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On the Growth of Solutions of Nonhomogeneous Higher order Complex Linear Differential Equations

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Abstract

The nonhomogeneous higher order linear complex differential equation (HOLCDE) with meromorphic (or entire) functions is considered in this paper. The results are obtained by putting some conditions on the coefficients to prove that the hyper order of any nonzero solution of this equation equals the order of one of its coefficients in case the coefficients are meromorphic functions. In this case, the conditions were put are that the lower order of one of the coefficients dominates the maximum of the convergence exponent of the zeros sequence of it, the lower order of both of the other coefficients and the nonhomogeneous part and that the solution has infinite order. Whiles in case the coefficients are entire functions, any nonzero solution with finite order has hyper order equals to the lower order of one of its coefficients is proved. In this case, the condition that the lower order of one of the coefficients is greater than the maximum of the lower order of the other coefficients and the lower order of the nonhomogeneous part is assumed.

Keywords: Complex linear differential equations; meromorphic functions; entire functions; order of growth; lower order of growth.

1. Introduction

The theory of meromorphic functions due to Nevanlinna is a good tool in the complex differential equations field. At the forefront of the application of Nevanlinna's theory of meromorphic functions to the complex differential equations was Yoseida in 1932, and since then complex differential equations have become an active field of study by researchers. We refer the reader to, for instance [1], for more information about the differential equations theory in the complex plane. One of the aims of studying this type of equation is the order of growth of its solutions of it. In our work, we shall study the hyper order of the solutions of Eq. (1) below considering the lower orders of coefficients and exponent of convergence of the zeros sequence of one of them. In this paper, we assume that the reader is familiar with the basic concepts and the results regarding the Nevanlinna value distribution theory of meromorphic functions such as $M(r,f), T(r,f)$ and $N(r,f)$ etc., see [2]. A nonhomogeneous HOLCDE is given by

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_2(z)f'' + A_1(z)f' + A_0(z)f = F(z) \quad (1)$$

where $f = f^{(0)}$ is unknown and $A_j(z), 0 \le j \le n-1,\ F(z)$ are given functions.

Many authors studied Eq. (1) and obtained some results. Here we shall mention some of them. The following two results study the hyper order of f when the order of one coefficient dominates the order of F and exponent of convergence of the zeros sequence of that coefficient.

Theorem 1 [3] Let $E \subseteq \mathbb{C}$ satisfy $m_l(\{|z|: z \in E\}) = \infty$ and $A_j(z), j = 0,1,\dots,n-1, F(z)$ be meromorphic functions. Suppose that it is $s, 0 \le s \le n-1$, satisfies

$$\max_{\substack{0\le j\le n-1\\ j\ne s}} \left\{\rho(A_j), \lambda\left(\frac{1}{A_s}\right), \rho(F)\right\} \le \rho(A_s) = \rho < \infty \quad (2)$$

and for some constants $0 \le \beta < \alpha$, we have

$$\left|A_j(z)\right| \le \exp(\beta|z|^{\rho-\varepsilon}),\ \ j \ne s \quad (3)$$

$$|A_s(z)| \ge \exp(\alpha|z|^{\rho-\varepsilon}) \quad (4)$$

hold for $\varepsilon > 0$ and as $|z| \to \infty, z \in E$. Then any meromorphic solution $f \ne 0$ of Eq. (1) with poles of uniformly bounded multiplicities satisfies $\rho_2(f) \ge \rho(A_s)$.

Theorem 2 [3] Let $A_j(z), j = 0,1,\dots,n-1, F(z)$ are satisfied (2). Then every meromorphic solution $f \ne 0$ with $\rho(f) = \infty$ has poles are of uniformly bounded multiplicities of Eq. (1) satisfies $\rho_2(f) \le \rho(A_s)$.

The following result studies the property of f when the order of one coefficient dominates the maximum orders of F and the other coefficients.

Theorem 3 [4] Let $A_j(z), j = 0,1,\dots,n-1, F(z)$ entire functions. Suppose there is $0 \le s \le n-1$, such that

$$\max\left\{\max_{\substack{0\le j\le n-1\\ j\ne s}} \rho(A_j), \rho(F)\right\} < \rho(A_s) \le \frac{1}{2}$$

Then every solution of Eq. (1) is either a polynomial or infinite order entire function.

