Ibn Al-Haitham Jour. for Pure & Appl. Sci. 33 (1) 2020

Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

Certain Family of Multivalent Functions Associated With Subordination

Zainab H. Mahmood
Department of physical ,College of Science , University of Baghdad,Baghdad,Iraq
zainab_hd@yahoo.com

Buthyna N. Shihab
Department of Mathematics,College of Education for Pure Science, Ibn-Al– Haitham/ University of Baghdad, Iraq
dr.buthyan@yahoo.com

Kassim A. Jassim
Department of Mathematics , College of Science, University of Baghdad, Baghdad,Iraq
kasimmmathphd@gmail.com

Article history: Received 18 April 2019, Accepted 26 May 2019, Publish January 2020.

Doi: 10.30526/33.1.2377

Abstract

The main objectives of this pepper are to introduce new classes. We have attempted to obtain coefficient estimates, radius of convexity, Distortion and Growth theorem and other related results for the classes $\mathcal{M}(A,B,\alpha,\delta,p)$ *and* $K\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$.

Keywords: multivalent function ,subordination, starlike function, Growth theorem , Schwarz function.

1. Introduction

Let $A(p)$ be the set of all function $f(w)$ having the form

$$f(w) = w^p - \sum_{c=p+1}^{\infty} a_c\, w^c, \qquad a_c \geq 0 \qquad (1)$$

Where $\in \mathbb{N}$, a set of natural numbers which are p-valent in $\mathcal{U}$ for $p \in \mathbb{N}$

Definition 1 [1]: A function $f(w) \in A(p)$ is in the subclass $\mathcal{H}(\propto)$ of starlike function if $\mathcal{R}\left(\frac{wf'(w)}{f(w)}\right) > \propto$, $z \in \mathcal{U}$, $0 \leq \propto \leq 1$.

Definition 2 [2]: A function $f(w) \in A(p)$ is in the subclass $G(\propto)$ of convex function if $\mathcal{R}\left(1+\frac{wf'(w)}{f(w)}\right) > \propto$, $z \in \mathcal{U}$.

Definition 3 [3]: A function $f(w) \in A(p)$ is in the subclass $\mathcal{M}(A,B,\alpha,\delta,p)$ if it satisfy

$$1+\frac{1}{\alpha}\left\{\frac{\frac{w^2f''(w)}{zf'(w)}+1-p}{\frac{w^2f''(w)}{wf'(w)}+1+p-2\delta}\right\} \prec \frac{1+Aw}{1+Bw} \qquad (2)$$

For $0 < \mathcal{R}e(\alpha), 0 < \delta \leq 1, -1 \leq B < A \leq 1, z \in \mathcal{U}$.

Furthermore a function $f(w) \in A(p)$is in the class $K\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ if $wf'(w) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

Theorem 1: A function given by (4.1) is in $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$.S.S. Miller, P.T. Mocanu [1]. If and only if $\sum_{c=p+1}^{\infty} \frac{1}{k(c)} a_c \leq 1$

Proof: Let $f(w) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$Therefore from (2) we have

$$P(w) = 1 + \frac{1}{\alpha}\left\{\frac{\frac{w^2 f''(w)}{wf'(w)} + (1-p)}{\frac{w^2 f''(w)}{wf'(w)} + (1+p-2\delta)}\right\} \prec \frac{1+Aw}{1+Bw}$$

$$P(w) = \frac{1+Ak(w)}{1+Bk(w)}$$

Where k(w) is Schwarz function

$$P(\mathrm{w})\text{=}(1+Bk(w)) = 1 + Ak(w)$$

$$k(w)(BP(w) - A) = 1 - P(w)$$

$$k(w) = \frac{P(w)-1}{A - BP(w)}$$

$$|k(w)| < 1$$

$$\left|\frac{\frac{1}{\alpha}\left[\frac{\left\{\frac{w^2 f''(w)}{wf'(w)} + (1-p)\right\}}{\frac{w^2 f''(w)}{wf'(w)} + 1 + p - 2\delta}\right]}{A - B\left\{1 + \frac{1}{\alpha}\left[\frac{\left\{\frac{w^2 f''(w)}{wf'(w)} + (1-p)\right\}}{\frac{w^2 f''(w)}{wf'(w)} + (1+p-2\delta)}\right]\right\}}\right| < 1$$

$$\left|\frac{w^2 f''(w) + (1-p)wf'(w)}{\alpha(A-B)\{w^2 f''(w) + (1+p-2\delta)wf'(w)\} - B\{w^2 f''(w) + (1-p)wf'(z)\}}\right| < 1 \qquad (3)$$

$$w^2 f''(w) + (1-p)wf'(w) = -\sum_{c=p+1}^{\infty} c(c-p)a_c w^c$$

$$w^2 f''(w) + (1+p-2\delta)wf'(w) = 2p(p-\delta)w^p - \sum_{c=p+1}^{\infty} c(c+p-2\delta)a_n w^n$$

