Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

Pseudo Primary-2-Absorbing Submodules and Some Related Concepts

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Article history: Received 3 March 2019, Accepted 20 March 2019, Publish September 2019

Doi:10.30526/32.3.2290

Abstract

Let R be a commutative ring with identity. The aim of this paper is introduce the notion of a pseudo primary-2-absorbing submodule as generalization of 2-absorbing submodule and a pseudo-2-absorbing submodules. A proper submodule K of an R-module W is called pseudo primary-2-absorbing if whenever $rsx \in K$, for $r, s \in R$, $x \in W$, implies that either $rx \in W - rad(K) + soc(W)$ or $sx \in W - rad(K) + soc(W)$ or $rsW \subseteq K + soc(W)$. Many basic properties, examples and characterizations of these concepts are given. Furthermore, characterizations of pseudo primary-2-absorbing submodules in some classes of modules are introduced. Moreover, the behavior of a pseudo primary-2-absorbing submodule under R-homomorphism is studied.

Keywords: Primary submodules, pseudo-2-absorbing submodules, pseudo primary-2-absorbing submodules, multiplication modules, non-singular modules, socle of a modules.

1. Introduction and Basic Concepts

Throughout this paper, we assume that all rings are commutative with identity and all R-modules are left unitary. Among the famous concepts of modules theory is prime submodules, where a proper submodule K of an R-module W is said to be a prime submodule if whenever $rx \in K$ where $r \in R$, $x \in W$, implies that either $x \in W$ or $rW \subseteq K$[1]. Primary submodule was introduced in [2]. as a generalization of a prime submodule, where a proper submodule K of an R-module W is called a primary submodule if whenever $rx \in K$ for $r \in R$, $x \in W$, implies that either $x \in K$ or $r^n W \subseteq K$ for some $n \in Z^+$. Recently many generalizations of prime submodules were introduced such as (app-prime, φ-prime, Nearly-prime) submodules see[$3-5$]. Darani and Soheilinia in[6]. introduced the concept of 2-absorbing submodule as a generalization of prime submodule, where a proper submodule K of an R-module W is said to be 2-absorbing submodule if whenever $rsx \in K$ for $r, s \in R$, $x \in W$, implies that either $rx \in K$ or $sx \in K$ or $rsW \subseteq K$. In recent decades several generalization of 2-absorbing submodules were introduced such as nearly 2-absorbing submodule, nearly quasi-2-absorbing submodule, pseudo-2-absorbing submodule and pseudo quasi-2-absorbing submodule see$[6-9]$. Badwi et, in $[10]$. introduced the concept of 2-absorbing primary ideal, where a proper ideal I of a ring R is called 2-absorbing primary, if whenever $abc \in I$ for $a,b,c \in R$, implies that either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$ where $\sqrt{I} = \{r \in R: r^n \in I, \text{for some n} \in Z^+\}$. This led us to introduce the concept of a pseudo primary-2-absorbing submodule, which is generalization of 2-absorbing submodule and pseudo- 2-absorbing submodule. Many basic properties, characterization and examples of this concept are given. The residual of submodule K is denoted by $[K:W]$ is an ideal of R defined by $\{r \in R: rW \subseteq K\}[1]$. The radical of a submodule K of W denoted by $W-rad(K)$ or $rad_W(K)$ is defined to be the intersection of all prime submodule of W containing K, if W has no prime submodules containing K, then we say $W-rad(K) = W$ and $W \subseteq W-rad(K)[2]$. Socle of a module W defined by the intersection of all essential submodules of W, denoted by $soc(W)[11]$. Recall that an R-module W is multiplication, if every submodule L of W is of the form $L = IW$ for some ideal I of a ring [12]. Recall that an R-module W is called faithful if $ann(W) = (0)$. Recall that an R-module W is called non-singular if $Z(W) = W$ where $(W) = \{y \in W: yI = (0), \text{ for some essential ideal I of R}\}$ [11].

2. Pseudo Primary-2-Absorbing Submodules

In this section we define the concept of a pseudo primary-2-absorbing submodule and give some basic results of these types of submodules and discuss on the relationships with class of 2-absorbing submodules and pseudo-2-absorbing submodules.

Definition (1)

A proper submodule K of an R-module W is said to be a pseudo primary-2-absorbing submodule of W, if whenever $rsx \in K$, for $r,s \in R, x \in W$, implies that either $rx \in W-rad(K)+soc(W)$ or $sx \in W-rad(K)+soc(W)$ or $rs \in [K+soc(W):_R W]$. And a proper ideal I of a ring R is called a pseudo primary-2-absorbing ideal of R, if I is pseudo primary-2-absorbing submodules of an R-module R.