Theorem 4 [5] Let $A_j(z), j = 0,1,\dots,n-1, F(z)$ be defined as in Theorem 3 such that

$$\max\{\rho(A_j), \rho(F)\} < \rho(A_s) < \frac{1}{2}$$

Then every transcendental solution of Eq. (1) satisfies $\rho_2(f) = \rho(A_s)$. Furthermore, if $F \ne 0$ then $\lambda_2(f) = \rho(A_s)$.

2. Material

We recall the following definitions.

Definition 1 [6] Let $E \subseteq [0,\infty)$. Then the linear measure of E is

$$m(E) = \int_E dt$$

Definition 2 [7, 8] Let $E \subseteq [1, \infty)$, then the logarithmic measure of E is

$$m_l(E) = \int_E \frac{dt}{t}$$

Definition 3 [9, 10] Let f be a meromorphic function. We define the order of growth $\rho(f)$ (respectively) and lower order of growth $\mu(f)$, by

$$\rho(f) = \lim_{r\to\infty} sup \frac{log^+ T(r,f)}{log\, r}$$

and

$$\mu(f) = \lim_{r\to\infty} inf \frac{log^+ T(r,f)}{log\, r}$$

If f is entire function, then $T(r,f)$ is replaced with $log^+ M(r,f)$, where

$$M(r,f) = \max_{|z|=r} |f(z)|$$

Definition 4 [11, 12] We define the hyper-order $\rho_2(f)$ of meromorphic function f by

$$\rho_2(f) = \lim_{r\to\infty} sup \frac{log^+ log^+ T(r,f)}{log\, r}$$

If f is entire function, then

$$\rho_2(f) = \lim_{r\to\infty} sup \frac{log^+ log^+ log^+ M(r,f)}{log\, r}$$

Definition 5 [3] Let f be meromorphic function. By

$$\lambda(f) = \lim_{r\to\infty} \sup \frac{log^+ N(r, \frac{1}{f})}{logr}$$

we meant the convergence exponent of the zeros sequence of f, while

$$\lambda\left(\frac{1}{f}\right) = \lim_{r\to\infty} \sup \frac{log^+ N(r,f)}{logr}$$

is called the convergence exponent of the poles sequence of f.

Definition 6 [3] The lower and upper logarithmic densities of $E \subseteq [1, \infty)$ are as follows

$$\underline{logdens}E = \lim_{r\to\infty} inf\ \frac{m_l(E \cap [1,r])}{logr}$$

and

$$\overline{logdens}E = \lim_{r\to\infty} sup\ \frac{m_l(E \cap [1,r])}{logr}$$

respectively. We say that E has logarithmic density if

$$\underline{logdens}E = \overline{logdens}E$$

3. Methods of Work

This section has introduced some results that will help us to prove our results.

Lemma 1 [13] Let (f, Γ) denote a pair that consists of transcendental meromorphic function $f(z)$ and a finite set

$$\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_q, j_q)\}$$

of distinct pairs in $\mathbb{Z}^+$ $k_i > j_i \geq 0$ for $i = 1,2,\dots,q$, let $\alpha > 0$, $\varepsilon > 0$. Then the following hold:

i) There is $E_1 \subseteq [0, 2\pi)$ with $m(E_1) = 0$, and there is $c > 0$ that depends only on α and Γ such that if $\varphi_0 \in [0, 2\pi)\backslash \mathrm{E}_1$, there is $R_0 = R_0(\varphi_0) > 1$ such that for $arg\ z = \varphi_0$ and $|z| = r \geq R_0$, & for $(k, j) \in \Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c\left(\frac{T(\alpha r, f)}{r}\ log\alpha r \log T(\alpha r, f)\right)^{k-j} \tag{5}$$

In particular, if $f(z)$ with $\rho(f\ ) < \infty$, then (5) is replaced by:

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c|z|^{(k-j)(\rho(f)-1+\varepsilon)} \tag{6}$$

ii) There is $E_2 \subseteq [1, \infty)$ with $m_l(E_2) < \infty$, and $c > 0$ that depends only on α and Γ such that for $|z| = r \notin E_2 \cup [0, 1]$ and for $(k, j) \in \Gamma$, (5) holds. In particular, if $f(z)$ is with $\rho(f) < \infty$, then (6) holds.

iii) There is $E_3 \subset [0, \infty)$ with $m(E_3) < \infty$, and $c > 0$ that depends only on α and Γ such that for $|z| = r \notin E_3$ and $(k, j) \in \Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c\ (T(\alpha r, f)r^{\varepsilon}\ log\ T(\alpha r, f))^{k-j} \tag{7}$$

