

Solving Linear Boundary Value Problem Using Shooting Continuous Explicit Runge-Kutta Method

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Abstract

In this paper we shall generalize fifth explicit Runge-Kutta Feldberg(ERKF(5)) and Continuous explicit Runge-Kutta (CERK) method using shooting method to solve second order boundary value problem which can be reduced to order one. These methods we shall call them as shooting Continuous Explicit Runge-Kutta method, the results are computed using matlab program.

Keyword: Boundary Value problems, Shooting Method ,Continuous Explicit Runge-Kutta Method

Introduction

The boundary value problem with second order describe many phenomena in the applied sciences, found in the theory of nonlinear diffusion generated by nonlinear sources, thermal ignition of gases and concentration in chemical or biological problems.[1]

Consider second order linear two-point boundary value problem of the form: -[2]

$$y''(x) + f(x)y' + g(x)y(x) = h(x) \tag{1}$$

with the following boundary conditions:-

$$y(a) = \alpha \quad , \quad y(b) = \beta$$

where $a < x < b$ and $f(x), g(x), h(x)$ are given continues functions.

The boundary value problem of second order have been studied by a number of researchers, for example, Min and Salvador using spline collocation methods to solve fractional second order Boundary Value Problems [3], while Guoju and Xiuli investigate the existence of continues solutions for the second order boundary value problems with integral boundary conditions[4].

In this paper, we generalize the Shooting Continuous Runge-Kutta method to solve the boundary value problem of second order.

Explicit Runge-Kutta Method (ERK)[5,6,7,8]

Explicit Rung-Kutta method is one of the methods to solve initial value problems,Bucher put on the basic rules of using the explicit Rung-Kutta method to solve ODE'S and DDE's, which is completed by Jeltsch and Torrilhon.

The general form of s-stage explicit Runge-Kutta formula, to compute the numerical solution of ordinary differential equations:-

$$y'(x) = f(x, y) \tag{2}$$

With initial value condition $y(x_0) = y_0$ at $(x_n + h_n)$ is defined by:-

$$y_{n+1} = y_n + h_n \sum_{i=1}^s b_i k_i \tag{3}$$

$$\text{where } k_i = f(x_n + c_i h_n, y_n + h_n \sum_{j=1}^{i-1} a_{ij} k_j) \quad , \quad c_i = \sum_{j=1}^{i-1} a_{ij} \tag{4}$$

for $i=1,2,\dots,s$ and a_{ij} , b_i and c_i are the coefficients of Runge-Kutta formula and y_n is the associated approximation to $y(x_n)$.

In this paper, we use one important type of Runge-Kutta method when $a_{ij} = 0$, $f(i \leq j)$ and $c_1 = 0$ which is called Explicit Runge-Kutta method.

The interval $[x_0, x_N]$ can be divided into a so called mesh points $x_0 < x_1 < \dots < x_N = x_N$ and generate a discrete approximation $y_i \cong y(x_i)$ for each associated distribution of these mesh points is usually determined adaptively by the method in an attempt to deliver acceptable accuracy at minimum cost.

These methods generally accomplish this objective by keeping N as small as possible subject to constraint at an indirect measure of $\max_{i=1,2,\dots,N} \|y(x_i) - y_i\|$ would be kept small (relation to an accuracy parameter TOL).

Continuous Explicit Runge-Kutta Method (CERK)[9, 10,11]

The ERK methods can be developed to be called Continuous Explicit Runge-Kutta method to solve ordinary differential equation (ODE) and Delay differential equation (DDE). The first of research in this subject, Enright and Hayashi.

A Continuous Explicit Runge-Kutta (CERK) method provides an approximation to an initial value problem. Such method may be obtained by appending additional stage to a discrete method, or alternatively by solving the appropriate order conditions directly. So, for some application, an approximation to the solution of the ODE:-

$$y' = f(x, y(x)) \tag{5}$$

$$y(x_0) = y_0$$

on the interval $[x_0, x_N]$ is needed, to obtain dense, output for plotting the solution of a standard ODE, to find the roots of the function associated with the solution of a standard ODE.

Consequently, continuous (CERK) formula, which produces continuous approximation to the solution of an ODE on $[x_0, x_N]$ has been developed Enright and have the form:-

$$z_n(x_n + rh_n) = z_{n-1}(x_n) + h_n \sum_{i=1}^{\bar{s}+1} b_i(r)k_i;$$

$$k_i = f(x_n + c_i h_{ni-1}, y_i) \tag{6}$$

$$y_i = z_{n-1}(x_n) + h_n \sum_{j=1}^{i-1} a_{ij}k_j$$

for $i=1,2,\dots,\bar{s} + 1$, and $r \in [0,1]$ where $b_1(r), b_2(r), \dots, b_{\bar{s}+1}(r)$ are polynomial with respect to r and $z_{n-1}(x_0) = y_0$

This class of (CERK) formula is constructed by adding extra stage such that (s+1) becomes $(\bar{s} + 1)$ with $c_{s+1} = 1, a_{s+1,j} = b_j$ for $j=1,2,\dots,s$ so that the approximation can easily be developed as a c' interpolate. Note that since $k_{s+1} = f(x_{n+1}, z_n(x_{n+1}))$ will be the first stage on the next attempted stops the actual cost per successful step is \bar{s} derivative evaluation.

Shooting Method[12,13,14]

Arciniega and Allen applied a shooting method procedure to numerically solve linear systems of stratonovich boundary value problems.

