Small Monoform Modules

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Abstract

Let R be a commutative ring with unity, let M be a left R-module. In this paper we introduce the concept small monoform module as a generalization of monoform module. A module M is called small monoform if for each non zero submodule N of M and for each $f \in \text{Hom}(N,M)$, $f \neq 0$ implies $\ker f$ is small submodule in N. We give the fundamental properties of small monoform modules. Also we present some relationships between small monoform modules and some related modules.

Key Words: Monoform module, small monoform module, small submodule, prime module, small prime module, uniform module, non singular module, quasi-Dedekind module.

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Introduction

Throughout this article, R denotes a commutative ring with identity, and modules are unitary left R-module.

We write $N \leq M$ to denote that N is a submodule of M. A proper submodule L of M ($L < M$) is called small in M (denoted by $L \ll M$) if, for every proper submodule K of M, $L + K \neq M$. A submodule N of M is called essential in M (denoted by $N \leq_e M$) if $N \cap K \neq 0$ for each $K \leq M$, $K \neq 0$, [1].

An R-module M is called monoform module if for each non zero submodule N of M and for each $f \in Hom(N,M)$, $f \neq 0$ implies $\ker f = 0$ (i.e. f is monomorphism, [2]). Equivalently M is monoform R-module if and only if M is uniform and prime module [3,theorem(2.3)], where M is uniform if every nonzero submodule N of M, $N \leq_e M$, [1]. M is called prime R-module if $ann_R M = ann_R N$, for each nonzero submodule N of M, [4], where $ann_R M = \{r \in R: rM = 0\}$.

In this paper, we introduce the concept small monoform as a generalization of monoform module, where M is called small monoform if for each $N \neq 0$, $N \leq M$, $f \in Hom(N,M)$, $f \neq 0$ implies $\ker f \ll N$. It is clear that every monoform module is small monoform, however the converse is not true (see Rem. and Ex. 1.2 (1)). We give many properties of small monoform. Also we see that under certain class of modules small monoform and monoform modules are equivalent.

Moreover, we introduce many relationships between small monoform module and other related modules such as small quasi-Dedekind modules, quasi-Dedekined module, compressible modules.

1- Main Results

Definition 1.1:

Let M be an R-module. M is called small monoform if for each non-zero submodule N and for each $f \in Hom(N,M)$, $f \neq 0$ implies $\ker f \ll N$.

Remarks and Examples 1.2:

(1) It is clear that every monoform module is small monoform. However the converse is not true in general for example:

The Z-module Z_4 is not monoform because there exists Z-homomorphism, $f: Z_4 \longrightarrow Z_4$ such that $f(\bar{x}) = 2\bar{x}$ for each $\bar{x} \in Z_4$ and $\ker f = <\bar{2}> \neq (\bar{0})$. But Z_4 is small monoform Z-module since the only non zero submodule of Z_4 are $<\bar{2}>$ and Z_4 and the only non zero Z-homomorphism from $<\bar{2}>$ in Z_4 is the inclusion mapping i and $\ker(i) = <\bar{0}>$.

Also there are three nonzero homomorphism from Z_4 in to Z_4 which are f_1 = identity mapping, $f_2(\bar{x}) = 2\bar{x}$ and $f_3(\bar{x}) = 3\bar{x}$. Hence $\ker(f_i) \ll Z_4$, $\forall$ i = 1,2,3.

(2) It is clear that every chained module is small monoform, where an R-module is called chained module if the lattice of submodules is linearly ordered.

In particular, each of the Z-module, Z_{p^∞}, Z_4, Z_8, Z_{16}, … is small monoform.

(3) The epimorphic image of small monoform module is not necessarily small monoform, for example

Z as Z-module is monoform since Z is uniform and prime. But $\pi: Z \longrightarrow Z/12Z \cong Z_{12}$, where π is the natural projection. However Z_{12} as Z-module is not small monoform, since if we take $N = <\bar{2}>$ and $f : N \longrightarrow Z_{12}$ defined by $f(\bar{x}) = 2\bar{x}$ for each $\bar{x} \in N$, $\ker f = \{\bar{0}, \bar{6}\} \subseteq N$. But $\{\bar{0}, \bar{6}\} + \{\bar{0}, \bar{4}, \bar{8}\} = N$. Thus $\ker f \not\ll N$.