From (1) we have $\left|\frac{-\sum_{c=p+1}^{\infty} c(c-p)a_c w^c}{\alpha(A-B)\left\{2p(p-\delta)w^p - \sum_{c=p+1}^{\infty} c(c+p-2\delta)a_c w^c\right\} + B\left\{\sum_{c=p+1}^{\infty} c(c-p)a_c w^c\right\}}\right| < 1$

$$\left|\frac{-\sum_{c=p+1}^{\infty} c(c-p)a_c w^c}{2\alpha p(A-B)(p-\delta)w^p - \sum_{c=p+1}^{\infty}\{c(c+p-2\delta) - Bc(c-p)\}a_c w^c}\right| < 1$$

Since $\mathcal{R}e(w) < |w|$. We obtain after considering on real axis and letting w→ 1 we get

$$\sum_{c=p+1}^{\infty} c(c-p)a_c \leq 2|\alpha|p(A-B)(p-\delta) - \sum_{c=p+1}^{\infty} |c(c+p-2\delta) - Bc(c-p)|a_c w^c$$

$$\sum_{c=p+1}^{\infty} c(c-p) + |c\{(c+p-2\delta) - B(c-p)\}| \leq 2|\alpha|p(A-B)(p-\delta)$$

That is $\sum_{c=p+1}^{\infty} \frac{1}{k(c)} a_c \leq 1$ Where

$$k(c) = \frac{2|\alpha|p(A-B)(p-\delta)}{\sum_{c=p+1}^{\infty} c(c-p) + |c\{(c+p-2\delta) - B(c-p)\}|}$$

Corollary 1: If $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$, then $a_c \leq k(c)$

and the equality holds for $f(w) = w^p - k(c)w^c$

Theoremd 2 : $f(w) = w^p - \sum_{c=p+1}^{\infty} a_c w^c$, $a_c \geq 0$ is in $\boldsymbol{k\mathcal{M}}(A,B,\alpha,\delta,p)$ if and only if $\sum_{c=p+1}^{\infty} \frac{c}{k(c)} a_c \leq p$

Proof: Suppose $f(w) \in \boldsymbol{k\mathcal{M}}(A,B,\alpha,\delta,p)$ If $wf'(w) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$
Let $g(w) = wf'(w)$ Therefore from (1) we have

$$P(w) = 1 + \frac{1}{\alpha}\left\{\frac{\frac{w^2 f''(w)}{f'(w)} + (1-p)}{\frac{w^2 f''(w)}{wf'(w)} + (1+p-2\delta)}\right\} \prec \frac{1+Aw}{1+Bw}$$

This is equivalent to (since $|k(w)| < 1$)

$$\left|\frac{\frac{1}{\alpha}\left[\frac{\left\{\frac{w^2 g''(w)}{wg(w)} + (1-p)\right\}}{\frac{w^2 g''(w)}{wg'(w)} + 1 + p - 2\delta}\right]}{A - B\left\{1 + \frac{1}{\alpha}\left[\frac{\left\{\frac{w^2 g''(w)}{wg'(w)} + (1-p)\right\}}{\frac{w^2 g''(w)}{wg(w)} + (1+p-2\delta)}\right]\right\}}\right| < 1$$

$$\left|\frac{w^2 g''(w) + (1-p)wg'(w)}{\alpha(A-B)\{w^2 g''(w) + (1+p-2\delta)wg'(w)\} - B\{w^2 g''(w) + (1-p)zg'(w)\}}\right| < 1$$

$$w^2 g''(w) + (1-p)wg'(w) = -\sum_{c=p+1}^{\infty} c^2(c-p)a_c w^c$$

$$w^2 g''(w) + (1+p-2\delta)wg'(w) = 2p^2(p-\delta)z^p - \sum_{c=p+1}^{\infty} c^2(c+p-2\delta)a_c w^c$$