Remarks and Examples (2)

1. It is clear that every 2-absorbing submodule of an R-module W is a pseudo primary-2-absorbing submodule, while the converse is not true in general, the following example shows that: Let $W = Z_{12}$, $R = Z$ and $K = \langle \bar{0} \rangle$. K is not 2-absorbing submodule since $2.3.\bar{2} \in K$ where $2,3 \in Z$, $\bar{2} \in Z_{12}$, then $2.\bar{2} = \bar{4} \notin K$ and $3.\bar{2} = \bar{6} \notin K$ and $2.3 = 6 \notin [K:Z_{12}] = 12Z$. But K is a pseudo primary-2-absorbing submodule of Z_{12}, since $soc(Z_{12}) = \langle \bar{2} \rangle$ and $W-rad(K) = \langle \bar{6} \rangle$ for all $r,s \in Z$, $x \in Z_{12}$ with $rsx \in \langle \bar{0} \rangle$, implies that either $rx \in \langle \bar{6} \rangle + soc(Z_{12}) = \langle \bar{2} \rangle$ or $sx \in \langle \bar{6} \rangle + soc(Z_{12}) = \langle \bar{2} \rangle$ or $rs \in [\langle \bar{0} \rangle + soc(Z_{12}): Z_{12}] = [\langle \bar{2} \rangle: Z_{12}] = 2Z$. That is $2.3.\bar{2} \in K$, implies that $2.\bar{2} = \bar{4} \in \langle \bar{6} \rangle + \langle \bar{2} \rangle = \langle \bar{2} \rangle$ or $3.\bar{2} = \bar{6} \in \langle \bar{6} \rangle + \langle \bar{2} \rangle = \langle \bar{2} \rangle$ or $2.3 = 6 \in [\langle \bar{0} \rangle + \langle \bar{2} \rangle: Z_{12}] = 2Z$.
2. It is clear that every pseudo-2-absorbing submodule of an R-module W is a pseudo primary-2-absorbing submodule, while the converse is not true in general, the following example shows that: Let $W = Z$, $R = Z$ and $K = 8Z$ where K be a submodule of W. K is not pseudo-2-absorbing submodule since $2.2.2 \in 8Z$ but $2.2 \notin 8Z + soc(Z) = 8Z + (0) = 8Z$ and $2.2 = 4 \notin [8Z + soc(Z): Z] = 8Z$. But K is a pseudo primary-2-absorbing submodule of W since $2.2.2 \in 8Z$, then $2.2 = 4 \in W - rad(8Z) + soc(Z) = 2Z + (0) = 2Z$. That is for all $r,s \in R$, $x \in W$ with $rsx \in K$, implies that either $rx \in W-rad(K)+soc(W) = 2Z$ or $sx \in 2Z$ or $rs \in [8Z:_Z Z] = 8Z$.

3. It is clear that every primary submodule of an R-module W is a pseudo primary-2-absorbing submodule, while the converse is not true in general, the following example shows that: Let $W = Z_{12}$, $R = Z$ and $K = \langle\bar{0}\rangle$ is a submodule of W. K is a pseudo primary-2-absorbing submodule of W but not primary submodule, since $3 \in Z$, $\bar{4} \in Z_{12}$ such that $3.\bar{4} \in K$, but $\bar{4} \notin K = \langle\bar{0}\rangle$ and $3 \notin \sqrt{[\langle\bar{0}\rangle : Z_{12}]} = \sqrt{12Z} = 6Z$.
4. It is clear that every prime submodule of an R-module W is a pseudo primary-2-absorbing submodule, while the converse is not true in general, the following example shows that: Let $W = Z$, $R = Z$ and $K = 6Z$. K is not prime submodule of W, since $2,3Z$ with $2.3 \in K$, but $3 \notin K$ and $2 \notin [K :_Z Z] = 6Z$. But K is a pseudo primary-2-absorbing submodule of W, since $2,3,1 \in Z$ with $2.3.1 \in K$, implies that $2.3 \in [K + soc(W) : W] = 6Z$ because $soc(W) = (0)$. $2.1 \notin W - rad(K) + soc(W) = 6Z$ and $3.1 \notin W - rad(K) + soc(W) = 6Z$. That is for all $r, s \in R$, $x \in W$, with $rsx \in K$, implies that either $rx \in W - rad(K) + soc(W)$ or $sx \in W - rad(K) + soc(W)$ or $rs \in [K + soc(W) :_R W]$.

The following results are characterizations of pseudo primary-2-absorbing submodules.

Proposition (3)

Let W be an R-module and K is a proper submodule of W. Then K is a pseudo primary-2-absorbing submodule of W if and only if for each $r, s \in R$ with $rs \notin [K + soc(W) :_R W]$, $[K :_W rs] \subseteq [W - rad(K) + soc(W) :_W r] \cup [W - rad(K) + soc(W) :_W s]$.

Proof:

($\Longrightarrow$) Let $x \in [K :_W rs]$, where $r, s \in R$ and $rs \notin [K + soc(W) :_R W]$, implies that $rsx \in K$. But K is a pseudo primary-2-absorbing submodule of W, and $rs \notin [K + soc(W) :_R W]$, then $rx \in W - rad(K) + soc(W)$ or $sx \in W - rad(K) + soc(W)$. That is either $x \in [W - rad(K) + soc(W) :_W r]$ or $x \in [W - rad(K) + soc(W) :_W s]$, thus $x \in [W - rad(K) + soc(W) :_W r] \cup [W - rad(K) + soc(W) :_W s]$. Hence $[K :_W rs] \subseteq [W - rad(K) :_W r] \cup [W - rad(K) :_W s]$.