(4) Every non zero submodule of small monoform module is small monoform module.
Proof: Let M be a small monoform R-module and let $N \le M$. For any $K \le N$, $K \ne 0$, let $f: K \longrightarrow N, f \ne 0$.
$K \xrightarrow{f} N \xrightarrow{i} M$, $i \circ f \ne 0$. But $\ker(i \circ f) = \ker f$, hence $\ker f \ll K$.
Thus N is small monoform.

Recall that: If M is an R-module, then M is an $\bar{R}$-submodule of M, where $\bar{R} = R/\text{ann } M$ by using the definition
$(r + \underset{R}{\text{ann}}\, M)x = rx$, for each $x \in M$. Hence every R-submodule of M is an $\bar{R}$-submodule of M, and conversely.

(5) Let M be an R-module. Then M is small monoform R-module if and only if M is small monoform $\bar{R}$-module
Proof: $(\Rightarrow)$
Let N be an $\bar{R}$-submodule of M, let $f: N \longrightarrow M, f \ne 0$ be $\bar{R}$-homomorphism. It is clear that N is R-submodule of M. To show that f is an R-homomorphism.
Let $r \in R, f(rx) = f[(r + \text{annM})x]$
$= (r + \text{annM}) + f(x)$ since f is an $\bar{R}$-homomorphism
$= r f(x)$
Thus f is an R-homomorphism. But M is small monoform, so $\ker f$ is small R-submodule of N. Hence $\ker f$ is small $\bar{R}$-submodule of N.
The proof of the converse is similarly.

Remark 1.3:
Let M be a semisimple R-module. Then the following statements are equivalent:
(1) M is small monoform.
(2) M is monoform.
(3) M is simple.

Proof: (1) ⇒ (2) Let $N \le M$, let $f: N \longrightarrow M, f \ne 0$. Since M is small monoform, then $\ker f \ll N$. But M is semisimple, so N is semisimple and hence N has only small submodule namely (0). Thus $\ker f = (0)$ and so M is monoform.

(2) ⇒ (1) It is clear by (Rem. and Ex. 1.2(1)).

(2) ⇒ (3) Let $x \in M$, $x \ne 0$. Since M is semisimple, then $<x>$ is a direct summand of M. So $<x> \oplus K = M$, for some $K \le M$. But M is monoform, so for each homomorphism $f: <x> \longrightarrow M, f \ne 0, \ker f = 0$. Define $g : M \longrightarrow M$, by $g(rx + K) = f(rx)$. We can show that g is well-defined as follows:
Let $r_1x + k_1 = r_2x + k_2$ where $r_1, r_2 \in R$, $k_1, k_2 \in K$
$(r_1 - r_2)x = k_2 - k_1 \in <x> \cap K = (0)$.
Hence $(r_1 - r_2)x = 0 = k_2 - k_1$. Thus implies $r_1x = r_2x$ and $k_1 = k_2$.
Thus $f(r_1x) = f(r_2x)$ and $g(r_1x + k_1) = g(r_2x + k_2)$.
Now let $rx + k \in \ker g$, then $g(rx + k) = f(rx) = 0$.
It follows that $\ker g = \ker f \oplus K = 0 \oplus K = K$. But $\ker g = 0$, so $K = 0$.

Thus <x> = M and therefore M is simple.

(3) ⇒ (2) If M is simple, then M has only two submodules (0), M. So that for each $f : M \longrightarrow M, f \neq 0$ $\ker f \leq M$, hence $\ker f = 0$. Thus M is monoform.

Recall that an R-module M is called free if it has a basis, [1].

Theorem 1.4:

Let M be a free Z-module. Then M is small monoform if and only if M is monoform.

