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A New Technique for Solving Fractional Nonlinear Equations by Sumudu Transform and Adomian Decomposition Method

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Abstract

A novel technique Sumudu transform Adomian decomposition method (STADM), is employed to handle some kinds of nonlinear time-fractional equations. We demonstrate that this method finds the solution without discretization or restrictive assumptions. This method is efficient, simple to implement, and produces good results. The fractional derivative is described in the Caputo sense. The solutions are obtained using STADM, and the results show that the suggested technique is valid and applicable and provides a more refined convergent series solution. The MATLAB software carried out all the computations and graphics. Moreover, a graphical representation was made for the solution of some examples. For integer and fractional order problems, solution graphs are shown. The results confirmed that the accuracy of this technique converges to the integer order of the issues.

Keywords: *Caputo* derivative, Fractional Calculus, Sumudu Transformation, Analytical the solution, Adomian method.

1. Introduction

Nonlinear problems are used to describe a variety of phenomena. Fractional differential equations (FDEs) have gained much attention from researchers due to their ability and are used in various fields of engineering and physics. The exact solution to any fractional differential equation is extremely difficult to find, and no general method provides the same solution for any fractional differential equations. Several analytical and approximate ways have been suggested for solving nonlinear problems of fractional order using the Sumudu variational iteration method [1,2], decomposition method, and variational iteration method [3,4,]. Many numerical and analytical techniques have been suggested for the solutions of non-integer differential equations of fractional order, such as the Adomian decomposition method, Variational iteration method, homotopy analysis method, and homotopy perturbation method [5-9].

2. Derivation of Sumudu Transform and Decomposition Approach:

Consider a general fractional order partial differential equations as the form:

$$D_t^{\alpha} y(x,t) + R[y(x,t)] + N[y(x,t)] = g(x,t) \qquad (1)$$

As $\quad y(x,0) = f(x)$

Since, $D_t^{\alpha} y(x,t)$ represents *Caputo* fractional derivative of $y(x,t)$ which is defined as:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} y(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-s)^{n-\alpha-1} \ \dfrac{\partial^n y(x,s)}{\partial t^n} & n-1 < \alpha < n \\ \dfrac{\partial^n y(x,t)}{\partial t^n} & \alpha = n \in N \end{cases}$$

By taking *Sumudu* transform of the Eq. (1), get :

$$S[D_t^{\alpha} y(x,t)] + S\big[R[y(x,t)]\big] + S\big[N[y(x,t)]\big] = S[g(x,t)]$$

Using the property of *Sumudu* transform of function derivatives is defined, we get:

$$S[y(x,t)] = y(x,0) + s^{\alpha} S[g(x,t)] - s^{\alpha} S\big[R\big(y(x,t)\big) + N\big(y(x,t)\big)\big] \quad (2)$$

Application of *Sumudu* inverse transform on Eq. (2) yields:

$$y(x,t) = f(x) + S^{-1}(s^{\alpha} S\,[g(x,t)]) \\ - S^{-1}(s^{\alpha} S[R(y(x,t) + N(y(x,t)]) \qquad (3)$$

$$\frac{S[y(x,t)]}{S^{\alpha}} = \sum_{k=0}^{n-1} \frac{y(x,0)^k}{S^{\alpha-k}} + S[g(x,t)] - S[R(y(x,t)) + N(y(x,t))]$$

Now, the representation of the solution for Eq. (3) is given below:

$$y(x,t) = \sum_{i=0}^{\infty} y_i(x,t) \qquad (4)$$

And

$$N[y(x,t)] = \sum_{i=0}^{\infty} A_i(y_0, y_1, \ldots, y_n) \qquad (5)$$

Where, A_i are the *Adomian polynomials* of functions $y_0, y_1, \ldots, y_n$ that can be calculated by the formula given as:

$$A_i = \frac{1}{i!} \frac{\partial^i}{\partial \lambda^i} \left[N\left(\sum_{i=0}^{\infty} \lambda^i y_i \right) \right]_{\lambda=0}$$

Substituting Eqs. (4) and (5) in Eq. (3):

$$\sum_{i=0}^{\infty} y_i(x,t) = \sum_{k=0}^{\infty} \frac{y(x,0)^{(k)}}{S^{\alpha-k}} \\ + \ S^{-1}(s^{\alpha} S[g(x,t)]) S^{-1} \left[s^{\alpha} S \left[R\left(\sum_{i=0}^{\infty} y_i(x,t) \right) + \sum_{i=0}^{\infty} A_i \right] \right] \qquad (6)$$

Simplification of Eq. (6) as many times as possible resulted in a series solutions, we get:

$$y_0(x,t) = \sum_{k=0}^{\infty} \frac{y(x,0)^{(k)}}{S^{\alpha-k}} + S^{-1}(s^{\alpha} S[g(x,t)])$$

$$y_1(x,t) = -S^{-1}\left[s^{\alpha} S[R(y_0(x,t)) + A_0]\right]$$

$$y_2(x,t) = -S^{-1}\left[s^{\alpha} S\left[R\left(y_1(x,t)\right) + A_1\right]\right]$$

$$\vdots$$

$$y_i(x,t) = -S^{-1}\left[s^{\alpha} S[R(y_{n-1}(x,t)) + A_n]\right]$$

Finally, the iteration $y_0, y_1, \ldots, y_i$ were obtained and we get $y(x,t) = \sum_{i=0}^{\infty} y_i(x,t)$.