($\Longleftarrow$) Let $rsx \in K$, where $x \in W$ and $r, s \in R$ with $rs \notin [K + soc(W) :_R W]$. It follows that $x \in [K :_W rs]$, by hypothesis $x \in [W - rad(K) + soc(W) :_W r] \cup [W - rad(K) + soc(W) :_W s]$. Hence $x \in [W - rad(K) + soc(W) :_W r]$ or $x \in [W - rad(K) + soc(W) :_W s]$. Therefore $rx \in W - rad(K) + soc(W)$ or $sx \in W - rad(K) + soc(W)$, that is K is a pseudo primary-2-absorbing submodule of W.

Proposition (4)

Let W be an R-module and L be a proper submodule of W. Then L is a pseudo primary-2-absorbing submodule of W if and only if $rsK \subseteq L$ for $r, s \in R$ and K is a submodule of W, with $rs \notin [L + soc(W) :_R W]$, implies that $rK \subseteq W - rad(L) + soc(W)$ or $sK \subseteq W - rad(L) + soc(W)$.

Proof

($\Longrightarrow$) Let L be a pseudo primary-2-absorbing submodule of W, and $rsK \subseteq L$, with r, $s \in R$ and K is a submodule of W with $rs \notin [L + soc(W) :_R W]$. Assume that $rK \nsubseteq W - rad(L) + soc(W)$ and $sK \nsubseteq W - rad(L) + soc(W)$, then $rk_1 \notin W - rad(L) + soc(W)$ and $sk_2 \notin W - rad(L) + soc(W)$ for some $k_1, k_2 \in K$. Now we have $rsk_1 \in L$ and since L is a pseudo primary-2-absorbing submodule of W and $rs \notin [L + soc(W) :_R W]$ and $rk_1 \notin W - rad(L) + soc(W)$, then $sk_1 \in W - rad(L) + soc(W)$. Also, since $rsk_2 \in L$ and

$rs \notin [L + soc(W) :_R W]$ and $sk_2 \notin W - rad(L) + soc(W)$, then $rk_2 \in W - rad(L) + soc(W)$. Again since $rs(k_1 + k_2) \in L$ and $rs \notin [L + soc(W) :_R W]$ we have $r(k_1 + k_2) \in W - rad(L) + soc(W)$ or $s(k_1 + k_2) \in W - rad(L) + soc(W)$. Suppose that $r(k_1 + k_2) = rk_1 + rk_2 \in W - rad(L) + soc(W)$, but $rk_2 \in W - rad(L) + soc(W)$, it follows that $rk_1 \in W - rad(L) + soc(W)$ a contradiction. Suppose that $s(k_1 + k_2) = sk_1 + sk_2 \in W - rad(L) + soc(W)$, but $sk_1 \in W - rad(L) + soc(W)$, we have $sk_2 \in W - rad(L) + soc(W)$ a contradiction. Hence $rK \subseteq W - rad(L) + soc(W)$ or $sK \subseteq W - rad(L) + soc(W)$.

($\Leftarrow$) Let $rsx \in L$, where $x \in W$ and $r, s \in R$ with $rs \notin [L + soc(W) :_R W]$. So $rs(x) \subseteq L$, it follows by hypothesis $r(x) \subseteq W - rad(L) + soc(W)$ or $s(x) \subseteq W - rad(L) + soc(W)$. That is $rx \in W - rad(L) + soc(W)$ or $sx \in W - rad(L) + soc(W)$. Hence L is a pseudo primary-2-absorbing submodule of W.

Proposition (5)

Let W be an R-module and L is a proper submodule of W. Then L is a pseudo primary-2-absorbing submodule of W if and only if $IJK \subseteq L$, where I, J are ideals of R and K is a submodule of W, implies that either $IJ \subseteq [L + soc(W) :_R W]$ or $IK \subseteq W - rad(L) + soc(W)$ or $JK \subseteq W - rad(L) + soc(W)$.

Proof

($\Rightarrow$)Assume that L is a pseudo primary-2-absorbing submodule of W, and $IJK \subseteq L$, where I, J are ideals of R and K is a submodule of W and $IJ \nsubseteq [L + soc(W) :_R W]$. We must prove that $IK \subseteq W - rad(L) + soc(W)$ or $JK \subseteq W - rad(L) + soc(W)$. Suppose that $IK \nsubseteq W - rad(L) + soc(W)$ and $JK \nsubseteq W - rad(L) + soc(W)$, it follows that there exists $r_1 \in I$ and $r_2 \in J$ such that $r_1K \nsubseteq W - rad(L) + soc(W)$ and $r_2K \nsubseteq W - rad(L) + soc(W)$. Now $r_1r_2K \subseteq L$ with $r_1K \nsubseteq W - rad(L) + soc(W)$ and $r_2K \nsubseteq W - rad(L) + soc(W)$ and L is a pseudo primary-2-absorbing submodule of W, implies that by Proposition(4) $r_1r_2 \in [L + soc(W) :_R W]$. Since $IJ \nsubseteq [L + soc(W) :_R W]$, it follows that there exists $s_1 \in I$, $s_2 \in J$ such that $s_1s_2 \notin [L + soc(W) :_R W]$. Since $s_1s_2K \subseteq L$, and $s_1s_2 \notin [L + soc(W) :_R W]$, we have by Proposition (4) either $s_1K \subseteq W - rad(L) + soc(W)$ or $s_2K \subseteq W - rad(L) + soc(W)$.

