Ibn Al-Haitham Jour. for Pure & Appl. Sci. 33 (4) 2020

Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

Solving Nonlinear Second Order Delay Eigenvalue Problems by Least Square Method

Israa M. Salman **Eman A. Abdul-Razzaq**

Department of Mathematics, College of Education for Pure science- Ibn Al- Haitham, University of Baghdad, Baghdad, Iraq.

israa2m1s@gmail.com eman.l@ihcoedu.uobaghdad.edu.iq

Article history: Received 11 February 2020, Accepted 15 March 2020, Published in October 2020

Doi: 10.30526/33.4.2509

Abstract

The aim of this paper is to study the nonlinear delay second order eigenvalue problems which consists of delay ordinary differential equations, in fact one of the expansion methods that is called the least square method which will be developed to solve this kind of problems.

Keywords: Nonlinear Second Order Sturm-Liouville Problems, the Least Square Method

1.Introduction

The nonlinear delay second order eigenvalue problems consist of delay nonlinear ordinary differential equations with the boundary conditions defined on some intervals, this kind of equations has many applications in different scientific fields, such as physical, biological and engineering science. Also, it is one of the most important application referred to as a delay nonlinear eigenvalue problem, [1].

The delay eigenvalue problem belongs to a wide class of problems whose eigenvalues and eigen-functions have particularly nice properties, [2].

In this paper, we study and solve this kind of problems by least square method.

2. Basic Definitions and Remarks

This section recalls some basic definitions and remarks that needed in this work. We start with the following definition.

Definition 2.1

The delay differential equation is the equation that the unknown function and some of its derivatives, evaluated at cases which are different by any of fixed number or function of values.

Consider the n-th order delay differential equation:

$E(k, f(k), f(k-\tau_1), \ldots, f(k-\tau_m), f'(k), f'(k-\tau_1), \ldots, f'(kx-\tau_m),$

$f^{(n)}(k), \ldots, f^{(n)}(k-\tau_m)) = h(k),\ k \in [a,b]$ (1)

where E is a given function and $\tau_1, \tau_2, \ldots, \tau_m$ are given fixed positive numbers called the time delays, [1].

We say that equation (1) is homogenous delay differential equation in case $f(x) \geq 0$, which we handle in this paper, otherwise it is called non-homogenous delay differential equation [2].

Definition 2.2

The delay differential equation is said to be nonlinear when it is nonlinear with respect to the unknown function that enter with different arguments and their derivatives that appeared in it, [1].

Hence, the new concepts of this work are given by the following definition.

Definition 2.3

The delay eigen-value problem consist of delay ordinary differential equation is said to be nonlinear when it is nonlinear with respect to the unknown eigen-function enter with different arguments and their derivatives that appeared in it.

Next, consider the following nonlinear delay second order eigen-value problem:

$$-(p(k)f'(k))' + q(k)f(k-\tau) - h(k, \lambda, f(k-\tau)) = 0 \qquad (2)$$

with the associated conditions:

$$a_1 f(a) + a_2 f'(a) = 0 \quad, k \in [a-\tau, a]$$

$$(3)$$

$$b_1 f(b) + b_2 f'(b) = 0 \quad, k \in [b-\tau, b]$$

$$f(k-\tau) = \varphi(k-\tau), \text{if } \ k-\tau < a$$

Where, a_1, a_2, b_1, b_2, p, p' and q are given real continuous functions defined on the interval $[a,b]$, p is positive, not both coefficients in one condition are zero, $\tau > 0$ is the time delay, h is a well-defined nonlinear function with respect to f. φ is the initial function defined on $k \in [k_0-\tau, k_0]$. The problem here is to determine the eigen-value λ in which a nontrivial solution f for the problem given by equations (2) & (3) occurs. In this case λ is said to be a delay eigenvalue and f is the associated delay eigen-function.

In other words f is an eigen function for the variable k and the nonlinear function $h(k, \lambda, f(k-\tau))$ with respect to the eigen-value λ.

Like the linear second order eigenvalue problems, the problem given by equations (2) & (3) satisfies the following remarks, [2].

