Fixed Point for Asymptotically Non-Expansive Mappings in 2-Banach Space

Salwa S. Abed
Rafah S. Abed Ali
Department of Mathematics/College of Education for Pure Science(Ibn AL-Haitham) /Baghdad University

Received in : 23 June 2013 , Accepted in : 4 December 2013

Abstract

In this paper, we introduced some fact in 2-Banach space. Also, we define asymptotically non-expansive mappings in the setting of 2-normed spaces analogous to asymptotically non-expansive mappings in usual normed spaces. And then prove the existence of fixed points for this type of mappings in 2-Banach spaces.

keywords: 2-Banach space, Non-expansive mapping, Asymptotically non-expansive mapping, Fixed point.

Introduction and Preliminaries

The concept of linear 2-normed spaces (breviary, 2-normed space) was initiated by S-Gahler in 1965 [1]. Other papers dealing with 2-normed spaces are [2], [3] and [4]. later on, several researchers studied 2-normed spaces using contractive map-pings (see [5] and [6]). Also, Mukti, Sahu and Baisnab [7] proved some fixed point theorems in 2-Banach spaces where mappings involved are of caristis type. The study of 2-normed spaces using asymptotically non-expansive mappings was not initiated by researcher. The purpose of this paper is to continue studying 2-normed spaces using asymptotically non-expansive mappings.

Now, we recall the following definitions:

Definition [8]

Let X be a real linear space and $\|.,.\|$ be a nonnegative real valued function defined on X × X satisfying the following conditions:

$\|x, y\| = 0$ if and only if x and y are linearly dependent in X. 1)

$\|x, y\| = \|y, x\|$, for all x ,y ϵX . 2)

3) $\|x, \alpha y\| = |\alpha|\|x, y\|$, α ϵ R ,x , y ϵ X.

4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$, for all x , y , z ϵ X.

Then (X ,$\|.,.\|$) is called a 2-normed space.

Note that the 2-normed space is Hausdorff space and $\|.,.\|$ is continuous function, for examples of 2-normed spaces see [1] .

The ball in 2-normed space X with center x, radius r > 0 and is defined by $B_r(x)$ = { y, u ϵ X, $\|$ x – y, u $\|$ < r}, and the open subset M of X is defined as follows: for any x ϵ M there is r > 0 such that $B_r(x) \subset M$. therefore M is called closed subset of X if its complement is open.

Definition [9]

A sequence (x_n) in a 2-normed space (X,$\|.,.\|$) is called a convergent sequence if there is, x ϵ X, such that

$\lim_{n\to\infty} \|x_n - x, u\| = 0$, for all u ϵ X.

Definition [9]

A sequence (x_n) in a 2-normed space (X,$\|.,.\|$) is called a Cauchy sequence if

$\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0$, for all $y \in$ X.

Definition [9]

A linear 2-normed space X is said to be complete if every Cauchy sequence is convergent to an element of X. Then X is called a 2-Banach space.

Definition [9]

Let X be a 2-Banach space and T:X→X be a Mapping T is said to be continuous at x if for every sequence (x_n) in X,$(x_n) \to x$ as n→∞ implies that $\{T(x_n)\} \to T(x)$ as n→∞.

We need to give some concepts in the setting of 2-normed space X as the first dual and the second dual of X are defined by

$X^* = \{f \mid f : X \to \mathbb{R}$, bounded linear function$\}$,

$X^{**} = \{ g \mid g : X^* \to \mathbb{R}$, bounded linear function$\}$.

respectively, then the mapping J: X→ X^* , where $J(x) = F_x(f) = f(x)$, f ϵ X^* is called a natural embedding, so we say that X is reflexive if the natural embedding is an onto mapping.

Definition [10]

Let X be a 2-normed space, (x_n) be a sequence in X , Then (x_n) is said to be converges weakly to x denoted by $x_n \to x$ as $n \to \infty$, if $f(x_n) \to f(x)$ as $n \to \infty$.