From (2) we have

$$\left|\frac{-\sum_{c=p+1}^{\infty} c^2(c-p)a_c w^c}{\alpha(A-B)\left\{2p^2(p-\delta)z^p - \sum_{c=p+1}^{\infty} c^2(c+p-2\delta)a_c w^c\right\} + B\left\{-\sum_{c=p+1}^{\infty} c^2(c-p)a_c w^c\right\}}\right| < 1$$

$$= \left|\frac{-\sum_{c=p+1}^{\infty} c^2(c-p)a_c w^c}{(2\alpha p^2(A-B)(p-\delta))w^p - \sum_{c=p+1}^{\infty}(c^2(c+p-2\delta) - Bc^2(c-p))a_c w^c}\right| < 1$$

Since $\mathcal{R}e(w) < |w|$. We obtain after considering on real axis and letting w$\rightarrow$ 1 we get

$$\sum_{c=p+1}^{\infty} c^2(c-p)a_c \leq 2|\alpha|p^2(A-B)(p-\delta) - \sum_{c=p+1}^{\infty} c^2\big((c+p-2\delta) - B(c-p)\big)a_c$$

$$\sum_{c=p+1}^{\infty} \left\{c^2(c-p) + \left|c^2\big((c+p-2\delta) - B(c-p)\big)\right|\right\}a_c \leq 2|\alpha|p^2(A-B)(p-\delta)$$

$$\sum_{c=p+1}^{\infty} \frac{c}{k(c)} a_c \leq p$$

Corollary 2: If $f(w) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ then $a_c \leq \frac{pk(c)}{c}$ and the equality holds for

$$f(w) = w^p - \sum_{c=p+1}^{\infty} \frac{pk(c)}{c} w^c$$

Theorem 3: $f(w) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ then

$$|w|^p - |w|^{p+1}k(p+1) \leq |f(w)| \leq |w|^p + |w|^{p+1}k(p+1)$$

With equality hold for $f(w) = w^p - w^{p+1}k(p+1)$

Proof: $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ Therefore from theorem 2 $\sum_{c=p+1}^{\infty} a_c \leq k(c)$

$$|f(w)| \geq |w|^p - \sum_{c=p+1}^{\infty} |a_n||w|^c \geq |w|^p - |w|^{p+1} \sum_{c=p+1}^{\infty} |a_c| \geq |w|^p - |w|^{p+1}k(p+1)$$

Similarly

$$|f(w)| \leq |w|^p + \sum_{c=p+1}^{\infty} |a_c||w|^c \leq |w|^p + |w|^{p+1} \sum_{c=p+1}^{\infty} |a_c| \leq |w|^p + |w|^{p+1}k(p+1)$$

Therefore

$$|w|^p - |w|^{p+1}k(p+1) \leq |f(w)| \leq |w|^p + |w|^{p+1}k(p+1)$$

Theorem 4 : $f(w) \in \boldsymbol{k\mathcal{M}}(A, B, \alpha, \delta, p)$ then

$$|w|^p - |w|^{p+1}\frac{pk(p+1)}{(p+1)} \leq |f(w)| \leq |w|^p + |w|^{p+1}\frac{pk(p+1)}{(p+1)}$$

With equality $f(w)=w^{p}-w^{p+1}\frac{pk(p+1)}{(p+1)}$

Proof: $f(w)\in \boldsymbol{k\mathcal{M}}(A,B,\alpha,\delta,p)$

Therefore from theorem 2 $\sum_{c=p+1}^{\infty}\frac{c}{k(c)}a_c\leq p$

$$|f(w)|\geq |w|^{p}-\sum_{c=p+1}^{\infty}|a_c||w|^{c}\geq |w|^{p}-|w|^{p+1}\sum_{c=p+1}^{\infty}|a_c|\geq |w|^{p}-|w|^{p+1}\frac{pk(p+1)}{(p+1)}$$

Similarly

$$|f(w)|\leq |w|^{p}-\sum_{c=p+1}^{\infty}|a_c||w|^{c}\leq |w|^{p}+|w|^{p+1}\sum_{c=p+1}^{\infty}|a_c|\leq |w|^{p}+|w|^{p+1}\frac{pk(p+1)}{(p+1)}$$

Therefore

$$|w|^{p}-|w|^{p+1}\frac{pk(p+1)}{(p+1)}\leq |f(w)|\leq |w|^{p}+|w|^{p+1}\frac{pk(p+1)}{(p+1)}$$

