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Common diskcyclic vectors

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Abstract

In this paper, we study the common diskcyclic vectors for a path of diskcyclic operators. In particular, if $\{T_t: t \in [a,b]\}$ is a path of diskcyclic operators, we show that under certain conditions the intersection of diskcyclic vectors for theses operators is a dense G_δ set.

Keywords: Diskcyclic operators, Common diskcyclic vectors, Weighted shift operators.

1. Introduction

$Orb(T,x) = \{T^n x: n \in \mathbb{N}\}$ which is dense in X. The study of hypercyclic operators on a Banach space goes back to a 1969 paper of Rolewi

Let X be a Banach space and $B(X)$ be the space of all bounded linear operators on$X$. An operator $T \in B(X)$ is called hypercyclic if there is a vector $x \in X$ called hypercyclic vector for T such that cz [1] that proves if B is the backward shift on the sequence space $l^p(\mathbb{N})$ of then λB is hypercyclic whenever λ is a scalar of modulus > 1. Perhaps, inspired by Rolewicz example, Hilden and Wallen [2] considered the scaled orbit of an operator. An operator T is supercyclic if there is a vector $x \in X$ called supercyclic vector for T such that $\mathbb{C}Orb(T,x) = \{\lambda T^n x: \lambda \in \mathbb{C},\ n \in \mathbb{N}\}$ is dense in X. Also, an operator T is called diskcyclic if there is a vector $x \in X$ called diskcyclic vector for T such that the disk orbit $\mathbb{D}Orb(T,x) = \{\lambda T^n x: \lambda \in \mathbb{C}, |\lambda| \leq 1,\ n \in \mathbb{N}\}$ is dense in X [3]. For more information on these operators, the reader may refer to [4- 6].

Recently, the orbit of an operator in subspaces was studied. More precisely, if the orbit of an operator is dense in a subspace, then such an operator is called subspace-hypercyclic. By, the same manner if the scaled orbit (disk orbit) of an operator is dense in a subspace, then such an operator is called subspace-supercyclic (subspace-diskcyclic) respectively. For more information on these operators, the reader may refer to [7- 10].

of real numbers, then, the set $\{T_t : t \in [a, b]\}$ is called a path of operators if $T_t \in B(X)$ for It has been studied that under certain conditions, an uncountable family of hypercyclic operators (or supercyclic operators) has a dense G_δ set of common hypercyclic vectors (or supercyclic vectors, respectively). For example, [11] ([12]) gave some conditions on a path of supercyclic (hypercyclic) operators to have a common supercyclic (or hypercyclic, respectively) vectors. For more information on common hypercyclic and supercyclic vectors, the reader may refer to [13- 19].

Now, since the set of all diskcyclic vectors is dense G_δ, then a countable collection of diskcyclic operators, by applying the Baire category theorem, has a dense G_δ set of common diskcyclic vectors. However, it is unknown in which cases an uncountable family of diskcyclic operators has a common diskcyclic vectors.

Therefore, in this paper, we study the common diskcyclic vectors for some uncountable families of diskcyclic operators. In particular, we give a sufficient condition for a path of operators to have a dense G_δ set of common diskcyclic vectors, every operator in this path satisfies diskcyclic criterion. Then, we study a path of unilateral weighted backward shifts with common diskcyclic vectors.

First, we recall the following definition from [12].

Definition 1.1. Let X be a Banach space and $[a, b]$ be an interval all $t \in [a, b]$ and if the map $T: [a, b] \to (B(X), \|.\|)$ defined by $T(t) = T_t$ is continuous with respect to both the operator norm topology on $B(X)$ and the usual topology on $\mathbb{R}$.

2. Main Results

Definition 2.1. Let $\{\alpha F_t^n : \alpha \in \mathbb{D},\ n \geq 1\} \subset B(X)$ for each $t \in [a, b]$. Let $t \to \alpha F_t^n$ be a path of operators on $[a, b]$, then the set of common diskcyclic vectors for the path of operators is defined as follows:

$$\bigcap_{t \in [a,b]} DC(F_t) = \{x \in X : Orb(F_t, x), t \in [a, b]\}$$

is dense in X.

Theorem 2.2. The set $\bigcap_{t \in [a,b]} DC(F_t)$ of common diskcyclic vectors is dense G_δ set in X if and only if for each nonempty open sets U and V, there exists a partition

$P = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of $[a, b]$, $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{D}$, $n_1, n_2, \ldots, n_k \in \mathbb{N}$, and an open set G such that if $1 \leq i \leq k$ and $t \in [t_{i-1}, t_i]$ then $G \subseteq U$ and $\alpha_i F_t^{(n_i)} G \subseteq V$.