Proof: (⇒)

Let $N \leq M$, $N \neq 0$, let $f : N \longrightarrow M$, $f \neq 0$. Since M is small monoform implies $\ker f \ll N$. But M is a free Z-module implies N is a free Z-module, [5,Corollary (5.5.3)]. So, N has only (0) small submodule. Thus $\ker f = 0$; that is M is monoform.

(⇐) It is clear by 1.2(1).

The following proposition gives a characterization of small monoform module under the class of Noetherian modules.

Proposition 1.5:

Let M be a non zero Noetherian R-module. Then M is small monoform if and only if every non zero 3-generated submodule of M is small monoform.

Proof: (⇒) It is clear.

(⇐) suppose every non zero 3-generated submodule of M is small monoform. Let $N \leq M$, $N \neq 0$ and let $f \in \text{Hom}(N,M)$, $f \neq 0$. To prove $\ker f \ll N$.

If $\ker f = (0)$ then $\ker f \ll N$.

If $\ker f \neq (0)$, let $x \neq 0$ and $x \in \ker f$. Let $y \in N$ and let $f(y) = z$. Put L = <x,y,z>, L is 3-generated submodule of M.

By hypothesis, L is small monoform, let H = <x,y>. Let $g = f|_H : H \longrightarrow L$. Hence $\ker g \ll H \leq N$, since L is small monoform. This implies $\ker g \ll N$. But $x \in \ker g$, so that $\langle x\rangle \subseteq \ker g \ll N$. Thus $\langle x\rangle \ll N$. Since M is Noetherian, $\ker f$ is finitely generated. Hence $\ker f = Rx_1 + Rx_2 + \dots + Rx_i = \langle x_1, x_2, \dots, x_n\rangle$ for $x_1, \dots, x_n \in M$. Since $\langle x_i\rangle \ll N$ for each $i = 1, \dots, n$. So $\ker f = \sum_{i=1}^{n} Rx_i \ll N$.

Thus M is small monoform.

Recall that an R-module M is called uniform if every non zero submodule is eesential, [1].

Recall that an R-module M is called quasi-Dedekind if for each $N \leq M$, $N \neq 0$, $\text{Hom}(\frac{M}{N}, M)=0$, that is every nonzero submodule N of M is quasi-invertible, [6].

Recall that an R-module M is called small quasi-Dedekind if for each $f \in \text{End}(M)$, $f \neq 0$, $\ker f \ll M$. Equivalently M is small quasi-Dedekind if $\text{Hom}((\frac{M}{N}, M) = 0$ for each $N \not\ll M$ [7], where End(M) = set of all homomorphism from M to M.

Let A be a submodule of an R-module M. A relative complement for A in M is any submodule B of M which is maximal with respect to the property $A \cap B = 0$ [8,p.17].

Proposition 1.6:
Let M be an R-module if M is small monoform, then M is uniform and M is small quasi-Dedekind.
Proof:
By [7, Rem. and Ex. (3.2.9),p.109] M is small quasi-Dedekind, let $N \leq M$, $N \neq (0)$. If $N \leq_e M$ nothing to prove.

Suppose $N \not\leq_e M$, then there exists ($H \leq M$) such that H is a relative complement of N. Hence $N \oplus H \leq_e M$ by [8,proposition 1.3,p.17].
Define $f : N \oplus H \longrightarrow M$ by $f(a+b) = a$ for each $a + b \in N \oplus H$.
Then $\ker f = (0) \oplus H$, but M is small monoform, so $\ker f = (0) \oplus H \ll N \oplus H$ and this implies $H \ll H$ (which is impossible) unless $H = (0)$ and hence $N \leq_e M$. Thus M is uniform.