Theorem 5: $f(w)\in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then

$$p|w|^{p-1}-(p+1)|w|^{p}k(p+1)\leq |f'(w)|\leq p|w|^{p-1}+(p+1)|w|^{p}k(p+1)$$

Proof: $f(w)\in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ Therefore from theorem 1 $\sum_{c=p+1}^{\infty}a_c\leq k(c)$

$$f'(w)=pw^{p-1}-\sum_{c=p+1}^{\infty}c\,a_c z^{c-1}$$

$$|f'(w)|\geq p|w|^{p-1}-\sum_{c=p+1}^{\infty}c\,|a_c||w|^{c-1}$$

$$\geq p|w|^{p-1}-(p+1)|w|^{p}\sum_{c=p+1}^{\infty}|a_c|\geq p|w|^{p-1}-(p+1)|w|^{p}k(p+1)$$

Similarly

$$|f'(w)|\leq p|w|^{p-1}+\sum_{c=p+1}^{\infty}c\,|a_c||w|^{c-1}$$

$$\leq p|w|^{p-1}+(p+1)|w|^{p}\sum_{c=p+1}^{\infty}|a_c|\leq p|w|^{p-1}+(p+1)|w|^{p}k(p+1)$$

Therefore

$$p|w|^{p-1}-(p+1)|w|^{p}k(p+1)\leq |f'(w)|\leq p|w|^{p-1}+(p+1)|w|^{p}k(p+1)$$

Theorem 6: $f(w)\in \boldsymbol{K\mathcal{M}}(A,B,\alpha,\delta,p)$ then

$$p|w|^{p-1}-|w|^{p}pk(p+1)\leq |f'(w)|\leq p|w|^{p-1}+|w|^{p}pk(p+1)$$

Proof: $f(z)\in \boldsymbol{K\mathcal{M}}(A,B,\alpha,\delta,p)$ then

Therefore from theorem 2 $\sum_{c=p+1}^{\infty}\frac{c}{k(c)}a_c\leq p$

$$f'(w) = pw^{p-1} + \sum_{c=p+1}^{\infty} ca_c w^{c-1}$$

$$|f'(w)| \geq p|w|^{p-1} - \sum_{c=p+1}^{\infty} c|a_c||w|^{c-1} \geq p|w|^{p-1} - (p+1)|w|^p \sum_{c=p+1}^{\infty} |a_c| \geq p|w|^p - |w|^{p+1}pk(p+1)$$

Similarly

$$|f'(w)| \leq p|w|^{p-1} + \sum_{c=p+1}^{\infty} c|a_c||w|^{c-1} \leq p|w|^{p-1} + (p+1)|w|^p \sum_{c=p+1}^{\infty} |a_c| \leq p|w|^p + |w|^{p+1}pk(p+1)$$

Therefore

$$p|w|^{p-1} - |w|^p pk(p+1) \leq |f'(w)| \leq p|w|^p + |w|^{p+1}pk(p+1)$$

$f(w)$ is function in $A(p)$ is called close to convex of order $\propto$ $(0 \leq\propto< 1)$ if $\mathcal{R}e(w)\{f'(w)\} > \propto$ for all $z \in \mathcal{U}$.

A function $f(w) \in A(p)$ is starlike of order $\propto$ $(0 \leq\propto< 1)$ if $\mathcal{R}e\left\{\frac{wf'(w)}{f(w)}\right\} >\propto$ for all $w \in \mathcal{U}$.

A function $f(w) \in A(p)$ is convex of order $\propto$ $(0 \leq\propto< 1)$ if $wf'(w)$ is starlike of order $\propto$, that is $\mathcal{R}e\left\{\frac{wf'(w)}{f(w)}\right\} >\propto$ for all $w \in \mathcal{U}$.

Theorem 7 : IF $f(w) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$,then $f \in K(\propto)$ if

$$|w| \leq r_1(A, B, \alpha, \delta, p) = \inf_n \left(\frac{p-\propto}{nk(n)}\right)^{\frac{1}{n-p}}$$

Proof: We need to show that $\left|\frac{wf'(w)}{w^{p-1}} - p\right| < p-\propto$ That is

$$\left|\frac{wf'(w)}{w^{p-1}} - p\right| \leq \sum_{n=p+1}^{\infty} n|a_n||w|^{n-p} < p-\propto \text{ ……(4)}$$

From theorem 1 we have $\sum_{c=p+1}^{\infty} \frac{1}{k(c)} a_c \leq 1$ Note that (4) is true if $\frac{c|w|^{c-p}}{p-\propto} \leq \frac{1}{k(c)}$

Therefore $|w| \leq \left(\frac{p-\propto}{nk(c)}\right)^{\frac{1}{c-p}}$ $(p \neq c\ , p, c \in \mathbb{N})$, thus we get required result.