In particular, if $f(z)$ with $\rho(f) < \infty$, then (7) is replaced by

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c|z|^{(k-j)(\rho(f)+\varepsilon)} \tag{8}$$

Lemma 2 [14] Let f be meromorphic with order $\rho = \rho(f\ ) < \infty$. Then, for $\varepsilon > 0$, there is $E \subseteq (1, \infty)$ with $m_l(E) < \infty$, $m(E) < \infty$, s. t.

$$|f(z)| \leq \exp(r^{\rho+\varepsilon})$$

for $|z| = r \notin [0,1] \cup E$.

Lemma 3 [3] Let $f(z) = \frac{g(z)}{d(z)}$ be meromorphic, where $g(z)$ and $d(z)$ are entire that satisfy

$$\mu(g) = \mu(f) = \mu \leq \rho(g) = \rho(f) \leq \infty$$
$$\lambda(d) = \rho(d) = \lambda\left(\frac{1}{f}\right) = \beta < \mu$$

Then there is $E \subseteq (1, \infty)$ with $m_l(E) < \infty$, s. t.

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{v(r, g)}{z}\right)^n \big(1 + o(1)\big), \qquad n \geq 1$$

holds for z with $|z| = r \notin [0,1] \cup E$, $|g(z)| = M(r, g)$, $M(r, g) = \max_{|z|=r}|g(z)|$, where $v(r, g)$ is the central index of g.

Lemma 4 [11] Let $g(r)$ and $h(r)$ be monotone nondecreasing functions on $[0, \infty)$, such that $g(r) \leq h(r)$ for $r \notin S$ where S is a set with $m_l(S) < \infty$, let $\alpha > 1$. Then there is $r_0 > 1$, such that $g(r) \leq h(\alpha r)$ for $r > r_0$.

Lemma 5 [15] Assume that $g(z)$ is an entire function with $0 \leq \mu(g) < 1$. Then, for $\alpha \in (\mu(g), 1)$, there is $E \subseteq [0, \infty)$ such that $\overline{logdens}(E)\ \geq 1 - \frac{\mu(g)}{\alpha}$, where

$$E = \{r \in [0, \infty) :\ m(r) > M(r)\ cos\ \pi\alpha\}$$

where

$$m(r) = inf_{|z|=r}\ log|g(z)|$$
$$M(r) = sup_{|z|=r}\ log|g(z)|$$

Lemma 6 [16] Let $g(z)$ satisfy the hypothesis of Lemma 5 with $\rho(g) = \rho$ instead of $\mu(g)$. If $\rho < \alpha < 1$, then

$$\underline{logdens}(E) \geq 1 - \frac{\rho}{\alpha}$$

Lemma 7 [15] Let $f(z)$ be an entire function with $\mu(f) = \mu < \frac{1}{2}$ and $\mu < \rho = \rho(f)$. If $\mu \leq \delta < min(\rho, \frac{1}{2})$ and $\delta < \alpha < \frac{1}{2}$, then

$$\overline{logdens}\{r \in [0, \infty): m(r) > (cos\pi\alpha)M(r) > r^{\delta}\} > C(\rho, \delta, \alpha)$$

where $C(\rho, \delta, \alpha) > 0$ and $m(r)$ and $M(r)$ are given as in Lemma 5.

Lemma 8 [5] Let $f(z)$ be transcendental entire function. Then there is $E \subseteq (1, \infty)$ with $m_l(E) < \infty$, s. t. for a point z with $|z| = r \notin [0,1] \cup E$ & $|f(z)| = M(r, f)$, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq 2r^s \ (s \in \mathbb{N})$$

Lemma 9 [3] Let f be an infinite order entire function, with $\rho_2(f) < \infty$, and let $\nu(r, g)$ be the central index of f. Then

$$\limsup_{r \to \infty} \frac{log^+ log^+ \nu(r, g)}{logr} = \rho_2(f)$$

Lemma 10 [3] Let $f(z) = \frac{g(z)}{d(z)}$ be given as in Lemma 3. If $0 \leq \rho(d) < \mu(f)$, then $\mu(g) = \mu(f),\ \rho(g) = \rho(f)$. Moreover, if $\rho(f) = \infty$, then $\rho_2(f) = \rho_2(g)$.