Corollary 1.7:
Let M be an R-module. If M is small monoform, then M is uniform and $\underset{R}{\text{ann}}\, M = \underset{R}{\text{ann}}\, N$ for each $N \not\ll M$.
Proof:
By proposition 1.6, M is uniform. Also M is small quasi-Dedekind, hence for each $N \not\ll M$, N is a quasi-invertible [7,Th. 3.1.3,p.95]. Thus $\underset{R}{\text{ann}}\, M = \underset{R}{\text{ann}}\, N$ for each $N \not\ll M$ [6, proposition 1.4,p.7]

Recall that an R-module $Z(M) = \{x \in M, \text{ann}_R(x) \leq_e R\}$ is called a singular submodule of M. If $Z(M) = M$, then M is a singular module. If $Z(M) = 0$, then M is called a non singular module, [8,p.31].

Proposition 1.8:
Let M be a non singular R-module. Then M is small monoform implies M is quasi-Dedekind.

Proof:
Let $N \leq M$. Since M is small monoform implies M is uniform (by proposition 1.6). Hence $N \leq_e M$, but $N \leq_e M$ and M is a non singular implies $\frac{M}{N}$ is singular [8,proposition 1.21,p.32]. Hence $\text{Hom}(\frac{M}{N}, M) = 0$ [8,Exc. 1,p.33]; that is N is quasi-invertible. Thus M is quasi-Dedekind.

Note 1.9:
The condition M is nonsingular in Proposition 1.8 is necessary. For example Z_4 as Z-module is small monoform, but is not quasi-Dedekind. Also Z_4 is not a nonsingular Z-module, since $Z(Z_4) \neq (0)$ (in fact $Z(Z_4) = Z_4$).

It is known that: A ring R is semisimple implies every R-module is a non singular. Hence we get the following corollary.

Corollary 1.10:

Let R be a semisimple ring, let M be an R-module, then M is small monoform implies M is quasi-Dedekind.

Proof:

Since R is semisimple, M is nonsingular. Hence the result follows by proposition 1.8.

Recall that an R-module M is called a prime R-module if $\underset{R}{\text{ann}}\, M = \underset{R}{\text{ann}}\, N$ for every non zero R-submodule N of M, [4].

Corollary 1.11:

Let M be a non singular small monoform, then M is prime.

Proof:

By Proposition 1.8, M is small monoform and non singular implies M is quasi-Dedekind. Thus M is prime [6,proposition 1.7, p.26].

Recall that an R-module M is called fully retractable, if for every non zero submodule N of M and every non zero element $g \in Hom_R(N,M)$ we have $Hom_R(M,N)g \neq 0$, [9].

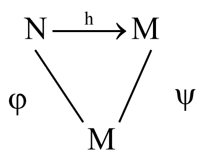
Proposition 1.12:

Let M be an R-module such that M is fully retractable and for each $N \leq M$, $N \neq (0)$, N is small quasi-Dedekind, then M is small monoform.

Proof:

Let $N \leq M$, $f : N \longrightarrow M$, $f \neq 0$. Since M is fully retractable, then there exists $g : M \longrightarrow N$, $g \neq 0$. Consider $N \xrightarrow{f} M \xrightarrow{g} N$. By M fully retractable, $g \circ f \neq 0$. Since N is small quasi-Dedekind, $\ker (g \circ f) \ll N$. But $\ker f \subseteq \ker (g \circ f)$ and this implies $\ker f \ll N$. Thus M is small monoform.

Recall that an R-module M is called a qusi-injective R-module if for each monomorphism $h: N \longrightarrow M$, where N is any R-submodule of M and any homomorphism $\varphi: N \longrightarrow M$, there is a homomorphism $\psi: M \longrightarrow M$ such that $\psi \circ h = \varphi$ i.e. the following diagram is commutative, [10,p.22].



Recall that A submodule N of M is called coclosed if whenever $K \leq N$ and $\frac{N}{K} \ll \frac{M}{K}$ implies $K = N$, [11].

We prove the following:

Proposition 1.13:

Let M be a quasi-injective R-module and every submodule of M is coclosed then M is small quasi-Dedekind if and only if M is small monoform.