Theorem 8 : IF $f(w) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$,then $f \in \boldsymbol{S}^*(\propto)$ if

$$|w| \leq r_2(A, B, \alpha, \delta, p) = \inf_c \left(\left(\left(\frac{p-\propto}{c-\propto}\right)\frac{1}{k(c)}\right)^{\frac{1}{c-p}}\right)$$

Proof: We must show that

$$\left|\frac{wf'(w)}{f(w)} - p\right| < p-\propto$$

We have

$$\left|\frac{wf'(w)}{f(w)} - p\right| \leq \frac{\sum_{c=p+1}^{\infty}(c-p)|a_c||w|^{c-p}}{1-\sum_{c=p+1}^{\infty}|a_c||w|^{c-p}} < p-\propto \qquad (5)$$

Hence (4.4.3) holds true if

$$\sum_{c=p+1}^{\infty} \frac{(c-\propto)}{(p-\propto)} |a_c||w|^{c-p} \leq 1 \qquad (6)$$

From theorem 1 we have

$$\sum_{c=p+1}^{\infty} \frac{1}{k(c)} a_c \leq 1 \tag{7}$$

Hence by using (6) and (7) we can obtain required result.

Theorem 9 : IF $f(w) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$,then $f \in \boldsymbol{C}(\propto)$ if

$$|z| \leq r_3(A,B,\alpha,\delta,p) = \inf_c \left(\left(\left(\frac{p(p-\propto)}{c(c-\propto)} \right) \frac{1}{k(c)} \right)^{\frac{1}{c-p}} \right)$$

Proof: We know that f is convex if and only if wf' is starlike We must show that

$$\left| \frac{wg'(w)}{g(w)} - p \right| p-\propto$$

Where $g(w) = wf'(w)$ Therefore we have

$$\sum_{c=p+1}^{\infty} \frac{c(c-\propto)}{c(c-\propto)} |a_c||w|^{c-p} \leq 1 \tag{8}$$

From theorem 1 we have

$$\sum_{c=p+1}^{\infty} \frac{1}{k(c)} a_c \leq 1 \tag{9}$$

Hence by using (8) and (9) we get

$$\left(\frac{p(p-\propto)}{c(c-\propto)} \right) |w|^{c-p} \leq \frac{1}{k(c)}$$

$$|z| \leq \left(\left(\left(\frac{p(p-\propto)}{c(c-\propto)} \right) \frac{1}{k(c)} \right)^{\frac{1}{c-p}} \right)$$

Theorem 10: Let $f_1(w) = w^n$ and $f_n(w) = w^p - k(n)w^p, for\ n \geq p+1$ then $f(w) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ if and only if $f(w)$ can be express in the form $f(w) = \lambda_1 f_1(w) + \sum_{c=p+1}^{\infty} \lambda_c f_c(w)$ where $\lambda_n \geq 0\ and\ \lambda_1 + \sum_{c=p+1}^{\infty} \lambda_c = 1$

Proof: Let $f(w) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ We have $a_c \leq k(c)$ If we take $\lambda_c = \frac{1}{k(c)} a_c$
$n \geq p+1$ and $\sum_{c=p+1}^{\infty} \lambda_c = 1 - \lambda_1$

Theorem 11: Let $f_i(w) = w^p - \sum_{c=p+1}^{\infty} a_{c,i} w^c, a_{c,i} \geq 0$ $(i = 1,2,3, \ldots .m)$ be the functions in the class $\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$, $(i = 1,2,3, \ldots .m)$then the function:
$G(w) = w^p - \frac{1}{m} \sum_{c=p+1}^{\infty} \sum_{i=1}^{m} a_{c,i} w^c$ is also in $\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ where $\delta = \min_{1 \leq i \leq m} \{\delta_i\}$ with $0 \leq \delta_i < 1$

Proof: since $f_i(w) = w^p - \sum_{c=p+1}^{\infty} a_{c,i} w^c,\ a_{c,i} \geq 0$ is in $\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$
So by theorem 2 we have $\sum_{c=p+1}^{\infty} a_c \leq k(c,\delta)$

$$k(c,\delta) = \frac{2|\alpha|p(A-B)(p-\delta)}{\{c(c-p) + |(c(c+p-2\delta) - Bc(c-p))|\}}$$

We have $\sum_{c=p+1}^{\infty} \frac{1}{k(c,\delta_i)} \left(\frac{1}{m} \sum_{i=1}^{m} a_{c,i} \right) = \frac{1}{m} \sum_{i=1}^{m} \sum_{c=p+1}^{\infty} \frac{1}{k(c,\delta_i)} a_{c,i} \leq \left(\frac{1}{m} \sum_{i=1}^{m} 1 \right) < 1$