4. Results and Discussion

The coefficients in the following two results are meromorphic and one of them has lower order that dominates the others on some subset of $\mathbb{C}$ with a finite order solution.

Theorem 5 Let $E \subseteq \mathbb{C}$ and define $S = \{|z|: z \in E\}$ and let E satisfies $m_l(S) = \infty$. Suppose that $A_j(z), j = 0,1, \dots, n-1$ and $F(z)$ are meromorphic functions. Suppose that it is $s, 0 \leq s \leq n-1$, such that

$$p = \max_{\substack{0 \leq j \leq n-1 \\ j \neq s}} \left\{\mu(A_j), \lambda\left(\frac{1}{A_s}\right), \mu(F)\right\} \leq \mu(A_s) = \mu < \infty, \qquad (9)$$

holds and for a constant $0 \leq \beta < \alpha$, the relations (3) and (4) hold, for $\varepsilon > 0$ and as $|z| \to \infty$, $z \in E$. Then any solution $f \neq 0$ of Eq. (1) with $\rho(f) < \infty$ satisfies $\rho_2(f) \geq \rho(A_s)$.

Proof From Eq. (1), we have

$$A_s = \frac{F}{f}\frac{f}{f^{(s)}} - \left\{\frac{f^{(n)}}{f^{(s)}} + A_{n-1}\frac{f^{(n-1)}}{f^{(s)}} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} + \frac{f}{f^{(s)}}\left(A_{s-1}\frac{f^{(s-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right)\right\} \qquad (10)$$

Using Lemma 1 (ii), with $\alpha = 2$, there is $E_1 \subseteq [1, \infty)$ with $m_l(E_1) < \infty$ and $B > 0$, such that

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leq Br\big(T(2r, f)\big)^{j-s}, j = s+1, s+2, \dots, n \qquad (11)$$

and

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq Br\big(T(2r, f)\big)^{j}, \quad j = 1, 2, \dots, s-1, \qquad (12)$$

hold for $|z| = r \notin ([0, R_1] \cup E_1)$, $R_1 > 1$.

Put $f(z)=\frac{g(z)}{d(z)}$ where $g(z)$ is entire, $d(z)$ is a product of poles sequence of f.
Let η be such that $p<\eta<\mu(A_s)$. Using Lemma 2, there is $E_2 \subseteq (1,\infty)$ with $m_l(E_2)<\infty$, such that

$$|F(z)d(z)| \leq \exp(r^{\eta}) \tag{13}$$

holds for $|z|=r \notin [0,1] \cup E_2$.
Thus, there is $R_2\ (>R_1)$, such that, for all z satisfying

$$|z|=r>R_2, v\,(r,g)>1, |1+o(1)|>\frac{1}{2}$$

and $|g(z)|=M(r,g)>1$ the following holds:

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq 2r^{s} \tag{14}$$

Set

$$H=\{|z| : z \in E\} \backslash ([0,R_2] \cup E_1 \cup E_2)$$

Then $m_l(H)=\infty$. It follows from (13) that

$$\left|\frac{F(z)}{f(z)}\right| = \left|\frac{F(z)}{g(z)}\ d(z)\right| = \left|\frac{F(z)}{M(r,g)}\right|\ |d(z)| \leq \exp(r^{\eta}) \tag{15}$$

for $|z|=r \in H, r>R_2$, and $|g(z)|=M(r,g)$.
It follows from (10), (11), (12), (14), (15) and (3), (4), that

$$\exp(\alpha r^{\rho-\varepsilon}) \leq 2(n+1)Br^{s+1}\big(T(2r,f)\big)^{n+1}\exp(\beta r^{\rho-\varepsilon})\exp(r^{\eta})$$

for z with $|z|=r \in H$, $\varepsilon \in (0,\frac{\mu(A_s)-\eta}{2})$ and $|g(z)|=M(r,g)$.
Since $\eta<\mu(A_s)$, we obtain

$$\rho_2(f\ ) \geq \rho(A_s)$$

In the following result, the same conditions described in the previous Theorem are given with a solution that has infinite order and obtains the opposite result.

Theorem 6 Let $A_j(z), j=0,1,\ldots,n-1$, $F(z)$ be defined as in Theorem 5 and satisfy inequality (9). Then every solution $f \neq 0$ with $\rho(f)=\infty$ of Eq. (1) satisfies $\rho_2(f\ ) \leq \rho(A_s)$.