Proof. The proof follows the same idea of [12, Theorem 2.1], therefore we omit the details.

The following theorem shows that in some cases, an uncountable family of diskcyclic operators has a common diskcyclic vectors which is a dense G_δ set.

Theorem 2.3. Let X be a separable, infinite dimensional Banach space, and let

$\{F_l: l \in [a,b]\}$ be a path of non-trivial bounded linear operators on X. If there exists a dense set D_1 such that for every $y \in D_1$ and $\varepsilon > 0$, there exists $\delta > 0$, a dense set D_2, an increasing sequence of positive integers $\{m_j\}_{j=1}^{\infty}$, and a set of maps $\{S_{l,j}: D_1 \to X: l \in [a,b],\ j \geq 1\}$such that

1. For each $p \in [a,b]$ and $x \in D_2$, the sequence $\left\|F_l^{(m_j)}x\right\| \left\|S_{p,j}y\right\| \to 0$ for all $l \in [a,b]$ and $j \to \infty$,
2. For each $p \in [a,b]$, $\left\|S_{p,j}y\right\| \to 0$ as $j \to \infty$,
3. For each $p \in [a,b]$ and integer $c \geq 1$, there exists $j \geq c$ such that if $|l-p| < \delta$ then $\left\|F_l^{(m_j)}S_{p,j}y - y\right\| < \varepsilon$.

Then $\bigcap_{t\in[a,b]} DC(F_t)$ of common diskcyclic vectors is dense G_δ.
Proof. Let $U_1$and U_2 be two non-trivial open sets in X. Pick $y \in D_1\{0\}$ and $\sigma > 0$ such that $B(y,\sigma) \subseteq U_2$. Then there is an increasing sequence $\{m_j\}_{j=1}^{\infty}$ of positive integers, a dense set D_2, $\delta > 0$ and a set of maps $S_{l,j}: D_1 \to X$ which satisfy the conditions (1), (2) and (3) with respect to the vector y and $\varepsilon = \min\left\{\frac{\sigma}{3}, \frac{\|y\|}{2}\right\}$. Let $P = \{a = l_0 < l_1 < \cdots < l_k = b\}$ be a partition of $[a,b]$ where $\max\{|l_i - l_{i-1}|: 1 \leq i \leq n\} < \delta$.

Claim. Let i be an integer such that $1 \leq i \leq n$, G be a nontrivial open set, and $c \geq 1$ be an integer. Then there exists a nonempty open set $G' \subseteq G$, a number $0 < \lambda \leq 1$, and an integer $j \geq c$ such that $\lambda F_l^{(m_j)} G' \subseteq U_2$, whenever $l \in [l_{i-1}, l_i]$.

Proof of Claim. Let $w \in D_2$ and k be a small enough positive integer such that $B(w,k) \subseteq G$.By putting $p = l_i$ in conditions (1), (2) and (3), one can see that there exists $j \geq c$ such that

$$\left\|F_l^{(m_j)}w\right\| \left\|S_{l_i,j}y\right\| < \frac{\varepsilon k}{2} \text{ for all } l \in [l_{i-1}, l_i], \quad (1)$$

$$\left\|S_{l_i,j}y\right\| < \frac{k}{2}, \quad (2)$$

and

$$\left\|F_l^{(m_j)}S_{l_i,j}y - y\right\| < \varepsilon \text{ for all } l[l_{i-1}, l_i]. \quad (3)$$

Since $\varepsilon = \min\left\{\frac{\sigma}{3}, \frac{\|y\|}{2}\right\}$ and $y \neq 0$, then $F_l^{(m_j)}S_{l_i,j}y \neq 0$ by (3). Now, let $\lambda = \frac{2}{k}\left\|S_{l_i,j}y\right\|$, then it is clear that $0 < \lambda \leq 1$ by (2). Let $= w + \frac{1}{\lambda}S_{l_i,j}y$, $\alpha = sup\left\{\lambda F_l^{(m_j)}: l \in [l_{i-1}, l_i]\right\} > 0$ and $G' = B(z, \frac{\varepsilon}{\alpha}) \cap G \subseteq G$. To prove that the open set G' is nonempty, it is clear that $\|z - w\| = \left\|\frac{1}{\lambda}S_{l_i,j}y\ \right\| = \frac{k}{2} < k$, therefore $z \in B\left(z, \frac{\varepsilon}{\alpha}\right) \cap B(w,k) \subseteq B\left(z, \frac{\varepsilon}{\alpha}\right) \cap G = G'$. Now, if $l \in [l_{i-1}, l_i]$, then $\left\|\lambda F_l^{(m_j)}z - y\right\| = \left\|\lambda F_l^{(m_j)}w + F_l^{(m_j)}S_{l_i,j}y - y\right\|$

