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Strongly Convergence of Two Iterations For a Common Fixed Point with an Application

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Abstract

In this paper, we study some cases of a common fixed point theorem for classes of firmly nonexpansive and generalized nonexpansive maps. In addition, we establish that the Picard-Mann iteration is faster than Noor iteration and we used Noor iteration to find the solution of delay differential equation.

Keywords: Banach space, common fixed point, strong convergence, nonexpansive map, condition (A).
MSC: 49J40; 47J20

1. Introduction

Let B be a non-empty subset of a Banach space M. A mapT on B is called nonexpansive [1]. if $\|Ta - Tb\| \leq \|a - b\|$ for all $a, b \in B$ and $F(T)$ denoted the set of all fixed points of T. In 1973, Bruck [2]. introduced a map called firmly nonexpansive map in Banach space. Certainly, every firmly nonexpansive is nonexpansive.

To discuss the convergence theorem for a pair of nonexpansive maps S and T on B to itself, a generalization of Mann and Ishikawa iterations was given by Das and Debata [3]. and Takahashi and Tamura [4]. This iteration dealt with two maps:

$$a_1 \in B$$
$$b_n = \beta_n a_n + (1 - \beta_n) T a_n$$
$$a_{n+1} = \alpha_n a_n + (1 - \alpha_n) S b_n, \qquad \forall n \in N$$

where (α_n) and $(\beta_n) \in [0, 1]$.

The aim of this paper is to prove some strongly convergenve theorems for approximating common fixed points of firmly nonexpansive and generalized nonexpansive.

2. Preliminaries

We will suppose that M is a Banach space and B is a non-empty closed convex subset of M. $F(T, S)$ denoted the set of all fixed points of S and T.

A sequences (a_n) in B is called :

Picard-Mann hybrid [5].

$$a_{n+1} = Sb_n$$
$$b_n = (1-\alpha_n)a_n + \alpha_n Ta_n \ , \quad \forall\, n \in N \qquad (1)$$

where (α_n) is a sequence in (0,1).

And a sequence (w_n) in B is called:

Noor iteration [6].

$$w_{n+1} = (1-\alpha_n)w_n + \alpha_n Su_n$$
$$u_n = (1-\beta_n)w_n + \beta_n Tv_n$$
$$v_n = (1-\gamma_n)w_n + \gamma_n Tw_n \quad ,\forall\, n \in N \qquad (2)$$

where $(\alpha_n), (\beta_n)$ and (γ_n) are sequences in [0,1].

Defintion(1)[2]: A map $T: B \to M$ is called firmly nonexpansive map if $\|Ta - Tb\| \le \|(1-t)(Ta - Tb) + t(a-b)\|, \forall\, a, b \in B$ and $t \ge 0$.

Defintion(2)[7]: A map $T: B \to M$ is said to be generalized nonexpansive map if there are nonnegative constants δ, μ and ω with $\delta + 2\mu + 2\omega \le 1$ such that $\forall\, a, b \in B$

$$\|Ta - Tb\| \le \delta\|a-b\| + \mu\{\|a - Ta\| + \|b - Tb\|\} + \omega\{\|a - Tb\| + \|b - Ta\|\}$$

Khan. And Fukhar-ud-din [8]. 2005, introduced the concept of condition (A') to prove the convergence of two-step iterative scheme with errors to common fixed points of two nonexpansive mappings, see also [9,10]. and [11].

Definition(3)[9]: Two maps are called satisfing condition (A) if there is a nondecreasing function $g: [0,\infty) \to [0,\infty)$ with $g(0) = 0, g(i) > 0, \forall\, i \,\epsilon\, (0,\infty)$ such that: Either $\|a - Ta\| \ge g\big(D(a,F)\big)\ or\ \|a - Sa\| \ge g\big(D(a,F)\big), \forall\, a \in B$ where $D(a,F) = inf\{\|a - a^*\|;\ a^* \,\epsilon\, F\}$ and $F = F(T) \cap F(S)$.

Definition(4)[12]: A map $T: B \to B$ is called affine if B is convex and $T(ra + (1-r)b) = rT(a) + (1-r)Tb, \forall\, a, b \in B$ and $r \in [0,1]$.

Definition(5)[5]: Let (f_n) and (g_n) be two sequences of real numbers that converge to f and g, resepectively. Assume that there exists a real number l such that:

$$\lim_{n\to\infty} \frac{\|f_n - f\|}{\|g_n - g\|} = l.$$

If $l = 0$, then we say that (f_n) converges faster to f than (g_n) to g.

Lemma(6)[13]: Let $(\Upsilon_n)^{\infty}_{n=0}$ and $(\varkappa_n)^{\infty}_{n=0}$ be nonnegative real sequences satisfying the inequality:

$$\Upsilon_{n+1} \le (1-\tau_n)\Upsilon_n + \varkappa_n$$

where $\tau_n \in (0,1), \forall\, n \ge n_0, \sum_{n=1}^{\infty} \tau_n = \infty$ and $\frac{\varkappa_n}{\tau_n} \to 0\ as\ n \to \infty$, then $\lim_{n\to\infty} \Upsilon_n = 0$.

Lemma(7)[14]: Let M be a uniformly convex Banach space and $0 < l \le t_n \le k < 1, \forall\, n \in N$. Suppose that (a_n) and (b_n) are two sequences of M such that $\lim_{n\to\infty}\|a_n\| \le m, \lim_{n\to\infty}\|b_n\| \le m$ and $\lim_{n\to\infty}\|t_n a_n + (1-t_n)b_n\| = m$ hold for some $m \ge 0$. Then $\lim_{n\to\infty}\|a_n - b_n\| = 0$.