To illustrate the approach, we use the following TPBVP:-

$$y''(x) + f(x)y' + g(x)y(x) = h(x)$$

$$y(a) = \alpha, \quad y(b) = \beta$$

(7)

The curve that represents y between these two points is desired, we anticipate that some such curve, such as the dotted line, its slope and curve true are interrelated to y and x the diff. eq.

(7)

If we assume the slope of the curve at $y(a)$, say $y'(a) = G_1$ then we solve the equation as an initial value problem using this assumed value, the result of this not the desired to $y(b) = \beta$, we assume another value of $y'(a)$ smaller for the slope, say $y'(a) = G_2$ and repeat. After these two trials, we linearly extrapolate for a third trial as follows:-

$$\text{Extrapolated estimate for Initial slope} = G_1 + \frac{G_2 - G_1}{R_2 - R_1} * (D - R_1)$$

(8)

where R_1 =first result at endpoint (using G_1)

R_2 =second result at endpoint (using G_2)

D = the desired value at the endpoint

so the correct value of $y(b)$ result.

Runge-Kutta Shooting Method for Solving TPBVP

Consider the following second order TPBVP equations:-

$$y''(x) + f(x)y' + g(x)y(x) = h(x)$$

Subject to the boundary conditions where f, g, h are function and α, β are constant:-

$$y(a) = \alpha, \quad y(b) = \beta$$

these equation changed into a first-order differential system:-

$$y' = \begin{bmatrix} 0 & 1 \\ g(x) & f(x) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -h(x) \end{bmatrix} \quad (9)$$

with assume the initial condition $y_1(a) = \alpha$, $y_1'(a) = \frac{\beta - \alpha}{b - a} = G_1$

applying explicit and continuous Runge-Kutta steps, the result we compare the computed value with the given conditions at the other boundary.

Repeat the solution with assume another value of $y_1'(a) = G_2$ and with extrapolated estimate until agreement is attained at the other boundary.

Algorithm

Example (1):-

Consider the following two-point boundary value Problem:-

$$y'' - 4y = 0$$

with boundary conditions:-

$$y(-1) = -1.1752, \quad y(1) = 10.0179$$

while the exact solution is :-

$$y(x) = \frac{1}{2} \sinh(2x+1)$$

Table (1) presents the results of the , Shooting fifth Explicit Runge-Kutta (SERKF(5)) and Shooting Continuous Explicit Runge-kutta method (SCERK) are used with differential value of r .

Example (2):-

Consider the following two-point boundary value Problem:-

$$y'' = y - x^2 - x$$

with boundary conditions:-

$$y(0)=2, \quad y(1)=4$$

while the exact solution is :-

$$y(x) = x^2 + x + 2 \square$$

Table (2) presents the results of the, Shooting fifth Explicit Runge-Kutta (SERKF(5)) and Shooting Continuous Explicit Runge-kutta method (SCERK) are used with differential value of r .

Conclusions

The obtains results show that the SCERK Method gives better accuracy than SERKF(5) of the same order, since the error in SCERK method is decreasing when $r \in [0,1)$ is increasing.

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Table (1) results the solution

H	x	SERKF(5)	SCERK				
			$r = 1$	$r = 0.25$	$ \text{error} ^*$	$r = 0.5$	$ \text{error} ^*$
0.1	-0.5	0.0070096	0.0070096	0.0067226	0.0069	0.0067526	0.0069
	0.5	3.6859483	3.6859483	3.6853062	0.0584	3.6854109	0.0583
	1	10.1787000	10.1787000	10.1787000	0.0608	10.1787000	0.0608
0.05	-0.5	0.0070095	0.0070095	0.0067941	0.0069	0.0068918	0.0069
	0.5	3.6859281	3.6859281	3.6853237	0.0584	3.6856672	0.0583
	1	10.1787000	10.1787000	10.1787000	0.0608	10.1787000	0.0608

Table (2) results the solution

h	x	SERKF(5)	SCERK				
			$r = 1$	$r = 0.25$	$ \text{error} ^*$	$r = 0.5$	$ \text{error} ^*$
0.1	0.1	2.1100000	2.1100000	2.1093142	7.0000e-004	2.1098533	1.0000e-004
	0.5	2.7500011	2.7500011	2.7477002	0.0023	2.7497009	3.0000e-004
	1	4.0000000	4.0000000	4.0000000	0.0000	4.0000000	0.0000
0.05	0.1	2.1100000	2.1100000	2.1093142	7.0000e-004	2.1098533	1.0000e-004
	0.5	2.7500011	2.7500011	2.7477002	0.0023	2.7497009	3.0000e-004
	1	4.0000000	4.0000000	4.0000000	0.0000	4.0000000	0.0000

* Where $|\text{error}|$ represent the difference between the exact solution and (SCERK) for $r=0.25, r=0.5$.

حل مسائل القيم الحدودية الخطية باستخدام طريقة تهديف رانج-كتا الصريحة المستمرة

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الخلاصة

في هذا البحث قمنا بتعميم طريقة رانج-كتا الصريحة من الرتبة الخامسة وطريقة رانج-كتا الصريحة المستمرة باستخدام طريقة التهديف لحل مسائل القيم الحدودية من الرتبة الثانية وذلك بتخفيضها الى الرتبة الاولى هذه الطرائق دمجت بأسم طريقة التهديف لرانج-كتا الصريحة المستمرة. وقد أُستخدِم برنامج الماتلاب لحساب النتائج.
الكلمات المفتاحية:-مسائل القيم الحدودية، طريقة التهديف، طريقة رانج-كتا الصريحة المستمرة.