Now we discussed the following cases:

Case one: Suppose that $s_1K \subseteq W - rad(L) + soc(W)$ but $s_2K \nsubseteq W - rad(L) + soc(W)$. Since $r_1s_2K \subseteq L$ and L is a pseudo primary-2-absorbing submodule of W with $s_2K \nsubseteq W - rad(L) + soc(W)$ and $r_1K \nsubseteq W - rad(L) + soc(W)$, implies that $r_1s_2 \in [L + soc(W) :_R W]$ by Proposition(4). Also since $s_1K \subseteq W - rad(L) + soc(W)$ but $r_1K \nsubseteq W - rad(L) + soc(W)$, it follows that $(r_1 + s_1)K \nsubseteq W - rad(L) + soc(W)$. Since $(r_1 + s_1)s_2K \subseteq L$ and $s_2K \nsubseteq W - rad(L) + soc(W)$and $(r_1 + s_1)K \nsubseteq W - rad(L) + soc(W)$ implies that by Proposition(4) $(r_1 + s_1)s_2 \in [L + soc(W) :_R W]$. That is $(r_1 + s_1)s_2 = r_1s_2 + s_1s_2 \in [L + soc(W) :_R W]$ and $r_1s_2 \in [L + soc(W) :_R W]$, implies that $s_1s_2 \in [L + soc(W) :_R W]$ a contradiction.

Case two: If $s_2K \subseteq W - rad(L) + soc(W)$ but $s_1K \nsubseteq W - rad(L) + soc(W)$ in similarly steps of Case one we get a contradiction.

Case three: Assume that $s_1K \subseteq W - rad(L) + soc(W)$ but $s_2K \subseteq W - rad(L) + soc(W)$. Now since $s_2K \subseteq W - rad(L) + soc(W)$ and $r_2K \nsubseteq W - rad(L) + soc(W)$, it follows that $(r_2 + s_2)K \nsubseteq W - rad(L) + soc(W)$. We have $r_1(r_2 + s_2)K \subseteq L$ and $r_1K \nsubseteq W - rad(L) +$

$soc(W)$ and $(r_2+s_2)K \nsubseteq W-rad(L)+soc(W)$, by Proposition(4) $r_1(r_2+s_2)=r_1r_2+r_1s_2 \in [L+soc(W):_R W]$. But $r_1r_2 \in [L+soc(W):_R W]$ and $r_1r_2+r_1s_2 \in [L+soc(W):_R W]$, it follows that $r_1s_2 \in [L+soc(W):_R W]$. Now, since $s_1K \subseteq W-rad(L)+soc(W)$ and $r_1K \nsubseteq W-rad(L)+soc(W)$, implies that $(r_1+s_1)K \nsubseteq W-rad(L)+soc(W)$ since $(r_1+s_1)r_2K \subseteq L$ and $r_2K \nsubseteq W-rad(L)+soc(W)$ and $(r_1+s_1)K \nsubseteq W-rad(L)+soc(W)$, it follows that $(r_1+s_1)r_2=r_1r_2+s_1r_2 \in [L+soc(W):_R W]$ by Proposition(4). Now, since $r_1r_2 \in [L+soc(W):_R W]$ and $r_1r_2+s_1r_2 \in [L+soc(W):_R W]$, implies that $s_1r_2 \in [L+soc(W):_R W]$. Also , since $(r_1+s_1)(r_2+s_2)K \subseteq L$ and $(r_1+s_1)K \nsubseteq W-rad(L)+soc(W)$ and $(r_2+s_2)K \nsubseteq W-rad(L)+soc(W)$, it follows that $(r_1+s_1)(r_2+s_2)=r_1r_2+r_1s_2+s_1r_2+s_1s_2 \in [L+soc(W):_R W]$ by Proposition(4). Again since $r_1r_2, r_1s_2, s_1r_2 \in [L+soc(W):_R W]$, we get that $s_1s_2 \in [L+soc(W):_R W]$ a contradiction. Thus we have either $IK \subseteq W-rad(L)+soc(W)$ or $JK \subseteq W-rad(L)+soc(W)$.

($\Longleftarrow$) Obvious.

Proposition (6)

Let L be a proper submodule of an R-module W, with $W-rad(L)$ is a prime submodule of W. Then L is a pseudo primary-2-absorbing submodule of W.