$$\leq \lambda\left\|F_l^{(m_j)}w\right\| + \left\|F_l^{(m_j)}S_{l_i,j}y - y\right\|$$

$$< \lambda\left\|F_l^{(m_j)}w\right\| + \varepsilon \text{ by (3)}$$

$$= \frac{2}{k}\left\|F_l^{(m_j)}w\right\| \left\|S_{l_i,j}y\right\| + \varepsilon$$

$$< \frac{2}{k}\frac{\varepsilon k}{2} + \varepsilon \text{ by (1)}$$

$$= 2\varepsilon.$$

And so if $g \in G'$ and $l \in [l_{i-1}, l_i]$, then

$$\begin{aligned}\left\|\lambda F_l^{(m_j)}g-y\right\| &= \left\|\lambda F_l^{(m_j)}g-\lambda F_l^{(m_j)}z+\lambda F_l^{(m_j)}z-y\right\| \\ &\leq \lambda \left\| F_l^{(m_j)}\right\| \|g-z\| + \left\|\lambda F_l^{(m_j)}z-y\right\| \\ &< sup\left\{\lambda \left\| F_l^{(m_j)}\right\| : l \in [\mathrm{l_{i-1}}, \mathrm{l_i}]\right\}\frac{\varepsilon}{\alpha}+2\varepsilon \\ &< 3\varepsilon \\ &< \sigma.\end{aligned}$$

Therefore, $\lambda F_l^{(m_j)}(G') \subseteq B(y,\sigma) \subseteq U_2$, whenever $l \in [\mathrm{l_{i-1}}, \mathrm{l_i}]$. The claim is proved.

Returning to the proof of the theorem, Let G_0 be an open ball with center $b, b \in D_2$ such that $G_0 \subseteq U_1$. Then by claim, there are a scalar $\lambda_1$such that $0 < \lambda_1 \leq 1$, a nonempty open set $G_1 \subseteq G_0$, and a positive integer $j_1 \geq 1$ such that
$\lambda_1 F_l^{(m_{j_1})}(G_1) \subseteq U_2$ whenever $l \in [\mathrm{l_0}, \mathrm{l_1}]$. Again, by the claim, there is a scalar $\lambda_2$such that $0 < \lambda_2 \leq 1$, a nonempty open set $G_2 \subseteq G_1$, and a positive integer $j_2 \geq j_1$ such that
$\lambda_2 F_l^{(m_{j_2})}(G_2) \subseteq U_2$ whenever $l \in [\mathrm{l_1}, \mathrm{l_2}]$. By applying this process n times, we get open sets $G_n \subseteq G_{n-1} \subseteq \cdots \subseteq G_1 \subseteq G_0$, scalars $0 < \lambda_1, \lambda_2, \ldots, \lambda_n \leq 1$, and integers $1 \leq j_1 < j_2 < \cdots < j_n$ such that $\lambda_i F_l^{\left(m_{j_i}\right)}(G_i) \subseteq U_2$ whenever $1 \leq i \leq n$ and $l \in [\mathrm{l_{i-1}}, \mathrm{l_i}]$. So, $G_n \subseteq U_1$ and $\lambda_i F_l^{\left(m_{j_i}\right)}(G_n) \subseteq \lambda_i F_l^{\left(m_{j_i}\right)}(G_i) \subseteq U_2$ whenever $l \in [\mathrm{l_{i-1}}, \mathrm{l_i}]$. The proof follows from Theorem 2.2.

Even though not every diskcyclic operator satisfies diskcyclic criterion, the following proposition shows that in some cases every diskcyclic operator satisfies diskcyclic criterion.

Proposition 2.4. If a path $\{F_t: l \in [a,b]\}$ satisfies the conditions of Theorem 2.3., then every that $\{\lambda_i \in \mathbb{D}\backslash\{0\}: i \geq 1\}$ be a countable set. Suppose that the sequence $\{T_n\}$ is an enumeration of the set $\{\lambda_i F_l^j: i,j \geq 1\}$ for each $l \in [a,b]$.Now, suppose that $\{T_n\}$ satisfies the operator F_l satisfies the diskcyclicity criterion.