Definition [11]

A 2-normed space $(X,\|.,.\|)$ is said to be uniformly convex if for every $\varepsilon \in (0,2]$ and $u \neq 0$ in X , there exists $\alpha > 0$ such that

$\| x , u \| \leq 1 ; \| y , u \| \leq 1$ and $\| x - y , u \| \geq \varepsilon$ implies that $\| \frac{1}{2}(x + y) , u \| \leq 1 - \alpha$.

Definition

Let X be a 2-normed space, Then we say that X satisfies Opial condition if for every bounded sequence $(x_n) \in X$ converges weakly to $x \in X$, Then $\lim_{n\to\infty} \inf\|x_n - x , u\| < \lim_{n\to\infty} \inf\|x_n - y , u\|$ for every $x \neq y$ & $y , u \in X$.

Definition

Let X be a 2-normed space. With the mapping $T : X \to X$

i- T is said to be Lipschitzian if there exists constant $\alpha \geq 0$ such that

$\|T(x) - T(y), u\| \leq \alpha \|x - y, u\|$ for all $x ,y ,u \in X$ …(1.1)

ii- If $\alpha = 1$ then T is said to be non-expansive mapping such that $\|Tx - Ty , u \| \leq \| x - y , u \|$; $x , y, u \in X$ …(1.2)

Definition

Let X be a 2-normed space, Then the mapping $T :X\to X$ is said to be Asymp-totically non-expansive mapping if there exists a positive sequence $(k_n) \in [1,\infty)$ with $\lim_{n\to\infty}(k_n) = 1$, such that

$\|T^n x - T^n y , u \| \leq k_n \| x - y , u \|$ …(1.3)

for all $x , y , u \in X$ and $n \geq 1$. .

Definition

i. If S a nonempty subset of a 2-normed space X and (x_n) a bounded sequence in X. consider the functional $r_a(\cdot, (x_n)) : X \times X \to R^+$ defined by

$r_a(x, (x_n)) = \lim_{n\to\infty} sup\|x_n - x, u\|$; $x, u \in X$.

ii. The infimum of $r_a(\cdot, (x_n))$ over S is said to be the asymptotic radius of (x_n) with respect to S and is denoted by $r_a(S, (x_n))$. A point $z \in S$ is said to be an asymptotic center of the sequence (x_n) with respect to S if

$r_a(z, (x_n)) = \inf \{r_a(x, (x_n)) : x \in S\}$

The set of all asymptotic centers of (x_n) with respect to S is denoted by $Z_a(S,(x_n))$.if (x_n) converges strongly to $x \in S$, then $Z_a(S, (x_n)) = \{x\}$.

Results in 2-Banach spaces

We begin with the following :

Theorem

Let S be a nonempty closed convex subset of a uniformly convex 2-Banach space X and (x_n) a bounded sequence in S such that $Z_a(S, (x_n)) = \{z\}$. If (y_m) is a sequence in S such that $\lim_{m\to\infty} r_a(y_m, (x_n)) = r_a(S, (x_n))$, then $\lim_{m\to\infty} y_m = z$.

Proof

Suppose that (y_m) does not converge strongly to z.
Then there exists a subsequence (y_{mi}) of (y_m) such that

$\|y_{mi} - z,u\| \geq d > 0$ for all $i \in N$, $u \in S$.

By the uniform convexity of X, there exists $\varepsilon > 0$ such that

$(r_a(S, (x_n)) + \varepsilon)[1 - \delta_x(d / r_a(S,(x_n)) + \varepsilon)] < r_a(S, (x_n))$.

Since $r_a(z, (x_n)) = r_a(S, (x_n))$, there exists $n_o \in N$ such that

$\|x_n - z ,u\| \leq r_a(S, (x_n)) + \varepsilon$ for all $n \geq n_o$.

Since $r_a(y_m, (x_n)) \to r_a(S, (x_n))$ as $m \to \infty$ and hence $r_a(y_{mi} , (x_n)) \to r_a(S, (x_n))$ as $i \to \infty$, then there exists an integer $m_o \in N$ such that

$\|x_n - y_{mi} ,u\| \leq r_a(S, (x_n)) + \varepsilon$ for all $n \geq m_o$.