Proof

Suppose that $rsx \in L$, where $r,s \in R$, $x \in W$ and $sx \notin W-rad(L)+soc(W)$. Since $L \subseteq W-rad(L)$, then $r(sx) \in W-rad(L)$, but $W-rad(L)$ is a prime submodule of W, then $rW \subseteq W-rad(L) \subseteq W-rad(L)+soc(W)$. That is $rx \in W-rad(L)+soc(W)$, for some $x \in W$. Thus L is a pseudo primary-2-absorbing submodule of W.

Lemma (7)[11, Ex. 10, p. 29]

Let L be an essential submodule of an R-module W, then $soc(L)=soc(W)$.

Proposition (8)

Let L and K are proper submodules of an R-module W such that $L \subsetneq K$ and K is an essential submodule of W. If L is a pseudo primary-2-absorbing submodule of W, then L is a pseudo primary-2-absorbing submodule of K.

Proof

Suppose that $rsx \in L$, where $r,s \in R$, $x \in K \subseteq W$. Since L is a pseudo primary-2-absorbing submodule of W, implies that either $rx \in W-rad(L)+soc(W)$ or $sx \in W-rad(L)+soc(W)$ or $rsW \subseteq L+soc(W)$. But K is an essential submodule of W, then by Lemma(7) $soc(K)=soc(W)$. Hence we have either $rx \in W-rad(L)+soc(K)$ or $sx \in W-rad(L)+soc(K)$ or $rsW \subseteq L+soc(K)$. Thus L is a pseudo primary-2-absorbing submodule of K.

Before we introduce the next result we need to recall the following lemmas.

Lemma (9)[13, Lemma(2.3.15)]

Let L, K and D are submodules of an R-module W with $K \subseteq D$, then $(L+K)\cap D=(L\cap D)+K=(L\cap D)+(K\cap D)$.

Lemma (10)[14, Coro(9.9)]

Let K be a submodule of an R-module W, then $soc(K)=K\cap soc(W)$.

Proposition (11)

Let L and K be a proper submodules of an R-module W with $L \subsetneq K$ and $soc(W) \subseteq K$. If L is a pseudo primary-2-absorbing submodule of W, then L is a pseudo primary-2-absorbing submodule of K.

Proof

Let $rsx \in L$, where $r, s \in R$, $x \in K \subseteq W$. Since L is a pseudo primary-2-absorbing submodule of W, implies that either $rx \in W-rad(L)+soc(W)$ or $sx \in W-rad(L)+soc(W)$ or $rsW \subseteq L+soc(W)$. That is either $rx \in (W-rad(L)+soc(W)) \cap K$ or $sx \in (W-rad(L)+soc(W)) \cap K$ or $rsW \subseteq (L+soc(W)) \cap K$. But by Lemma (9) $\big(W-rad(L)+soc(W)\big) \cap K = (W-rad(L) \cap K) + (soc(W) \cap K) = (W-rad(L) \cap K) + soc(K)$ by Lemma(10). Thus we have either $rx \in (W-rad(L) \cap K)+soc(K) \subseteq W-rad(L)+soc(K)$ or $sx \in (W-rad(L) \cap K)+soc(K) \subseteq W-rad(L)+soc(K)$ or $rsW \subseteq (L \cap K)+(soc(W) \cap K) = (L \cap K)+soc(K) \subseteq L+soc(K)$. Hence L is a pseudo primary-2-absorbing submodule of K.

Recall that for any submodules L, K of a multiplication R-module W with $L = IW$, $K = JW$ for some ideals I and J of R. The product $LK = IW.JW = IJW$. That is $LK = IK$, in particular $LW = IWW = IW = L$. Also for any $x \in W$ we have $Lx = Ix$[15].

The following result gives a characterization of pseudo primary-2-absorbing submodules in class of multiplication modules.

Proposition (12)

Let W be a multiplication R-module and L is a proper submodule of W. Then L is a pseudo primary-2-absorbing submodule of W if and only if, whenever $L_1L_2L_3 \subseteq L$ for L_1, L_2, L_3 are submodules of W, implies that either $L_1L_3 \subseteq W-rad(L)+soc(W)$ or $L_2L_3 \subseteq W-rad(L)+soc(W)$ or $L_1L_2W \subseteq L+soc(W)$.

Proof

($\Longrightarrow$) Let L be is a pseudo primary-2-absorbing submodule of W and $L_1L_2L_3 \subseteq L$ for L_1, L_2, L_3 are submodules of W, with $L_1L_2W \nsubseteq L+soc(W)$. Since W is a multiplication, then $L_1 = I_1W$ and $L_2 = I_2W$ for some ideals I_1, I_2, I_3 of R. Clearly $I_1I_2L_3 \subseteq L$ and $I_1I_2 \nsubseteq [L+soc(W):_R W]$. Since L is a pseudo primary-2-absorbing submodule of W, implies that either $I_1L_3 \subseteq W-rad(L)+soc(W)$ or $I_2L_3 \subseteq W-rad(L)+soc(W)$, it follows that either $L_1L_3 \subseteq W-rad(L)+soc(W)$ or $L_2L_3 \subseteq W-rad(L)+soc(W)$.