Proof. Suppose universality criterion, and then each F_l satisfies the diskcyclic criterion; see [20, Definition 1.1], [5, Theorem 1.2, Theorem 2.6, Proposition 2.8]. Thus, it is enough to prove that $\{T_n\}$ satisfies the universality criterion which is equivalent to showing that for every nonempty open sets U, G, W with $0 \in W$, we have $T_n(U) \cap W \neq \emptyset$ and $T_n(W) \cap G \neq \emptyset$; see [20, Theorem 3.4]. Now, by Theorem 2.3, for any dense set D_1, we can choose $y \in D_1\{0\}$ and $0 < \varepsilon < \|y\|$ such that $B(y,\varepsilon) \subseteq G \, and \, B(0,\varepsilon) \subseteq W$. Then there exists an increasing sequence $\{\mathrm{m_j}\}_{\mathrm{j}=1}^{\infty}$of positive integers, a dense set D_2, a small positive number $\delta > 0$, and maps $\{S_{c,j}: D_1 \to X: j \geq 1, c \in [a,b]\}$ satisfying all conditions of Theorem 2.3. Thus, we can choose $x \in D_2 \cap U$, an integer $j \geq 1$ such that
$\left\|F_l^{(m_j)}x\right\| \|S_{l,j}y\| < \frac{\varepsilon^2}{2}$, $\left\|F_l^{(m_j)}S_{l,j}y-y\right\| < \varepsilon$ and $\|S_{l,j}y\| \to 0$
Let $\lambda \in \{\lambda_i: i \geq 1\}$, since $\lambda < 1$ we can assume that

$$\varepsilon\lambda < 2\|S_{l,j}y\| < 2\varepsilon\lambda \tag{4}$$

Then $\lambda F_l^{(m_j)} = T_n$ for some $n \geq 1$. Since $x \in U$, then by Equation (4), we get

$\|T_n x\| = \left\|\lambda F_l^{(m_j)} x\right\| < \frac{2}{\varepsilon}\left\|F_l^{(m_j)} x\right\| \left\|S_{l,j} y\right\| < \frac{2}{\varepsilon}\frac{\varepsilon^2}{2} = \varepsilon$

Then, $T_n(U) \cap W \neq \emptyset$. Again from Equation (4), we have $\frac{1}{\lambda}\left\|S_{l,j} y\right\| < \varepsilon$ which means that $S_{l,j} y \in W$. Also, we have

$$\left\|T_n(S_{l,j} y) - y\right\| = \left\|F_l^{(m_j)} S_{l,j} y - y\right\| < \varepsilon$$

So, $T_n(W) \cap G \neq \emptyset$, which gives the proof.

Corollary 2.5. An operator $T \in B(X)$ satisfies diskcyclic criterion if and only if there exists a dense set D_1 such that for each $y \in D_1$and a small positive number $\varepsilon > 0$, there is an increasing sequence $\{m_j\}_{j=1}^{\infty}$of positive integers, a dense set D_2 and maps $S_j: D_1 \to X$ satisfying

1. For each $x \in D_2$, the sequence $\| T^{m_j} x\| \left\|S_j y\right\| \to 0$ as $j \to \infty$,
2. $\left\|S_j y\right\| \to 0$ as $j \to \infty$,
3. For each integer $c \geq 1$, there exists $j \geq c$ such that $\left\|T^{m_j} S_j y - y\right\| < \varepsilon$.

Since a unilateral shift T is diskcyclic if and only if it is hypercyclic with $HC(T) = DC(T)$ [5, Corollary 3.6], then the following propositions follow immediately by [12, Theorem 3.5] and [12, Theorem 4.1] respectively.

Proposition 2.6. Suppose that T and S are two diskcyclic weighted backward shifts. Then , there exists a path of unilateral weighted backward shifts between T and S , such a path has a dense G_δ set of common diskcyclic vectors.

Proposition 2.7. Suppose that T and S are two diskcyclic weighted backward shifts. Then , there exists a path of unilateral weighted backward shifts between T and S , such a path has no any common diskcyclic vector.

Conclusion

We have given a sufficient condition for a path of operators to have a dense G_δ set of common diskcyclic vectors such that every operator in that path satisfies diskcyclic criterion.

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