Proof: ($\Rightarrow$)

Let $N \leq M$, $N \neq (0)$, let $f \in \text{Hom}(N,M)$, $f \neq 0$ consider the following diagram

$$\begin{array}{ccc} N & \xrightarrow{i} & M \\ & f \searrow & \swarrow \exists g \\ & M & \end{array}$$

Since M is quasi-injective, then there exists $g \in \text{End}(M)$ such that $g \circ i = f$. Hence $g(n) = f(n)$ for each $n \in N$, which implies $\ker f \leq \ker g$. But M is small quasi-Dedekind, so $\ker g \ll M$. Thus implies $\ker f \ll M$.

But every submodule of M is coclosed, then N is coclosed. Thus $\ker f \subseteq N$ and $\ker f \ll M$ which implies $\ker f \ll N$, [12, Lemma 1.1]. Therefore M is small monoform.

($\Leftarrow$) It is clear.

Under the class of non singular modules, we have the following:

Proposition 1.14:

Let M be a non singular R-module. Then the following statements are equivalent:

(1) M is small monoform.
(2) M is uniform quasi-Dedekind
(3) M is uniform prime.
(4) M is uniform.
(5) M is monoform.

Proof:

(1) → (2) By Proposition 1.6 M is uniform. But M is small monoform and non singular implies M is quasi-Dedekind by Proposition 1.8.

(2) → (3) It is follows by [6, Proposition 1.7, p.26].

(3) → (4) It is clear.

(4) → (5) It follows by [3, Theorem 2.2].

(5) → (1) It is clear by 1.2(1).

Recall that an R-module M is called compressible if for each $N \leq M$, $N \neq 0$ M can be embedded in M (i.e. there exists $f : M \longrightarrow N$, f is monomorphism), [13].

Consider the following statement (∗):

(∗) Let M be an R-module such that $\text{ann}_R \dfrac{M}{N} \not\subseteq \text{ann}M$, for each $N \leq M$, $N \neq 0$.

We prove the following:

Proposition 1.15:

Let M be a nonsingular R-module such that M satisfies (∗). Then the following statements are equivalent

(1) M is small monoform.
(2) M is quasi-Dedekind.
(3) M is prime.
(4) M is compressible.
(5) M is monoform
(6) M is uniform
(7) $End_R(M)$ is an integral domain.
(8) $R/ann_R M$ is an integral domain.
(9) $ann_R M$ is a prime ideal in R.

Proof:

(1) → (2) By Proposition 1.8.
(2) → (3) It is follows by [6, proposition 1.7, p.26].
(3) ↔ (4) ↔ (5) It is follows by [14, proposition 1.7].
(5) ↔ (6) It is follows by [3, theorem 2.2].
(5) → (1) It is clear.
i.e. (1) ↔ (2) → (3) ↔ (4) ↔ (5) ↔ (6)
(3) ↔ (9) It is follows by [14, proposition 1.9].
(4) ↔ (7) ↔ (8) It is follows by [14, theorem 2.5].
i.e. (6) ↔ (3) ↔ (9) ↔ (4) ↔ (7) ↔ (8).
Thus all statement (1) through (9) are equivalent.

Corollary 1.16:

Let M be a multiplication non singular R-module. Then the statements from 1 to 9 in proposition 1.15 are equivalent.

Proof:

It follows directly by proposition 1.15, since every multiplication module satisfies (∗).

Recall that an R-module M is called retractable if $Hom_R(M,N) \neq 0$ for all $0 \neq N \subseteq M$, [15].

Proposition 1.17:

Let M be retractable and nonsingular R-module, then the following statements are equivalent:

(1) M is monoform.
(2) M is uniform.
(3) M is small monoform.
(4) M is compressible.

Proof:

(1) ↔ (2) It follows by [3, theorem 2.2].
(1) → (3) It is clear by 1.2(1).
(3) → (2) It follows by Proposition 1.6.
(2) → (4) It follows by [9, Proposition 1.7].
(4) → (1) It follows by [3,corollary 2.5].

Recall that an R-module M is called small prime if $\underset{R}{ann}\,M = \underset{R}{ann}\,N$ for each $N \ll M$, [16].

Proposition 1.18:

Let M be small monoform and small prime R-module. Then M is monoform.