($\Longleftarrow$) Assume that $I_1I_2K \subseteq L$, where I_1, I_2 are ideals of R, and K is a submodule of W. Since W is multiplication, then $I_1I_2K = L_1L_2K \subseteq L$, by hypothesis either $L_1K \subseteq W-rad(L)+soc(W)$ or $L_2K \subseteq W-rad(L)+soc(W)$ or $L_1L_2 \subseteq [L+soc(W):_R W]$. That is either $I_1K \subseteq W-rad(L)+soc(W)$ or $I_2K \subseteq W-rad(L)+soc(W)$ or $I_1I_2 \subseteq [L+soc(W):_R W]$. Then by Proposition (5) L is a pseudo primary-2-absorbing submodule of W.

Lemma (13)[2].

Let $f: W \longrightarrow \overline{W}$ be an R-epimorphism and L is a submodule of $\overline{W}$ with $\ker(f) \subseteq L$, then $f\big(W-rad(L)\big) = \overline{W}-rad(f(L))$.

Proposition (14)

Let $f: W \longrightarrow \bar{W}$ be an R-epimorphism and $\bar{L}$ is a pseudo primary-2-absorbing submodule of $\bar{W}$. Then $f^{-1}(\bar{L})$ is a pseudo primary-2-absorbing submodule of W.

Proof

Let $\in f^{-1}(\bar{L})$, where $r, s \in R$, $x \in W$, implies that $rsf(x) \in \bar{L}$. Since $\bar{L}$ is a pseudo primary-2-absorbing submodule of $\bar{W}$, it follows that either $rf(x) \in \bar{W} - rad(\bar{L}) + soc(\bar{W})$ or $sf(x) \in \bar{W} - rad(\bar{L}) + soc(\bar{W})$ or $sr\bar{W} \subseteq \bar{L} + soc(\bar{W})$.Thus either $rx \in f^{-1}(\bar{W} - rad(\bar{L})) + f^{-1}(soc(\bar{W})) \subseteq W - rad(f^{-1}(\bar{L})) + soc(W)$ or $sx \in f^{-1}(\bar{W} - rad(\bar{L})) + f^{-1}(soc(\bar{W})) \subseteq W - rad(f^{-1}(\bar{L})) + soc(W)$ or $rsW \subseteq f^{-1}(\bar{L}) + soc(W)$. Hence $f^{-1}(\bar{L})$ be a pseudo primary-2-absorbing submodule of W.

Proposition (15)

Let $f: W \longrightarrow \bar{W}$ be an R-epimorphism and L is a pseudo primary-2-absorbing submodule of W with $\ker(f) \subseteq L$. Then $f(L)$ is a pseudo primary-2-absorbing submodule of $\bar{W}$.

Proof

Let $rs\bar{x} \in f(L)$, where $r, s \in R$, $\bar{x} \in \bar{W}$. Since f is onto, then $f(x) = \bar{x}$ for some$x \in W$. Thus $rsf(x) \in f(L)$, implies that $rsf(x) = f(l)$ for some $l \in L$, it follows that $f(rsx - l) = 0$, implies that $rsx - l \in \ker(f) \subseteq L$, then $rsx \in L$. But L be a pseudo primary-2-absorbing submodule of W, then either $rx \in W - rad(L) + soc(W)$ or $sx \in W - rad(L) + soc(W)$ or $rsW \subseteq L + soc(W)$, it follows that by Lemma(13) either $rf(x) \in f(W - rad(L)) + f(soc(W)) \subseteq \bar{W} - rad(f(L)) + soc(\bar{W})$ or $sf(x) \in f(W - rad(L)) + f(soc(W)) \subseteq \bar{W} - rad(f(L)) + soc(\bar{W})$ or $rsf(W) \subseteq f(L) + f(soc(W)) \subseteq f(L) + soc(\bar{W})$. That is either $r\bar{x} \in \bar{W} - rad(f(L)) + soc(\bar{W})$ or $s\bar{x} \in \bar{W} - rad(f(L)) + soc(\bar{W})$ or $rs\bar{W} \subseteq f(L) + soc(\bar{W})$. Hence $f(L)$ is a pseudo primary-2-absorbing submodule of $\bar{W}$.

Lemma (16)[12, Theo(2.12)]**.**

Let R be a commutative ring with identity, L be a proper submodule of a multiplication R-module W and $A = [L:_R W]$. Then$W - rad(L) = \sqrt{A}.W = \sqrt{[L:_R W]}.W$.

Lemma (17)[12, Coro(2.14)]**.**

Let W be faithful multiplication R-module, then $soc(R)W = soc(W)$.

Proposition (18)

Let W be a faithful multiplication R-module and L is a proper submodule of W. Then L is a pseudo primary-2-absorbing submodule of W if and only if $[L:_R W]$ is a pseudo primary-2-absorbing ideal of R.