Proof From Eq. (1), we have

$$-\frac{f^{(n)}}{f} = A_{n-1}\frac{f^{(n-1)}}{f} + \cdots + A_s\frac{f^{(s)}}{f} + \cdots + A_1\frac{f'}{f} + A_0 - \frac{F}{f} \tag{16}$$

Using Lemma 2, for $\varepsilon>0$, there is $E_3 \subseteq (1,\infty)$ with $m_l(E_3)<\infty$, such that

$$\left|A_j(z)\right| \leq \exp\left(r^{\rho(A_s)+\varepsilon}\right),\ \ j=0,1,\ldots,n-1 \tag{17}$$

and

$$|F(z)| \leq \exp\left(r^{\rho(A_s)+\varepsilon}\right), \tag{18}$$

holds for $|z|=r \notin [0,1] \cup E_3$.
Put $f(z)=\frac{g(z)}{d(z)}$ as in the proof of Theorem 6. Thus by Lemma 3, there is $E_4 \subseteq (1,\infty)$ with $m_l(E_4)<\infty$, such that

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{v(r,g)}{z}\right)^{j}\big(1+o(1)\big), \qquad j=1,\ldots,n \tag{19}$$

holds for $|z|=r \notin [0,1] \cup E_4$, $|g(z)|=M(r,g)>1$.
Hence from (16), (17) and (19) there is $R>1$, such that

$$\left|\left(\frac{v(r,g)}{z}\right)^{n}\left(1+o(1)\right)\right|\left\{\left|\left(\frac{v(r,g)}{z}\right)^{n-1}\left(1+o(1)\right)\right|+\cdots+\left|\left(\frac{v(r,g)}{z}\right)^{s}\left(1+o(1)\right)\right|+\cdots\right.$$
$$\left.+\left|\left(\frac{v(r,g)}{z}\right)\left(1+o(1)\right)\right|+1\right\}\exp\left(r^{\rho(A_s)+\varepsilon}\right)+\left|\frac{F(z)}{f(z)}\right| \tag{20}$$

holds for $|z|=r\notin[0,1]\cup E_3\cup E_4$.
Since $\rho(d)<\rho(A_s)$, we have for $r>R$,

$$|d(z)|\le\exp\left(r^{\rho(A_s)+\varepsilon}\right) \tag{21}$$

Then from (18) and (21) we have

$$\left|\frac{F}{f}\right|=\left|\frac{F}{g}\,d\right|=\left|\frac{F}{M(r,g)}\,d\right|\le\exp\left(2r^{\rho(A_s)+\varepsilon}\right)$$

Combining (20) and the above inequality, we get

$$\left(v(r,g)\right)^{n}|1+o(1)|\le(n+1)\exp\left(2r^{\rho(A_s)+\varepsilon}\right)|z|^{n}\left(v(r,g)\right)^{n-1}|1+o(1)|$$

Hence

$$v_g(r)\le(n+1)\exp\left(2r^{\rho(A_s)+\varepsilon}\right)|z|^{n}$$

Combining Lemma 9 and Lemma 4 and the above inequality, we get

$$\rho_2(g)\le\rho(A_s)$$

Combining Lemma 10 and the above inequality we get

$$\rho_2(f)\le\rho(A_s)$$

This completes the proof.

Theorem 7 Under the assumptions of the previous two Theorems we have $\rho_2(f)=\rho(A_s)$.

In what follows we shall consider Eq. (1) when $A_j(z)$ and $F(z)$ are entire.

Theorem 8 Assume that $A_j(z), j=0,1,\dots,n-1$ and $F(z)$ are entire functions and it is $s, 0\le s\le n-1$, such that

$$q=\max_{\substack{0\le j\le n-1\\ j\ne s}}\{\mu(A_j),\mu(F)\}<\mu(A_s)<\frac{1}{2}$$

Let $f\ne 0$ be any solution with $\rho(f)<\infty$. Then $\rho_2(f)=\mu(A_s)$.