Proof:

Since M is small monoform then M is uniform by Proposition 1.6. Also $\underset{R}{\text{ann}}\, M = \underset{R}{\text{ann}}\, N$ for each $N \not\ll M$ by proposition 1.7. But by hypothesis M is small prime, so for each $N \ll M$, $N \neq (0)$, $\underset{R}{\text{ann}}\, M = \underset{R}{\text{ann}}\, N$. Thus ann M = ann N for each $N \leq M$, $N \neq (0)$, that is M is a prime R-module. But M is uniform and prime implies M is monoform [3,theorem 2.3].

Under the class of finitely generated modules, we have the following result.

Corollary 1.19:

Let M be a finitely generated R-module, then the following statements are equivalent:

(1) M is monoform.
(2) M is uniform prime.
(3) M is quasi-Dedekind.
(4) M is small monoform and small prime.
(5) M is is compressible.

Proof:

(1) ↔ (2) It follows by [3, Theorem 2.3].
(2) ↔ (3) It follows by [6, Corollary 3.13].
(1) → (4) It is clear.
(4) → (1) It follows by Proposition 1.18.
(5) ↔ (1) It follows by [3,Lemma 1.9 and Theorem 2.3].

Next we turn our attention to direct sum of small monoform R-modules

Remark 1.20:

$M = M_1 \oplus M_2$, M_1 and M_2 submodule of M, M is small monoform. Then M_1 and M_2 are small monoform. But the converse is not true in general.

Proof: (⇒)

It is clear by Rem. and Ex. 1.2 (5).

Now, consider the following example:

Let $M = Z_4 \oplus Z_4$ as Z-module, Z_4 as Z-module is small monoform (by Remarks and Examples 1.2 (1)), let $N = Z_4 \oplus <\bar{2}>$. Let $f : Z_4 \oplus <\bar{2}> \longrightarrow Z_4 \oplus Z_4$ defined by $f(\bar{x},\bar{y}) = (\bar{x}, 2\bar{y})$, $\ker f = \{(\bar{0},\bar{0}),(\bar{0},\bar{2})\} = (\bar{0}) \oplus <\bar{2}> \not\ll Z_4 \oplus (\bar{2})$ since $<\bar{0}> \oplus <\bar{2}> + (Z_4 \oplus <\bar{0}>) = Z_4 \oplus <\bar{2}>$

Thus the direct sum of small monoform modules need not be small monoform.

Recall that an R-module M is called fully stable if for each $N \leq M$. N is stable; that is for each $f : N \longrightarrow M$, f is R-homomorphism, $f(N) \subseteq N$, [17].

Now we show that under certain condition, the direct sum of small monoform is small monoform.

Theorem 1.21:

Let M be a fully stable R-module, such that $M = M_1 \oplus M_2$, $M_1, M_2 \leq M$ and for each R-homomorphism $f: N_1 \oplus N_2 \longrightarrow M$, $f \neq 0$ implies $f(N_1) \neq 0, f(N_2) \neq 0$ (i.e. $f/N_1 \neq 0, f/N_2 \neq 0$). Then M_1 and M_2 are small monoform if and only if M is small monoform.

Proof: ($\Rightarrow$)

Let $N \leq M$, $N \neq (0)$, $f: N \longrightarrow M$, $f \neq 0$, to prove $\ker f \ll N$. Snce M is fully stable, every submodule of M is stable so, N is stable and this implies $N = (N \cap M_1) \oplus (N \cap M_2)$ by [17,Prop.4.5,p.29].

Consider $(N \cap M_1) \xrightarrow{i_1} N \xrightarrow{f} M \xrightarrow{\rho_1} M_1$,

$(N \cap M_2) \xrightarrow{i_2} N \xrightarrow{f} M \xrightarrow{\rho_2} M_2$

Where i_1, i_2 are inclusion mappings and ρ_1, ρ_2 are projection mappings. Then $\rho_1 \circ f \circ i_1: (N \cap M_1) \longrightarrow M_1$, $\rho_2 \circ f \circ i_2: (N \cap M_2) \longrightarrow M_2$, let $N_1 = N \cap M_1$ $N_2 = N \cap M_2$. By hypothesis $f/N_1 \neq 0$, so there exists $n_1 \in N \cap M_1$, $n_1 \neq 0$, $f(n_1) \neq 0$ and $f/N_2 \neq 0$, so there exists $n_2 \in N \cap M_2$, $n_2 \neq 0$, $f(n_2) \neq 0$.