Since X is uniformly convex,

$$\|x_n - (z + y_{mi}) / 2 ,u\| \leq [1-\delta_x(d / (r_a(S,(x_n)))] (r_a(S,(x_n)) + \varepsilon) < r_a(S,(x_n))$$

for all $n \geq \max \{n_o,m_o\}$ This implies that

$r_a((z + y_{mi} / 2), (x_n)) < r_a(S, (x_n))$,

which contradicts the uniqueness of the asymptotic center z. ■

Theorem

Let S be a nonempty closed convex subset of a uniformly convex 2- Banach space. Then every bounded sequence (x_n) in X has a unique asymptotic center with respect to S, i.e., $Z_a(S, (x_n)) = \{z\}$ and

$$\lim_{n\to\infty} \sup\|x_n - z,u\| < \lim_{n\to\infty} \sup\|x_n - x,u\| \;\; for\; x \neq z , u \in S.$$

Theorem

Let X be a uniformly convex 2-Banach space satisfying the Opial condition and S a nonempty closed convex subset of X. If (x_n) is a sequence in S such that $x_n \to z$, then z is the asymptotic center of (x_n) in S.

Proof

From Theorem 2-2, $Z_a(S, (x_n))$ is singleton. Let $Z_a(S, (x_n)) = \{x\}$, $x \neq z$ Since $x_n \to z$, by the Opial condition,

$\lim_{n\to\infty} sup\, \|x_n - z,u\| < \lim_{n\to\infty} sup\, \|x_n - x,u\|$, $u \in S$.

By theorem 2-2, we obtain

$\lim_{n\to\infty} \sup\|x_n - x,u\| < \lim_{n\to\infty} sup\|x_n - z,u\|$.

Therefore, z = x. ■

Theorem

Let X be a uniformly convex 2-Banach space, let S be a nonempty closed convex subset of X and $T:S\to S$ an asymptotically non-expansive mapping. If (x_n) a bounded sequence in S such that $\lim_{n\to\infty}\|x_n - Tx_n,u\| = 0 ; u \in S$ and $Z_a(S,(x_n)) = \{ v \}$,then v is the fixed point in S.

Proof

Define a sequence (y_m) in S by $y_m = T^m v$, $m \in N$. for integers n, $m \in N$, we have

$\| y_m - x_n , u\| \leq \|T^m v - T^m x_n , u\| + \|T^m x_n - T^{m-1} x_n , u\| + \ldots + \|Tx_n - x_n , u\|$

$\leq k_m \|v - x_n , u\| + (\|Tx_n - x_n , u\| + \sum_{i=1}^{m-1} k_i \, \| x_n - Tx_n ,u\| \ldots (2.1)$

Then by condition (2.1) we have $r_a (y_m , (x_n)) = \lim_{n\to\infty} sup\|x_n - y_m,u\|$

$k_m r_a(v , (x_n))$ $\leq$

$k_m\, r_a\,(S\,,(x_n))$. =

Hence

$r_a\,(y_m\,,(x_n)) - r_a\,(S\,,(x_n))| \le k_m\, r_a\,(S\,,(x_n)) - r_a\,(S\,,(x_n))$
$\le (k_m - 1)\, r_a(S,(x_n)) \to 0$ as $m \to \infty$.

it follows from 2-1 that $T^m v \to v$ By the continuity of T we have $Ty = T(\lim_{m\to\infty} T^m\, y) = \lim_{m\to\infty} T^{m+1}\, y = y$.

Theorem (Fixed Point Theorem)

Let X be a uniformly convex 2-Banach space, S a nonempty closed convex bounded subset of X and $T : S \to S$ an asymptotically non-expansive mapping, Then T has a fixed point in S.

Proof

For fixed $y \in S$ and $r > 0$, set

$R_y = \{r :$ there exists $k \in N$ with $S \cap (\bigcap_{i=k}^{\infty} B_r[T^i\, y]) \neq \emptyset\}$ and $d = \text{diam}\,(S)$.

Then $d \in R_y$. Hence $R_y \neq \emptyset$.