3. Numerical Example

The following example demonstrates the efficiency and reliability of the Sumudu transform Adomian decomposition method (STADM). The software MATLAB R2021b is used to calculate all of the results.

Example 3.1: Consider $\frac{\partial^{\alpha} y}{\partial t^{\alpha}} - \frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2} = 0 \qquad 0 < \alpha \le 1$ (7)

And $y(x,0) = x$

Solution:

By taking *Sumudu* transform for Eq. (7) and using the initial condition, we get:

$$\frac{S[y]}{u^{\alpha}} = x + S\left[\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\right]$$

$$S\,[y] = x + u^{\alpha} S\left[\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\right]$$

And applying the inverse *Sumudu* transform for the above equation

$$y(x,t) = x + S^{-1}\left[u^{\alpha} S\left[\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\right]\right] \tag{8}$$

That assumes a series solution of the function $y(x,t)$ and is given by:

$$y(x,t) = \sum_{n=0}^{\infty} y_n(x,t) \tag{9}$$

Using Eqs. (9) and (8), we get:

$$y(x,t) = x + S^{-1}\left[u^{\alpha} S\left[\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} y_n(x,t)\right) + \frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} y_n(x,t)\right)\right]\right]$$

Then, we get:

$$y_0(x,t) = x$$

$$y_{n+1}(x,t) = S^{-1}\left[u^{\alpha} S[y_{nx} + y_{nxx}]\right]$$

For $n = 0$

$$y_1(x,t) = S^{-1}\left[u^{\alpha} S[1]\right] = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

For $n = 1$

$$y_2(x,t) = S^{-1}\left[u^{\alpha} S[y_{1x} + y_{1xx}]\right] = 0$$

$$y_3 = y_4 = 0$$

Hence,

$$y(x,t)=\sum_{n=0}^{\infty} y_n(x,t)=y_0+y_1+y_2+\cdots$$
$$=x+\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

When $\alpha=1$,

$$y(x,t)=x+t$$
$$y_0(x,t)=x$$
$$y_1(x,t)=\frac{t^{\alpha}}{\Gamma(\alpha+1)} \qquad \text{and} \qquad y_2(x,t)=y_3(x,t)=0$$

4. Results and Discussion:

The following Tables and Figures present the Absolute error between the same solution and approximate solutions for Example (3.1) at various values of t = 0,0.1. Here, we use **several terms to approximate** the exact solution, and the proposed method, FSTADM, has a high convergence order and higher accuracy we get. Similarly, **Figure4.1–Figure4.6** show the 3D exact and obtained results are plotted at $\alpha=0.75, 0.9$ and $\alpha=1$. All the exact and approximate results on the Graphs have shown are much closed and explain the reliability of the present technique.

Table 1: The Data for y-Scales at x= 1 and α=0.9.

T	y0	yexact	y1	Abs0	Abs1
0	1	1	0	0	1
0.1	1	1.1	0.130897	0.1	0.969103
0.2	1	1.2	0.244263	0.2	0.955737
0.3	1	1.3	0.351836	0.3	0.948164
0.4	1	1.4	0.455811	0.4	0.944189
0.5	1	1.5	0.55719	0.5	0.94281
0.6	1	1.6	0.656548	0.6	0.943452
0.7	1	1.7	0.754256	0.7	0.945744
0.8	1	1.8	0.850573	0.8	0.949427
0.9	1	1.9	0.94569	0.9	0.95431
1	1	2	1.039754	1	0.960246

Table 2: The Data for y-Scales at x= 1 and α=1.

t	y0	yexact	y1	Abs0	Abs1
0	1	1	0	0	1
0.1	1	1.1	0.1	0.1	1
0.2	1	1.2	0.2	0.2	1
0.3	1	1.3	0.3	0.3	1
0.4	1	1.4	0.4	0.4	1
0.5	1	1.5	0.5	0.5	1
0.6	1	1.6	0.6	0.6	1
0.7	1	1.7	0.7	0.7	1
0.8	1	1.8	0.8	0.8	1
0.9	1	1.9	0.9	0.9	1
1	1	2	1	1	1

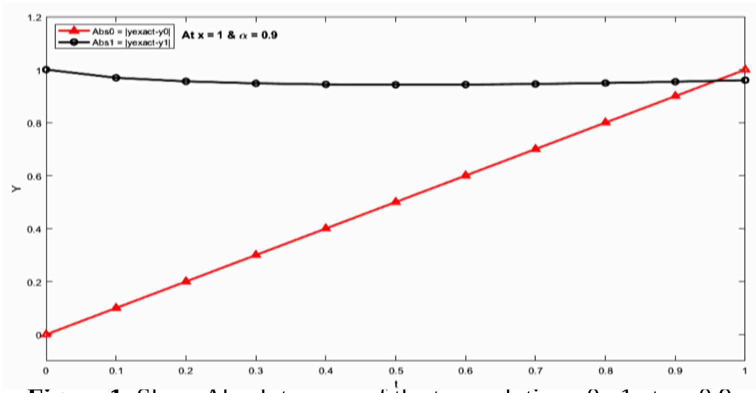


Figure 1. Show Absolute error of the true solution y0,y1 at α=0.9.