Remarks 2.4

1. The linear delay operator: $L=-\frac{d^2}{dk^2}p(k)-\frac{d}{dk}p'(k)+A(k)q(k)$, where $A(x)$ is an operator defined by $A(k)f(k)=f(k-\tau)$, is self- adjoint.
2. The delay eigen-functions are orthogonal.
3. There are infinite number of delay eigenvalues forming a monotone increasing sequence with $\lambda_j \to \infty$ as $j\to\infty$**.** Moreover, the delay eigen-functions corresponding to the delay eigenvalues has exactly j roots on the interval (a,b)**.**
4. The delay eigen-functions are complete and normal in $L^2[a,b]$.
5. Each delay eigenvalue corresponds only one delay eigen-function in $L^2[a,b]$.

To check remarks (2-4), see [3,4]**.**

3. The Least-Square Method

This method is one of the expansion methods that used to solve the linear (nonlinear) differential equations and equations with or without delays, [5, 6].

Here we develop this method to solve the problem given by equations (2) & (3).

The method is based on approximating the unknown function f as a linear combination of n linearly independent functions $\{\phi_i\}^n_{i=1}$, that is write:

$$f=\sum_{i=1}^{n}\phi_i(k) \tag{4}$$

Which implies that, $f(k-\tau)=\sum_{i=1}^{n}\phi_i(k-\tau)$

This approximated solution must satisfy the boundary conditions given by equations (3) to get a new approximated solution. By substituting this approximated solution into equation (2) one can get:

$$R(k,\lambda,\vec{c})=-(p(k)\sum_{i=1}^{n}\phi_i'(k))'+q(k)\sum_{i=1}^{n}\phi_i(k-\tau)-h(k,\lambda,\sum_{i=1}^{n}\phi_i(k-\tau)) \tag{5}$$

where R is the error in the approximation of equation (2) and $\vec{c}$ is the vector of n–2 elements of c_i, $i=1,2,\ldots,n$, [7-8]

Thus, to minimize the functional:

$$J(\lambda,\vec{c})=\int_a^b (R(k,\lambda,\vec{c}))^2\,dk \tag{6}$$

put $\frac{\partial J}{\partial\lambda}=\frac{\partial J}{\partial c_i}=0$, i = 1,2,…,n, to get a system of n–1 nonlinear equations with n–1 unknowns which can be solved by any suitable method to get the values of λ and $\vec{c}$,[9,10].

To check this method, look at the following examples:

Example 3.1

Consider the following nonlinear delay eigenvalue problem:

$$-(kf'(k))'+2kf(k-1)-\lambda(f^2(k-1)-0.5)=0,\ k\in[1,2] \tag{7}$$

with the associated boundary conditions:

$$f(1) = f'(1) \quad , \ k \in [0,1]$$
$$f(2) = 2f'(2) \ , k \in [1,2] \qquad (8)$$
$$f(k-1) = k-1$$

We use the least-square method to solve this problem. To do this, we approximate the unknown function y as a polynomial of degree three, that is, write:

$$f(k) = \sum_{i=1}^{4} c_i k^{i-1}$$

But this approximated solution must satisfy the boundary conditions given by equations [3.5], thus this approximated solution reduces to $f(k) = c_1 + c_1 k - 6c_4 k^2 + c_4 k^3$

From which, we have:

$$f(k-1) = c_1 + c_1(k-1) - 6c_4(k-1)^2 + c_4(k-1)^3$$

By substituting this approximated solution into equation [3.4], we obtain

$$\begin{aligned} R(k,\lambda,c_1,c_4) = {} & -k(-18c_4 + 6c_4 k) - (c_1 - 18c_4 k - 9c_4 + 3c_4 k^2) \\ & + 2k(c_1 + c_1(k-1) - 6c_4(k-1)^2 + c_4(k-1)^3) \\ & - \lambda[(c_1 + c_1(k-1) - 6c_4(k-1)^2 + c_4(k-1)^3)^2 - \frac{1}{2}] \end{aligned}$$

Thus, if we minimize the functional:

$$J(\lambda,c_1,c_4) = \int_1^2 (R(k,\lambda,c_1,c_4))^2 \, dx$$

set $\frac{\partial J}{\partial \lambda} = \frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_4} = 0$ to get the following system of nonlinear equations:

$$\frac{\partial}{\partial \lambda} \int_1^2 (R(k,\lambda,c_1,c_4))^2 \, dx = 0$$
$$\frac{\partial}{\partial c_1} \int_1^2 (R(k,\lambda,c_1,c_4))^2 \, dx = 0$$
$$\frac{\partial}{\partial c_4} \int_1^2 (R(k,\lambda,c_1,c_4))^2 \, dx = 0$$

Solving the above system by any suitable method, to find that the nontrivial solution is $\lambda = 2$, $c_1 = 1$ and $c_4 = 0$. Therefore 2 is delay eigenvalue with the corresponding delay eigen-function

$f(k-1) = 1 + (k-1), \ k \in [1,2]$.

