈ <5>.

(2) Consider the Z-module Z_8. Let $N = <\overline{4}>$. N is not coprime submodule (in the sense of J.Abu.), because $Z_8/<\overline{4}> \simeq Z_4$ which is not coprime Z-module. Now $N \subseteq <\overline{2}> + <\overline{0}>$, and by lemma 3.3, $<\overline{2}> + <\overline{0}> \subseteq (<\overline{2}> \underset{Z_8}{:} <\overline{0}>)$, so $N \subseteq (<\overline{2}> \underset{Z_8}{:} <\overline{0}>)$ and $N \subseteq <\overline{2}>$.

Similarly, $N \subseteq <\overline{4}> + <\overline{0}> \subseteq (<\overline{4}> \underset{Z_8}{:} <\overline{0}>)$ and $N \subseteq <\overline{4}>$. Thus, for any L, L' ≤ Z_8, $N \subseteq (L \underset{M}{:} L')$, implies $N \subseteq L$ or $N \subseteq L'$. Then N is coprime submodule (in the sense of J.Rios).

3.5 Remark:

The concept of coprime module (in the sense of S.Annin,2002) and (in the sense of J.Rios, 2005) are independent as the following example shows:
The Z-module Z_4 is not coprime module (in the sense of S.Annin), see Remark 3.1(2). But Z_4 is coprime Z-moduule (in sense of J.Rios) as follows:
Suppose $Z_4 \subseteq (L \underset{Z_4}{:} L')$. Since $(L \underset{Z_4}{:} L')$ is a submodule of Z_4, then $(L \underset{Z_4}{:} L') = Z_4$. But by simple calculation, we have:
$(<\overline{0}>:<\overline{0}>) = <\overline{0}>, (<\overline{0}>:<\overline{2}>) = <\overline{2}>, (<\overline{0}>: Z_4) = Z_4, (Z_4 :<\overline{2}>) = Z_4$,
$(<\overline{2}>:<\overline{0}>) = <\overline{2}>$. Thus if $(L \underset{Z_4}{:} L') = Z_4$, then either $L = Z_4$ or $L' = Z_4$; that is Z_4 is coprime Z-module (in the sense of J.Rios).

3.6 Remark:

The concept of coprime module (in sense of J.Rios) and SH(qH)-module are independent as follows:
The Z-module Q is coprime (in sense of J.Rios) by [12, Remark 4.7(2)]. But Q is not SH(qH) as Z-module by [2, Remark 1.4(3)].

In 2005, [12] introduced the concept of V-coprime and bi-hollow submodules as follows :

3.7 Definition:

If K, L, L' are fully invariant submodules of M. Then:
(1) K is called V-coprime if $K \subseteq L + L'$ implies $K \subseteq L$ or $K \subseteq L'$.
(2) K is called bi-hollow if $K = L + L'$ implies $K = L$ or $K = L'$.
(3) M is called bi-hollow if $M = L + L'$ implies $M = L$ or $M = L'$.

3.8 Remark:

If K is V-coprime submodule of an R-module M, then K is bi-hollow.
Proof:
Let $K = L + L'$ where L, L' are fully invariant submodules of M. So $K \subseteq L + L'$ and since K is V-coprime, then $K \subseteq L$ or $K \subseteq L'$. But $K \supseteq L + L'$. Hence $K = L$ or $K = L'$.

3.9 Proposition:

M is bi-hollow R-module if and only if M is V-coprime.
Proof:
$\Rightarrow$ Let $M \leq L + L'$ where L, L' are fully invariant submodules of M. So $M = L + L'$, then $M = L$ or $M = L'$ since M is bi-hollow. Hence $M \subseteq L$ or $M \subseteq L'$.
$\Leftarrow$ Let $M = L + L'$, then $M \subseteq L + L'$. So $M \subseteq L$ or $M \subseteq L'$. Since M is V-coprime. Hence $M = L$ or $M = L'$.

3.10 Proposition:

If K is coprime submodule of an R-module M (in the sense of J.Rios), then K is V-coprime.

Proof:

Let $K \subseteq L + L'$ where L, L' are fully invariant submodules of M. But $L + L' \subseteq (L \underset{M}{:} L')$ by Lemma 3.3. So $K \subseteq (L \underset{M}{:} L')$ and since K is coprime submodule of M, hence either $K \subseteq L$ or $K \subseteq L'$. Thus K is V-coprime, hence K is bi-hollow.