Hence by theorem 1 , $G(w) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$

Theorem 12: Let the function $f(w) = w^p - \sum_{n=p+1}^{\infty} a_n w^n and \; g(w) = w^p - \sum_{n=p+1}^{\infty} b_n w^n$ be in the class (A,B,α,δ,p) . Then the function $F(w)$ defined by

$$F(w) = (1-y)f(w) + yg(w) = w^p - \sum_{c=p+1}^{\infty} c_c w^c$$

Where $c_c = (1-y)a_c + yb_c, 0 \leq y \leq 1$ is also in $\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$.
Proff: we have $F(w) = (1-y)f(w) + yg(w)$

$$= (1-y)\left(w^p - \sum_{c=p+1}^{\infty} a_c w^c\right) + y\left(w^p - \sum_{c=p+1}^{\infty} b_c w^c\right)$$

$$= w^p - \sum_{c=p+1}^{\infty} \big((1-y)a_c + yb_c\big)\, w^c$$

Since $f, g \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ so by theorem 1 we have

$$\sum_{c=p+1}^{\infty} \frac{1}{k(c)} a_c \leq 1 \text{ and } \sum_{c=p+1}^{\infty} \frac{1}{k(c)} b_c \leq 1$$

Therefore

$$\sum_{c=p+1}^{\infty} \frac{1}{k(c)} \big((1-y)a_c + yb_c\big) = (1-y) \sum_{c=p+1}^{\infty} \frac{1}{k(x)} a_c + y \sum_{n=p+1}^{\infty} \frac{1}{k(x)} b_c$$

$$\leq (1-y) \sum_{c=p+1}^{\infty} \frac{1}{k(x)} + y \sum_{c=p+1}^{\infty} \frac{1}{k(x)} = 1$$

Therefore

$$\boldsymbol{c_c} \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$$

Let $\in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$, $\tau \geq 0$ *then* $a(t,\tau)$ −neighborhood of the function $f \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ is defined by

$\aleph_\tau^t(w) = \left\{g \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p) : g(w) = w^p - \sum_{c=p+1}^{\infty} b_n w^n \; and \; \sum_{c=p+1}^{\infty} |a_c - b_c| c^{t+1} \leq \tau\right\}$ ….(4.6.1)

For the identity function if $e(w) = w^c, q \in \mathbb{N}$, then

$$\aleph_\tau^t(e) = \left\{g \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p) : g(w) = w^p - \sum_{c=p+1}^{\infty} b_c w^c \; and \; \sum_{c=p+1}^{\infty} |b_c| c^{t+1} \leq \tau\right\} \qquad (10)$$

Definition 4: A function $f(w) = w^p - \sum_{c=p+1}^{\infty} a_c w^c, a_c \geq 0$ is in the class $\boldsymbol{\mathcal{M}}^{\boldsymbol{\pi}}(A,B,\alpha,\delta,p)$ if there exist $g(w) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ such that

$$\left|\frac{f(w)}{g(w)} - 1\right| < p - \pi \, , z \in \mathcal{U}, 0 \leq \pi < 1 \qquad (11)$$

Theorem 13 : If $g(w) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ $\pi = p - \frac{\tau}{c^{t+1}}\left[\frac{1}{1-k(c)}\right]$ (12)

Then $\aleph_\tau^t(g) \subset \boldsymbol{\mathcal{M}}^{\boldsymbol{\pi}}(A,B,\alpha,\delta,p)$

PROOF: Let $\in \aleph_\tau^t(g)$, then by (4.6) $\sum_{c=p+1}^{\infty} n^{c+1}|a_c - b_c| \le \tau$

This implies that $\sum_{c=p+1}^{\infty}|a_c - b_c| \le \frac{\tau}{c^{t+1}}$ (13)

Therefore

$$\left|\frac{f(w)}{g(w)} - 1\right| \le \frac{\sum_{c=p+1}^{\infty}|a_c - b_c|}{1 - \sum_{c=p+1}^{\infty} b_c} \le \frac{\tau}{c^{t+1}}\left[\frac{1}{1-k(c)}\right] < \frac{\tau}{c^{t+1}}\left[\frac{1}{1-k(c)}\right] = p - \pi$$

Then by definition 13 , we get $f \in \boldsymbol{\mathcal{M}}^{\pi}(A,B,\alpha,\delta,p)$ Thus $\aleph_\tau^t(g) \subset \boldsymbol{\mathcal{M}}^{\boldsymbol{\pi}}(A,B,\alpha,\delta,p)$.