Let $r_o = \inf\{\, r : r \in R_y \}$, fot each $\varepsilon > 0$, we define

$S_\varepsilon = \bigcup_{k=1}^{\infty}(\bigcap_{i=k}^{\infty} B_{r+\varepsilon}[T^i\, y])$.

Thus, for each $\varepsilon > 0$, the set $S_\varepsilon \cap S$ is nonempty and convex. The reflexivity of X implies that

$\bigcap_{\varepsilon>0}\,(\overline{S_\varepsilon} \cap S) \neq \emptyset$

Let $x \in \bigcap_{\varepsilon>0}(\overline{S_\varepsilon} \cap S)$ and $\eta > 0$, there exists an integer n_o such that

$x - T^n y\,,u\| \le r_o + \eta$ for all $n \ge n_o$, $u \in S$. ||

Now let $x \in \bigcap_{\varepsilon>0}(\overline{S_\varepsilon} \cap S)$ and suppose that the sequence $(T^n x)$ does not converge strongly to x. Then there exists $\varepsilon > 0$ and a subse- quence $(T^{ni} x)$ of $(T^n x)$. such that

$\|T^{ni}\, x - x\,,u\| \ge \varepsilon$, for all i=1,2,... .

Suppose k_n is the Lipschitz constant of T^n. Then for $m > n$, we have

$\|T^n x - T^m x\,,u\| \le k_n\, \|x - T^{m-n} x\,,u\|$.

Suppose that $r_o > 0$ and choose $\alpha > 0$ such that

$(1 - \theta_x((\frac{\varepsilon}{r_o+\alpha}))\,(r_o+\alpha) < r_o$

Select n such that

$\|\, x - T^n\, x,u\| \ge \varepsilon$ and $k_n = (r_o + \frac{\alpha}{2}) \le r_o + \alpha$

If $n_o \ge n$, then $m > n_o$ implies

$\|x - T^{m-n} y\,,u\| \le r_o + \frac{\alpha}{2}$.

Since

$\|T^n x - T^m y\,,u\| \le k_n\, \|x - T^{m-n} y\,,u\|$
$k_n\,(r_o + \frac{\alpha}{2} \le)\quad r_o + \alpha \le$ And $\|x - T^m y\,,u\| \le r_o + \alpha$

It follows from the uniform convexity of X that for $m > n_o$

$\|\tfrac{1}{2}\,(x + T^n x) - T^m y,\, u\| \le (1 - \theta_x(\frac{\varepsilon}{r_o+\alpha}))\,(r_o + \alpha) < r_o$.

This contradicts the definition of r_o. Hence $r_o = 0$ or $Tx = x$. But $r_o = 0$ implies that $(T^n y)$ is a Cauchy sequence and hence $\lim_{n\to\infty} T^n y = x = Tx$.

Therefore, the set $\bigcap_{\varepsilon>0}(\overline{S_\varepsilon} \cap S)$ is a singleton that is a fixed point of T. ■

Theorem

Let X be a uniformly convex 2-Banach space, S a nonempty closed convex subset of X and T : S → S an asymptotically non-expansive mapping, u ∈ S, Then the follo-wing statements are equivalent:

1) T has a fixed point .

2) There exists a point x_o ϵ S such that the sequence $(T^n x_o)$ is bounded.

3) There exists a bounded sequence (y_n) in S such that $\lim_{n\to\infty} \|y_n - Ty_n, u\| = 0$.

Proof

(1) ⟹ (2) and (1) ⟹ (3) follows easily.

(3) ⟹ (1) Let (y_n) be a bounded sequence in S such that

$\lim_{n\to\infty} \|y_n - Ty_n, u\| = 0$

Let Z_a (S,(y_n)) = {v} ,therefore ,by theorem 2-4, implies that v is a fixed point of T. ■

Corollary

Let S be a nonempty closed convex subset of a strictly convex 2-Banach space X and T : S → X a non-expansive mapping. Then F(T) is closed and convex.

We have seen in a Corollary 2-7, that F(T) is closed and convex in strictly convex 2-Banach space for non-expansive mappings. However, we think that Corollary 2-7, is not true for asymptotically non-expansive mappings. In fact, we have:

Theorem

Let X be a uniformly convex 2-Banach space, S a nonempty closed convex bounded subset of X and T : S → S an asymptotically non-expansive mapping. Then F(T) is closed and convex.