Generally, if $f(k) = \sum_{i=1}^{n} c_i k^{i-1}$, then the same result can be obtained for all values of n, $n \in N$.

That is if $f(k-\tau)=\sum_{i=1}^{n} c_i (k-\tau)^{i-1}$, then $f(k-1)=1+(k-1),\ k\in[1,2]$ corresponding to the same delay eigenvalue.

Example 3.2

Consider the following nonlinear delay eigenvalue problem:

$$-f''(k)=\lambda(2f(k-\frac{\pi}{2})+\sin(k-\frac{\pi}{2})), \quad k\in[\frac{\pi}{2},\pi]$$

$$f(\frac{\pi}{2})+f'(\frac{\pi}{2})=1, \quad k\in[0,\frac{\pi}{2}]$$

$$f(\pi)\_f'(\pi)=1, \quad k\in[\frac{\pi}{2},\pi]$$

$$f(k-\frac{\pi}{2})=k-\frac{\pi}{2}$$

We use the least-square method to solve this problem. To do this, we follow the same arguments as in example (3.1).

One can get that the eigen- pair of this nonlinear delay eigenvalue problem $(f(k),\lambda)$ is $(\sin k,\frac{1}{3})$, thus, $(f(k-\frac{\pi}{2}),\lambda)=(\sin(k-\frac{\pi}{2}),\frac{1}{3})$

4. Conclusions

From this article we can conclude the following:

1. The nonlinear second order delay eigenvalue problems consist of delay nonlinear ordinary differential equations satisfying the same properties as those consist of delay nonlinear ordinary differential equations with or without delay.
2. The least square method has been developed to solve the above kind of problems.
3. In future we can use the methods in [11, 12] to solve the problems of this article.

References

1. Koivo, A., Estimation in Nonlinear Systems with Transport Delay, **1971**, IEEE *Conference on Decision and Control*.
2. Bhattacharyya, T.; Binding, P.; Seddighi, K. Multiparameter Sturm-Liouville Problems with Eigenparameter Dependent Boundary Conditions, *Journal of Mathematical Analysis and Applications.* **2001,** *264*, 560-576.
3. Bhattacharyya, T.; Kosir, T.; Plestenjak, B. Right Definite Multiparameter Sturm-Liouville Problems With Eigen parameter Dependent Boundary Conditions, *Math, Subj. Class.* **2001,** *204*, 1-13.
4. Bhattacharyya, T.; Binding, P.; Seddighi, K.Two parameter Right Definite Sturm-Liouville Problems with Eigen parameter Dependent Boundary Conditions, *Proc, Roy. Soc. Edinburgh Sect.,* **2001**. *131*, 45-58.
5. Vichnevetsky, R. Computer Methods for Partial Differential Equation, *Prentice-Hall, London.* **1981.**
6. Burden, R. Numerical Analysis*, PWS Publisher, New York.* **1985.**

7. Blanchard, P.; Devancy, L. R.; Hall, G. R. Differential Equations, 2nd. Ed. *The Wadsworth group, a division of Thomson Learning Inc., U.S.A* **2001.**
8. Edwards, C. H.; Penney D. E., Differential Equations and Boundary Value Problems 3rd Ed., *Person Education. Inc.* **2004**.
9. Burden, R.; Faires, J. Numerical Analysis, *PWS Publisher, New York.* **2001**.
10. Reinhold, K. QMC Methods for The Solution of Delay Differential Equation, *Department of Mathematics, Graz University of Technology, Steyrergasse 30.* **2002.**
11. Enadi, M. O.; Tawfiq, L.N.M. New Technique for Solving Autonomous Equations, *Ibn Al-Haitham Journal for Pure and Applied Science*. **2019**, *32,2*,123-130, Doi: 10.30526/32.2.2150.
12. Tawfiq LNM; Jabber AK. New Transform Fundamental Properties and its Applications. *Ibn Alhaitham Journal for Pure and Applied Science*. **2018**, *31,1*,151-12, doi: http://dx.doi.org/10.30526/31.2.1954