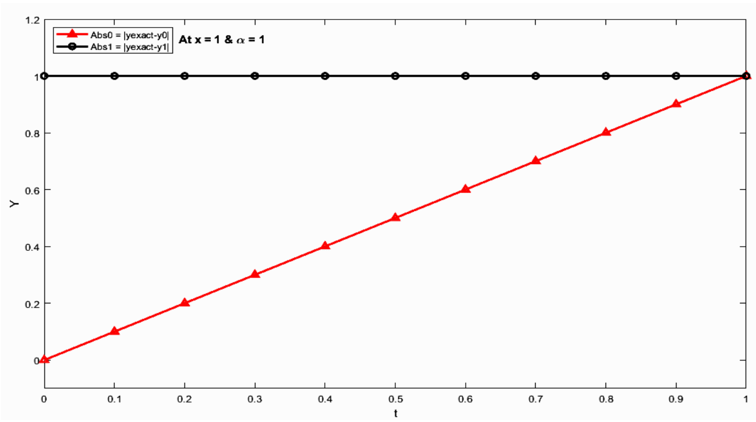


Figure 2. Show Absolute error of the true solution y0,y1 at α=1.

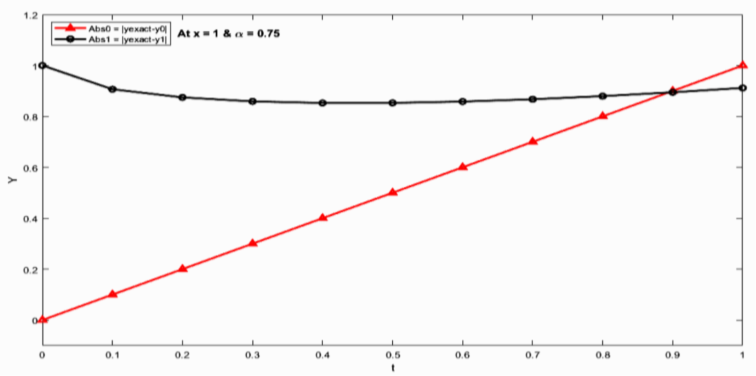
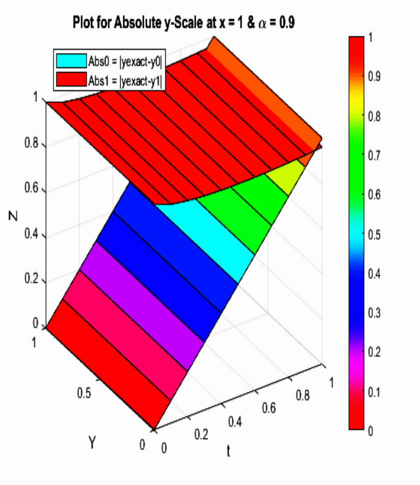
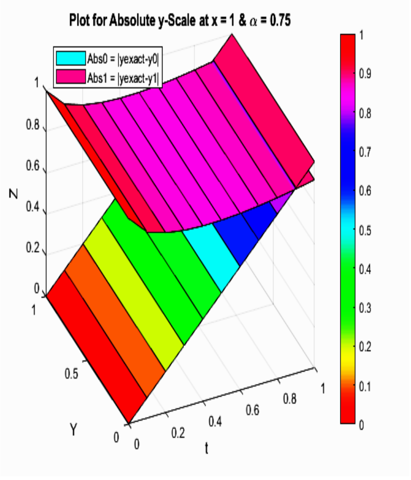


Figure 3. Show Absolute error of the true solution y0,y1 at α=0.75.





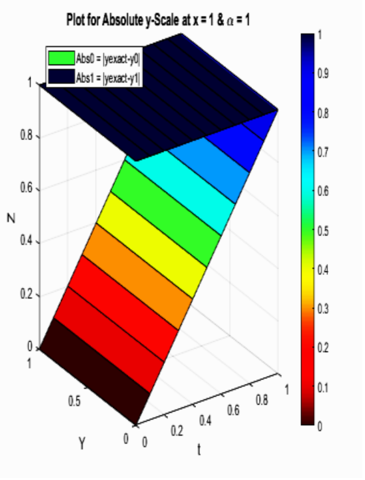


Figure 4 . Figure 1.6: Show the 3D absolute solution plots at α=0.75, 0.9 and α=1.

5. Conclusion

Fractional order non-linear differential equations with initial and boundary conditions are investigated analytically using STADM. Fractional derivatives are defined in the Caputo sense. The solution graphs are shown to demonstrate the current technique's best applicability. The graphs show that the proposed approach is a powerful tool for solving integer and fractional order problems. Some examples of the analytical solution are evaluated to confirm the accuracy and efficiency of the available approach.

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