Proof

($\Longrightarrow$) Let L is a pseudo primary-2-absorbing submodule of W, and $r.s.t \in [L:_R W]$ for $r,s,t \in R$, implies that $rstW \subseteq L$, that is $rst(x) \in L$ for all $x \in W$. But W is a multiplication R-module, then $(x) = IW$ for some ideal I of R. That is $rs(tIW) \subseteq L$. Since L is a pseudo primary-2-absorbing submodule of W, then by Proposition (4) either $r(tIW) \subseteq W - rad(L) + soc(W)$ or $s(tIW) \subseteq W - rad(L) + soc(W)$ or $rsW \subseteq L + soc(W)$ by Lemma (16) $W - rad(L) = \sqrt{[L:_R W]}.W$ and by Lemma (17) $soc(R)W = soc(W)$. Hence we get either $r(tIW) \subseteq \sqrt{[L:_R W]}.W + soc(R)W$ or $s(tIW) \subseteq \sqrt{[L:_R W]}.W + soc(R)W$ or

$rsW \subseteq [L:_R W]W + soc(R)W$. That is either $rtx \in \sqrt{[L:_R W]}.W + soc(R)W$ or $stx \in \sqrt{[L:_R W]}.W + soc(R)W$ for all $x \in W$ or $rsW \subseteq [L:_R W]W + soc(R)W$. It follows that either $rtW \subseteq \sqrt{[L:_R W]}.W + soc(R)W$ or $stW \subseteq \sqrt{[L:_R W]}.W + soc(R)W$ or $rsW \subseteq [L:_R W]W + soc(R)W$. Hence either $rt \in \sqrt{[L:_R W]} + soc(R)$ or $st \in \sqrt{[L:_R W]} + soc(R)$ or $rs \in [L:_R W] + soc(R)$. Therefore $[L:_R W]$ is pseudo primary-2-absorbing ideal of R.

($\Leftarrow$) Assume that $[L:_R W]$ is pseudo primary-2-absorbing ideal of R, and $rsx \in L$, for $r,s \in R$, $x \in W$, that is $rs(x) \subseteq L$. Since W is a multiplication R-module then $(x) = IW$ for some ideal I of R. Thus $rsIW \subseteq L$, implies that $rsI \subseteq [L:_R W]$. By hypothesis and Proposition (4) either $rI \subseteq \sqrt{[L:_R W]} + soc(R)$ or $sI \subseteq \sqrt{[L:_R W]} + soc(R)$ or $rs \in [L:_R W] + soc(R)$. That is either $rIW \subseteq \sqrt{[L:_R W]}W + soc(R)W$ or $sIW \subseteq \sqrt{[L:_R W]}W + soc(R)W$ or $rsW \subseteq [L:_R W]W + soc(R)W$. Thus by Lemma (16) and Lemma (17) we get $rx \in W - rad(L) + soc(W)$ or $sx \in W - rad(L) + soc(W)$ or $rsW \subseteq L + soc(W)$. Hence L is a pseudo primary-2-absorbing submodule of W.

We need to recall the following lemma before we introduce the next result.

Lemma (19)[11, Coro(1.26)].

If W is a non-singular R-modules, then $soc(R)W = soc(W)$.

Proposition (20)

Let W be a non-singular multiplication R-module and L is a proper submodule of W. Then L is a pseudo primary-2-absorbing submodule of W if and only if $[L:_R W]$ is a pseudo primary-2-absorbing ideal of R.

Proof

($\Leftarrow$) Let $[L:_R W]$ is a pseudo primary-2-absorbing ideal of R, and $aby \in L$, for $a,b \in R$, $y \in W$, that is $ab(y) \subseteq L$, it follows that $abJW \subseteq L$ for W is a multiplication R-module. Hence $abJ \subseteq [L:_R W]$, implies that by Proposition (4) either $aJ \subseteq \sqrt{[L:_R W]} + soc(R)$ or $bJ \subseteq \sqrt{[L:_R W]} + soc(R)$ or $ab \in \left[[L:_R W] + soc(R):R\right] = [L:_R W] + soc(R)$. Thus either $aJW \subseteq \sqrt{[L:_R W]}W + soc(R)W$ or $bJW \subseteq \sqrt{[L:_R W]}W + soc(R)W$ or $abW \subseteq [L:_R W]W + soc(R)W$. Hence by Lemma (16) and Lemma (19), we have either $ay \in W - rad(L) + soc(W)$ or $by \in W - rad(L) + soc(W)$ or $abW \subseteq L + soc(W)$. Therefore L is a pseudo primary-2-absorbing submodule of W.