3.11 Remark:

The concept of V-coprime (bi-hollow) does not implies SH(qH) modules as the following examples shows:
The Z-module Q is coprime (in sense J.Rios), then Q is V-coprime (bi-hollow) by previous Proposition. But Q is not SH (not qH) by [2, Remark 1.4(3)].

3.12 Lemma:

Let M be an R-module. If M is a multiplication (or cyclic or scalar or comultiplication). Then every submodule of M is fully invariant.

Proof:

It is known if M is a multiplication, then every submodule is fully invariant of M.
If M is cyclic, then M is multiplication, so every submodule is fully invariant. If M is scalar R-module, then for each $f \in End_R M$, there exists $r \in R$ such that $f(x) = rx$ for all $x \in M$. Hence, if $N \leq M$, then $f(N) = rN \subseteq N$; that is N is fully invariant.
If M is comultiplication, then every submodule is fully invariant, see [8, Theorem 3.17(a)].

3.13 Corollary:

Let M be a multiplication (or cyclic or scalar or comultiplication), let N be a non-zero submodule of M. Then
(1) N is V-coprime if and only if N is SH-submodule.
(2) N is bi-hollow if and only if N is qH-submodule.
(3) M is bi-hollow if and only if M is SH-submodule.

Proof:

It follows directly from the previous lemma and definitions of SH(qH) and V-coprime (bi-hollow) submodules.

3.14 Remark:

The concept of coprime submodule (in sense of J.Abu.) and V-coprime submodule are independent as the following examples show:
(1) Consider the Z-module Z. Let $N = \langle 3 \rangle \subseteq Z$. N is coprime submodule of Z (in sense of J.Abu) by Remark 3.4(1). But N is not SH. Hence N is not V-coprime by Corollary 3.13. Since Z is multiplication Z-module.
(2) Consider the Z-module Z_8. Let $K = \langle \bar{4} \rangle$ is a V-coprime submodule of Z_8 as Z-module. But K is not coprime submodule (in sense of J.Abu), because $Z_8/K \simeq Z_4$ which is not coprime module (in sense of S,Annin or Wij.).

Now we investigate the behaviour of SH(qH)-submodule under the Localization.

3.15 Proposition:

Let N be a submodule of an R-module M, and S is multiplicatively closed subset of R. Then $S^{-1}N$ is SH(qH-submodule) of $S^{-1}M$ as $S^{-1}R$-module if and only if N is SH(qH-submodule) of M. [Provided, $S^{-1}N \subseteq S^{-1}W \Leftrightarrow N \subseteq W$].

Proof:

$\Rightarrow$ If $S^{-1}N$ is qH, let $N = L_1 + L_2$ where $L_1, L_2 \leq M$. So $S^{-1}N = S^{-1}(L_1 + L_2)$. Then $S^{-1}N = S^{-1}L_1 + S^{-1}L_2$, so $S^{-1}N = S^{-1}L_1$ or $S^{-1}N = S^{-1}L_2$ since $S^{-1}N$ is qH. Hence $N = L_1$ or $N = L_2$.

$\Leftarrow$ If N is qH, let $S^{-1}N = S^{-1}L_1 + S^{-1}L_2$, where $S^{-1}L_1, S^{-1}L_2 \leq S^{-1}M$
$= S^{-1}(L_1 + L_2)$.

Then $S^{-1}N = S^{-1}(L_1 + L_2)$ implies $N = L_1 + L_2$. So $N = L_1$ or $N = L_2$ by hypothesis.
$S^{-1}N = S^{-1}L_1$ or $S^{-1}N = S^{-1}L_2$.

By a similar proof, N is SH iff $S^{-1}N$ is SH.

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المقاسات الجزئية المجوفة (التامة) بقوة II

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الخلاصة

ليكن M مقاساً على R حيث R حلقة ابدالية ذات محايد. في هذا البحث درسنا سلوك كل من المقاسات الجزئية المجوفة بقوة وشبه المجوفة في المقاسات الجدائية المضادة بقوة. اضافة الى هذا درسنا العلاقات بين المقاسات الجزئية المجوفة بقوة وشبه المجوفة مع المقاسات الجزئية الاولية المضادة في النمط V والمقاسات الجزئية المضادة والمقاسات المجوفة الثنائية.

الكلمات المفتاحية : المقاسات الجزئية المجوفة بقوة وشبه المجوفة ، المقاسات الجدائية المضادة بقوة، المقاسات الاولية المضادة من النمط V، المقاسات الجزئية الاولية المضادة من النمط V، المقاسات الجزئية المضادة، المقاسات المجوفة الثنائية، المقاسات الجزئية المجوفة الثنائية.