On the otherhand $f \circ i_1: (N \cap M_1) \longrightarrow M$ implies $f \circ i_1(n_1) = f(n_1) \neq 0$, $f \circ i_2: (N \cap M_2) \longrightarrow M$ implies $f \circ i_2(n_2) = f(n_2) \neq 0$.

Thus implies $f \circ i_1(N \cap M_1) \subseteq N \cap M_1$, since N_1, N_2 are stable. Hence $f(N \cap M_1) \subseteq N \cap M_1$. Similarly $f(N \cap M_2) \subseteq N \cap M_2$. But $f(n_1) \in N \cap M_1$ and $f(n_1) \neq 0$ and, so that $(\rho_1 \circ f \circ i_1)(n_1) = f(n_1) \neq 0$.

Similarly $(\rho_2 \circ f \circ i_2)(n_2) = f(n_2) \neq 0$. Thus $\rho_1 \circ f \circ i_1 \neq 0$, $\rho_2 \circ f \circ i_2 \neq 0$. Since M_1, M_2 are small monoform, then $\ker(\rho_1 \circ f \circ i_1) \oplus \ker(\rho_2 \circ f \circ i_2) \ll (N \cap M_1) \oplus (N \cap M_2) = N$. Let $x = n_1' + n_2' \in \ker f$, where $n_1' \in N \cap M_1$, $n_2' \in N \cap M_2$, $f(n_1') + f(n_2') = 0$. Hence $f(n_1') = -f(n_2') \in (N \cap M_1) \cap (N \cap M_2) = (0)$, it follows $f(n_1') = 0, f(n_2') = 0$. This implies $\rho_1 \circ f \circ i_1(n_1') = \rho_1 \circ f(n_1') = \rho_1(f(n_1')) = f(n_1') = 0$ $\rho_2 \circ f \circ i_2(n_2') = \rho_2 \circ f(n_2') = \rho_2(f(n_2')) = f(n_2') = 0$.

Hence $x = n_1' + n_2' \in \ker(\rho_1 \circ f \circ i_1) \oplus \ker(\rho_2 \circ f \circ i_2) \ll N$.

So that $\ker f \subseteq \ker(\rho_1 \circ f \circ i_1) \oplus \ker(\rho_2 \circ f \circ i_2) \ll N$. Thus $\ker f \ll N$.
Therefore M is small monoform.

($\Leftarrow$) It is clear by remarks and examples 1.2 (4).

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المقاسات ذات الصيغة المتباينة الصغيرة

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الخلاصة

لتكن R حلقة إبدالية ذات محايد وليكن M مقاساً أيسر على R. قدمنا في هذا البحث مفهوم المقاسات ذات الصيغة المتباينة الصغيرة تعميماً للمقاسات ذات الصيغة المتباينة. يسمى M مقاس ذي صيغة متباينة صغيرة اذا كان لكل مقاس جزئي غير صفري N في M ولكل تشاكل $f \in \text{Hom}(N,M)$ و $f \neq 0$ يؤدي الى $\ker f$ مقاس جزئي صغير في N. أعطينا الخواص الاساسية للمقاسات ذات الصيغة المتباينة الصغيرة. كذلك قدمنا بعض العلاقات بين المقاسات ذات الصيغة المتباينة الصغيرة مع بعض المقاسات المرتبطة معها.

الكلمات المفتاحية : المقاس ذي صيغة المتباينة ، المقاس ذي صيغة المتباينة الصغيرة، مقاس جزئي صغير ، مقاس أولي ، مقاس أولي صغير ، مقاس منتظم ، مقاس غير منفرد، مقاس شبه ديدكايند.