($\Rightarrow$) Let it be $abc \in [L:_R W]$ where $a,b,c \in R$, then $abcW \subseteq L$, so $abcy \in L$ for all $y \in W$. Since W is a multiplication$R$-module, then $(y) = JW$, thus $abc(y) \subseteq L$, it follows that $ab(cJW) \subseteq L$, implies that by hypothesis and by Proposition(4) either $a(cJW) \subseteq W - rad(L) + soc(W)$ or $b(cJW) \subseteq W - rad(L) + soc(W)$ or $abW \subseteq L + soc(W)$. It follows that by Lemma (16) and by Lemma (19) and W is multiplication either $acy \in \sqrt{[L:_R W]}W + soc(R)W$ or $bcy \in \sqrt{[L:_R W]}W + soc(R)W$ for all $y \in W$ or $abW \subseteq [L:_R W]W + soc(R)W$. Hence either $acW \subseteq \sqrt{[L:_R W]}W + soc(R)W$ or $bcW \subseteq \sqrt{[L:_R W]}W + soc(R)W$ or $abW \subseteq [L:_R W]W + soc(R)W$. That is either $ac \in \sqrt{[L:_R W]} + soc(R)$ or $bc \in \sqrt{[L:_R W]} + soc(R)$ or $ab \in [L:_R W] + soc(R) = \left[[L:_R W] + soc(R):R\right]$. That is $[L:_R W]$ is a pseudo primary-2-absorbing ideal of R.

We need to recall the following results before we introduce the next propositions.

Lemma (21)[16, Coro of Theo. 9]

Let I_1 and I_2 are ideals of a ring R and W is a finitely generated multiplication R-module. Then $I_1W \subseteq I_2W$ if and only if $I_1 \subseteq I_2 + ann_R(W)$.

Lemma (22)[17, Pro. (2.4)].

Let W be a multiplication R-module and I is an ideal of R such that $ann_R(W) \subseteq I$, then $W - rad(IW) = \sqrt{I}W$.

Proposition (23)

Let W be a faithful finitely generated multiplication R-module and I is a pseudo primary-2-absorbing ideal of R and $IW \neq W$. Then IW is a pseudo primary-2-absorbing submodule of W.

Proof

Let $abx \in IW$ for $a,b \in R$, $x \in W$, then $ab(x) \subseteq IW$, implies that $abJW \subseteq IW$ for some ideal J of R since W is a multiplication. Hence by Lemma(21) $abJ \subseteq I + ann_R(W)$, but W is a faithful. It follows that $ann_R(W) = (0)$, that is $abJ \subseteq I$. Since I is a pseudo primary-2-absorbing ideal of R, then by Proposition (4) either $aJ \subseteq \sqrt{I} + soc(R)$ or $bJ \subseteq \sqrt{I} + soc(R)$ or $ab \in [I + soc(R): R] = I + soc(R)$. It follows that $aJW \subseteq \sqrt{I}W + soc(R)W$ or $bJW \subseteq \sqrt{I}W + soc(R)W$ or $abW \subseteq IW + soc(R)W$. But by Lemma (17)$soc(R)W = soc(W)$ and by Lemma (22) $\sqrt{I}W =. W - rad(IW)$. Thus either $ax \in W - rad(IW) + soc(W)$ or $bx \in W - rad(IW) + soc(W)$ or $abW \subseteq IW + soc(W)$. Hence IW is a pseudo primary-2-absorbing submodule of W.

Proposition (24)

Let W be a faithful finitely generated multiplication R-module and K be a proper submodule of W. Then the following statements are equivalent .

1. K is a pseudo primary-2-absorbing submodule of W.
2. $[K:_R W]$ is a pseudo primary-2-absorbing ideal of R.
3. $K = JW$ for some pseudo primary-2-absorbing ideal of R.

Proof

(1) ⇔ (2) By Proposition (18).

(2) ⇒ (3) Since $[K:_R W]$ is a pseudo primary-2-absorbing ideal of R with $ann_R(W) = [0: W] \subseteq [K:_R W]$ and $K = [K:_R W]W$, implies that $K = IW$ where $I = [K:_R W]$ is a pseudo primary-2-absorbing ideal of R.

(3) ⟹ (2) Suppose that $K = JW$ for some a pseudo primary-2-absorbing ideal J of R. Since W is multiplication, then $K = [K:_R W]W = IW$. Since W is faithful finitely generated multiplication then we have $[K:_R W] = J$. Thus $[K:_R W]$ is a pseudo primary-2-absorbing ideal of R.

Proposition (25)

Let W be a finitely generated multiplication non-singular R-module and I be a pseudo primary-2-absorbing ideal of R with $ann_R(W) \subseteq I$. Then IW is a pseudo primary-2-absorbing submodule of W.

Proof

Similarly as in Proposition (23) and using Lemma (19).

Proposition (26)

Let W be a finitely generated multiplication non-singular R-module and K be a proper submodule of W. Then the following statements are equivalent.

1. K is a pseudo primary-2-absorbing submodule of W.
2. $[K:_R W]$ is a pseudo primary-2-absorbing ideal of R.
3. $K = JW$ for some pseudo primary-2-absorbing ideal of R, with $ann_R(W) \subseteq J$.

Proof

Similarly as in Proposition (24), by using Proposition (20).

3. Conclusion

In this article we introduce new generalization of (prime, primary, 2-absorbing, pseudo-2-absorbing) submodules called pseudo primary-2-absorbing submodules and we explain the converse implication of above by examples. Many characterizations of this generalization are introduced. Relationships of this generalization with other classes of modules are given.

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