The generalized Bernardi integral operator is given by

$$\mathcal{L}_c[f(w)] = \frac{c+p}{w^c}\int_0^z f(\zeta)\zeta^{c-1}d\zeta \qquad (\, c > -p, w \in \mathcal{U})$$

$$\mathcal{L}_c[f(w)] = z^p - \sum_{c=p+1}^{\infty} d a_c w^c \tag{14}$$

Where $d = \left(\frac{c+p}{c+c}\right)$

Theorem 14: Let $f \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then $\mathcal{L}_c[f(w)] \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$, S.S. Miller, P.T. Mocanu [4, 5].

Proof : We need to prove that $\sum_{c=p+1}^{\infty}\frac{d}{k(c)}a_c \le 1$

Since $f \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then from theorem 1 $\sum_{c=p+1}^{\infty} k(c)a_c \le 1$ But $d < 1$ therefore theorem 14 holds and the proof is over.

Theorem 15: Let $f \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then $\mathcal{L}_c[f(w)]$ is starlice of order $\sigma, 0 \le \sigma < 1$ in $|w| < r_1$ Where $|z| \le \left(\left(\frac{p-\sigma}{c-\sigma}\right)\left(\frac{1}{dk(c)}\right)\right)^{\frac{1}{c-p}}$

Proof: $\mathcal{L}_c[f(w)] = w^p - \sum_{c=p+1}^{\infty} d a_c w^c$

It is enough to prove

$$\left|\frac{w(\mathcal{L}_c[f(w)])'}{\mathcal{L}_c[f(w)]} - p\right| < p - \sigma$$

$$\left|\frac{w(\mathcal{L}_c[f(w)])'}{\mathcal{L}_c[f(w)]} - p\right| = \left|\frac{\sum_{c=p+1}^{\infty} d(c-p)a_c w^{c-p}}{1 - \sum_{c=p+1}^{\infty} d a_c w^{c-p}}\right|$$

$$\le \frac{\sum_{c=p+1}^{\infty} d(c-p)a_c|w|^{c-p}}{1 - \sum_{c=p+1}^{\infty} d a_c|w|^{c-p}} < p - \sigma$$

$$\sum_{c=p+1}^{\infty} d(c-p)a_c|w|^{c-p} < p - \sigma\left(1 - \sum_{c=p+1}^{\infty} d a_c|w|^{c-p}\right)$$

$$\sum_{c=p+1}^{\infty}\frac{(n-\sigma)}{(p-\sigma)} d a_c|w|^{c-p} \le 1 \tag{15}$$

From theorem 1 $\sum_{c=p+1}^{\infty}\frac{1}{k(c)}a_c \leq 1$ (16)

Hence by using (15) and (16) we get

$$\frac{(c-\sigma)}{(p-\sigma)}d|w|^{c-p} \leq \frac{1}{k(c)}$$

$$|w|^{c-p} \leq \left(\frac{p-\sigma}{c-\sigma}\right)\left(\frac{1}{dk(c)}\right)$$

Therefore

$$|w| \leq \left(\left(\frac{p-\sigma}{c-\sigma}\right)\left(\frac{1}{dk(c)}\right)\right)^{\frac{1}{c-p}}$$

The definitions given below are of the fractional calculus studied by , S. Ruscheweyh [4].

Definition 5 [6]: For a function f(w) which is analytic function in w − plane containing the origin which is a simply connected region , we define the fractional integral of order μ as

$$D_w^{-\mu}f(w) = \frac{1}{\Gamma(\mu)}\int_0^w \frac{f(\xi)}{(w-\xi)^{1-\mu}}d\xi \ where\ \mu > 0$$

Definition 6: For a function $f(w)$ which is analytic function in $w-$ plane containing the origin which is a simply connected region , we define the fractional integral of order μ as

$$D_z^{\mu}f(w) = \frac{1}{\Gamma(\mu)}\frac{d}{dz}\int_0^w \frac{f(\xi)}{(w-\xi)^{\mu}}d\xi \ where\ \ 1 > \mu \geq 0$$

Theorem 16: Let $f \in \mathcal{M}(A,B,\alpha,\delta,p)$ then

$$\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|w|^{p+\mu}\left(1-\frac{(p+1)k(n)}{(p+\mu+1)}|w|\right) \leq \left|D_w^{-\mu}f(w)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|w|^{p+\mu}\left(1+\frac{(p+1)k(n)}{(p+\mu+1)}|w|\right) \quad (17)$$