Proof From Eq. (1) we have Eq. (10). Using Lemma 1 (ii), there is $E_1\subseteq[1,\infty)$ with $m_l(E_1)<\infty$ such that

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right|\le Mr^{c}\left(T(2r,f)\right)^{2n},\ \ j=s+1,\dots,n \tag{22}$$

and

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\le Mr^{c}\left(T(2r,f)\right)^{2n},\qquad 1\le j\le s-1 \tag{23}$$

holds for $|z|=r\notin[0,1]\cup E_1$.
Choose α,β such that, $q<\alpha<\beta<\mu(A_s)$.
Then we have

$$\left|A_j(z)\right|\le\exp(r^{\alpha}),\ \ 0\le j\le n-1, j\ne s \tag{24}$$
$$|F(z)|\le\exp(r^{\alpha}) \tag{25}$$

for $r\to\infty$.
By Lemma 6 or Lemma 7 there is $H\subseteq(1,\infty)$ with $m_l(H)=\infty$, such that

$$|A_s(z)|>\exp\left(r^{\beta}\right) \tag{26}$$

holds for $|z|=r\in H$.

Because $M(r, f) > 1$, by (25) we have

$$\frac{|F(z)|}{M(r, f)} \le \exp(r^{\alpha}) \tag{27}$$

for $r \to \infty$. By Lemma 8, there is $E_2 \subseteq (1, \infty)$ with $m_l(E_2) < \infty$, such that

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \le 2r^{s} \tag{28}$$

for a point z with $|z| = r \notin [0,1] \cup E_2$ and $|f(z)| = M(r, f)$.
From (22)-(24), (26-28) and Eq. (10) we have

$$\exp\left(r^{\beta}\right) < Mr^{c}\left(T(2r, f)\right)^{2n}(n \exp(r^{\alpha}))2r^{s}$$

and

$$\exp\left(r^{\beta}(1 + o(1)\right) < \left(T(2r, f)\right)^{2n} \tag{29}$$

for a point z with $|z| = r \in H \setminus ([0,1] \cup E_1 \cup E_2)$.
Thus from (29) and since β is arbitrary, we deduce

$$\mu(A_s) \le \rho_2(f) \tag{30}$$

Now, we prove that $\rho_2(f) \le \mu(A_s)$. By Lemma 3, there is $E_3 \subseteq (1, \infty)$ with $m_l(E_3) < \infty$, such that

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{v(r, f)}{z}\right)^{j}\left(1 + o(1)\right), \quad 1 \le j \le n \tag{31}$$

holds for $|z| = r \notin [0,1] \cup E_3$. Hence for $\varepsilon > 0$, we have

$$\left|A_j(z)\right| \le \exp\left(r^{\mu(A_s)+\varepsilon}\right), 0 \le j \le n-1 \tag{32}$$

$$|F(z)| \le \exp\left(r^{\mu(A_s)+\varepsilon}\right) \tag{33}$$

for $r \to \infty$.
Because (33) and $|f(z)| = M(r, f) > 1$, we get

$$\left|\frac{F(z)}{f(z)}\right| \le \exp\left(r^{\mu(A_s)+\varepsilon}\right) \tag{34}$$

for $r \to \infty$.
Take z with $|z| = r \notin [0,1] \cup E_3$. From Eq. (1) we have

$$\left|\frac{f^{(n)}}{f}\right| \le |A_{n-1}(z)|\left|\frac{f^{(n-1)}}{f}\right| + \cdots + |A_2(z)|\left|\frac{f''}{f}\right| + |A_1(z)|\left|\frac{f'}{f}\right| + A_0(z) + \left|\frac{F}{f}\right| \tag{35}$$

Substituting (31), (32) and (34) into (35) yields

$$\left|\frac{v(r, f)}{z}\right|^{n}|1 + o(1)| \le (n+1)\left|\frac{v(r, f)}{z}\right|^{n-1}|1 + o(1)| \exp\left(r^{\mu(A_s)+\varepsilon}\right),$$

This gives

$$\rho_2(f) = \lim_{r\to\infty} \sup \frac{log^{+}log^{+}v(r, f)}{logr} \le \mu(A_s) + \varepsilon \tag{36}$$

Because ε is arbitrary, by Lemma 9 and Eq. (36), we get

$$\rho_2(f) \le \mu(A_s)$$

Combining this and (30) yields $\rho_2(f) = \mu(A_s)$. This completes the proof.

5. Conclusions

The hyper order of growth of the solutions of (HOLCDE) under some conditions related to the lower order of its coefficients is studied and considered in this paper. It is seen that the Nevanlinna theory of the meromorphic functions is a good and helpful tool that we have used to prove our results. our results have been obtained by extending some of the previous results in this direction.

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