Proof

The closedness of F(T) is obvious. To show convexity, it is sufficient to prove that z = $(x + y)/2$ ϵ F(T) for x, y and u ϵ F(T),for each n ϵ N, we have

$\|x - T^n z, u\| = \|T^n x - T^n z, u\| \le k_n \|x - z, u\| = \frac{1}{2} k_n \|x - y, u\|$

$\|y - T^n z, u\| = \|T^n y - T^n z, u\| \le k_n \|y - z, u\| = \frac{1}{2} k_n \|x - y, u\|$

By the uniform convexity of X , we have

$\|z - T^n z, u\| \le \frac{1}{2} [1 - \lambda_x(2 / k_n)]\, k_n \|x - y, u\| \le \frac{1}{2} [1 - \lambda_x (2 / k_n)]\, k_n \operatorname{diam}(S)$

Hence $T^n z \to z$ as $n \to \infty$

z is a fixed point of T , by the continuity of T. ■

References

1. Gahler S., (1965), Linear 2-Normietre Raume, Math. Nachr., 28, pp. 1 –43.
2. White A., (1969), 2-Banach Spaces, Math Nachr. 42, pp. 43 – 60.
3. Rezapour S., (2005), Quasi-Chebyshew Subspaces in Generalized Normed Space, International Journal of Pure and Applied Mathemetical Scinces, Vol. 2, No. 1, pp. 53 – 61.
4. Lawandowska Z., (2003), On 2-Normed Sets, Glasnik Math. Ser. III 38(58), No. 1, pp. 99 – 110.
5. . Khan M. S and Khan M. D., (1993), Involutions with Fixed Points in 2-Banach Spaces, Internet. J. Math. Sci, 16, 429 – 434.
6. Cho Y. J., Huang N. and Long X., (1995), Some Fixed Point Theorems for Non-linear Mappings in 2-Banach Spaces, Far East J. Math. Sci., 3(2), pp. 125 – 133.
7. Mukti G., Mantu S. and Baisanb A. P., (2009), " Fixed Point Theorems for a Class of Mappings in 2-Banach Space", Int. Journal of Math. Analysis, 3, No. 27, pp. 1339 – 1347.
8. Kuldip R. and Sunil K. S., (2011), " Some Sequence Spaces in 2-Normed Spaces Defined by Musielak-Orlisz Function", Acta Univ. Sapientiae, Mathematical, 3, 1, pp. 97 - 109.
9. Mukti G., Mantu S. and Baisnab A. P., (2012), "Caristi-Type Fixed Point Theor-ems in 2-Banach Space", Gen, Math. Notes, 8, No. 1, pp. 1 – 5.
10. George B. and Lawrence N., (2000), Functional Analysis, Canada, General Pub-lishing Company.
11. Dominic Y. and Marudai M., (2012), "Best Approximation in Uniformly Convex 2-Normed Space" Int. Journal of Math. Analysis, 6, No. 21, pp. 1015 – 1021.

النقطة الصامدة للتطبيقات شبة اللامتمددة في فضاء 2- بناخ

سلوى سلمان عبد
رفاه ساجد عبدعلي
قسم الرياضيات/ كلية التربية للعلوم الصرفة أبن الهيثم/جامعة بغداد

أستلم البحث في : 23حزيران2013 ، قبل البحث في: 4 كانون الاول2013

الخلاصة

خلال هذا البحث قدمنا بعض الحقائق في فضاء2-بناخ. ايضا،عرفنا تطبيقات شبه اللامتمددة asymptotically non-expansive mapping في الفضاءات2-المعيارية بطريقة مشابهة للتطبيقات غير المتمددة في الفضاءات المعيارية العادية، و من ثم البرهنة عن وجود نقاط صامدة لهذا النمط من التطبيقات في فضاء 2-بناخ.

الكلمات المفتاحية : فضاء 2-بناخ، تطبيق لا متمدد، تطبيق شبة لا متمدد، نقطة صامدة.