Proof: From definition 5 we have

$$D_w^{-\mu}f(w) = \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}w^{p+\mu} - \sum_{c=p+1}^{\infty}\frac{\Gamma(c+1)}{\Gamma(c+\mu+1)}a_c w^{c+\mu} \quad (18)$$

$$\mu > 0\,, c \geq p+1\,; p,c \in \mathbb{N}$$

Let $\phi(n) = \frac{\Gamma(c+1)}{\Gamma(c+\mu+1)}$

Clearly $\phi(n)$ is non – increasing function of n , $0 < \phi(n) \leq \phi(p+1) = \frac{\Gamma(p+2)}{\Gamma(p+\mu+2)}$

From theorem 1 we have

$$\sum_{c=p+1}^{\infty}|a_c| \leq k(c) \quad (19)$$

From (18) and (19) it follows that

$$\left|D_z^{-\mu} f(w)\right| \le |w|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} + \phi(p+1)|w| \sum_{c=p+1}^{\infty} |a_c|\right)$$

$$\le \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|w|^{p+\mu}\left(1+\frac{(p+1)k(n)}{(p+\mu+1)}|w|\right)$$

Similarly

$$\left|D_w^{-\mu} f(w)\right| \ge |w|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - \phi(p+1)|w| \sum_{c=p+1}^{\infty} |a_c|\right)$$

$$\ge \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|w|^{p+\mu}\left(1-\frac{(p+1)k(n)}{(p+\mu+1)}|w|\right)$$

This proves the theorem

Theorem 17: Let $f \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then

$$\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|w|^{p-\mu}\left(1-\frac{(p+1)k(n)}{(p-\mu+1)}|w|\right) \le \left|D_w^{\mu} f(w)\right| \le \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|w|^{p-\mu}\left(1+\frac{(p+1)k(n)}{(p-\mu+1)}|w|\right) \qquad (20)$$

Proof: From definition 6 we have

$$D_w^{\mu} f(w) = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} w^{p-\mu} - \sum_{c=p+1}^{\infty} \frac{\Gamma(c+1)}{\Gamma(c-\mu+1)} a_c w^{c-\mu} \qquad (21)$$

$$1 > \mu \ge 0\,, c \ge p+1\,; p, c \in \mathbb{N}$$

Let $\psi(n) = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}$

Clearly $\psi(n)$ is non – increasing function of n , $0 < \psi(n) \le \psi(p+1) = \frac{\Gamma(p+2)}{\Gamma(p+\mu+2)}$

From theorem 1 we have $\sum_{c=p+1}^{\infty}|a_c| \le k(c)$……(22)

From (21) and (22) it follows that $\left|D_w^{\mu} f(w)\right| \le |w|^{p-\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \psi(p+1)|w|\sum_{c=p+1}^{\infty}|a_c|\right)$

$$\le \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|w|^{p-\mu}\left(1+\frac{(p+1)k(n)}{(p-\mu+1)}|w|\right)$$

Similarly $\left|D_w^{\mu} f(w)\right| \ge |z|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \psi(p+1)|w|\sum_{c=p+1}^{\infty}|a_c|\right)$

$$\ge \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|w|^{p-\mu}\left(1-\frac{(p+1)k(n)}{(p-\mu+1)}|w|\right)$$

Conclusions

The main impact of this research work is to motivate to construct new Subclasses of Holomorphic (or analytic) multivalent functions belonging the disk and study their various geometrical properties. We have derived new Sub classes of Meromorphic (analytic except for isolated singularities i. e. poles) Holomorphic (an analytic) multivalent functions in the punctured disk. The well-known properties like distortion theorem, radii of star likeness, coefficient inequalities and convexity etc. by using Subordination.

References

1. Miller, S.S.; Mocanu, P.T. Differential subordinations and univalent functions. *Michigan Math. J.***1981**, *28*, 157–171.
2. Miller, S.S.; Mocanu, P.T. Differential Subordination: Theory and Applications. *Monographs and Textbooks in Pure and Applied Mathematics*.**2000**, *225*, *31-39.*
3. Ruscheweyh, S. New criteria for univalent functions. *Proc. Amer. Math. Soc.***1975**, *49*, 109–115.
4. Saitoh, H. A linear operator and its applications of first order differential subordinations. *Math. Japon.***1996**, *44*, 31–38.
5. Piejko, K.; Sokol, J. On the Dziok-Srivastava operator under multivalent analytic functions. *Appl. Math. and Compution*.**2006**, *177*, 839-843.
6. Jabber, A.K.; Tawfiq, L.N.M. New Transform Fundamental Properties and its Applications. *Ibn Alhaitham Journal for Pure and Applied Science*.**2018**, *31*, *1*, 151-163.