3. The Main Results

Lemma (3.1): Let M be a Banach space, $B \subseteq M$, $T: B \to B$ be a Lipschitzain and firmly nonexpansive map and $S: B \to B$ be Lipschitzain and generalized nonexpansive map. Let

1-(a_n) defined in (1) where $(\alpha_n) \in (0,1), n \in N$.

2-(w_n) defined in (2) where $(\alpha_n), (\beta_n)$ and $(\gamma_n) \in [0,1]$.

If $F(S,T) \ne \emptyset$, then $\lim_{n\to\infty}\|a_n - a^*\|$ and $\lim_{n\to\infty}\|w_n - a^*\|$ exist $\forall\, a^* \in F(S,T)$.

Proof: Let $a^* \in F(T,S)$.

1- Now, to proof $\lim_{n\to\infty}\|a_n - a^*\|$ exists. Let the sequence (a_n) be as shown in step (1), so

$\|a_{n+1} - a^*\| = \|Sb_n - a^*\|$

$\leq \delta\|b_n - a^*\| + \mu\{\|a^* - a^*\| + \|b_n - Sb_n\|\} + \omega\{\|a^* - Sb_n\| + \|b_n - a^*\|\}$
$\leq (\delta + 2K\mu + 2K\omega)\|b_n - a^*\|$
$\leq (\delta + 2\mu + 2\omega)\|b_n - a^*\|$
$= \|(1 - \alpha_n)a_n + \alpha_n Ta_n - a^*\|$
$\leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n(1 - t)\|Ta_n - a^*\| + \alpha_n t\|a_n - a^*\|$
$\leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n[(1 - t)k + t]\|a_n - a^*\|$
$\leq \|a_n - a^*\|$

Then, $\lim_{n\to\infty}\|a_n - a^*\|$ exists $\forall a^* \in F(T,S)$.

2- Now, to proof $\lim_{n\to\infty}\|w_n - a^*\|$ exists. Let the sequence (w_n) be as shown in step (2), so

$\|v_n - a^*\| \leq (1 - \gamma_n)\|w_n - a^*\| + \gamma_n\|Tw_n - a^*\|$
$\leq (1 - \gamma_n)\|w_n - a^*\| + \gamma_n[(1 - t)k + t]\|w_n - a^*\|$
$\leq (1 - \gamma_n + \gamma_n)\|w_n - a^*\|$
$\leq \|w_n - a^*\|$

$\|u_n - a^*\| \leq (1 - \beta_n)\|w_n - a^*\| + \beta_n\|Tv_n - a^*\|$
$\leq (1 - \beta_n)\|w_n - a^*\| + \beta_n[(1 - t)k + t]\|v_n - a^*\|$
$\leq (1 - \beta_n + \beta_n)\|w_n - a^*\|$
$\leq \|w_n - a^*\|$

Now,

$\|w_{n+1} - a^*\| \leq (1 - \alpha_n)\|w_n - a^*\| + \alpha_n\|Su_n - a^*\|$
$\leq (1 - \alpha_n)\|w_n - a^*\| + \alpha_n(\delta + 2\mu + 2\omega)\|u_n - a^*\|$
$\leq (1 - \alpha_n + \alpha_n)\|w_n - a^*\|$
$\leq \|w_n - a^*\|$

Then, $\lim_{n\to\infty}\|w_n - a^*\|$ exists $\forall a^* \in F(T,S)$.

Lemma(3.2): Let M be a uniformly convex Banach space and $B \subseteq M$. Let

1- $T: B \to B$ be a Lipschitzain and firmly nonexpansive map, $S: B \to B$ be affine, Lipschitzain and generalized nonexpansive map and (a_n) defined in (1) .

2- $T: B \to B$ be a Lipschitzain and firmly nonexpansive map, $S: B \to B$ be Lipschitzain and generalized nonexpansive map and (w_n) defined in (2). Suppose that $\|a - Tb\| \leq \|Sa - Tb\|, \forall\, a, b \in B$ holds. If $F(S,T) \neq \emptyset$, then:

$\lim_{n\to\infty}\|Ta_n - a_n\| = 0 = \lim_{n\to\infty}\|Sa_n - a_n\|$ and $\lim_{n\to\infty}\|Tw_n - w_n\| = 0 = \lim_{n\to\infty}\|Sw_n - w_n\|$.

Proof: Let $a^* \in F(T,S)$.

1-By Lemma (3.1) $\lim_{n\to\infty}\|a_n - a^*\|$ exists $\forall a^* \in F(T,S)$. Suppose that $\lim_{n\to\infty}\|a_n - a^*\| = c, \forall\, c \geq 0$.

If $c = 0$, no prove is needed.

Now suppose $c > 0$,

$$\|a_{n+1} - a^*\| = \|Sb_n - a^*\|$$
$$\leq \|b_n - a^*\|$$

By Lemma (3.1), we show that $\|b_n - a^*\| \leq \|a_n - a^*\|$.

This implies to:

$$\lim_{n\to\infty}\|b_n - a^*\| = c$$

Next consider.

$c = \|b_n - a^*\| \leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\|$

By applying Lemma (2.7), we get:

$$\lim_{n\to\infty}\|a_n - Ta_n\| = 0$$

$c = \lim_{n\to\infty}\|a_{n+1} - a^*\| = \lim_{n\to\infty}\|Sb_n - a^*\|$

and,

$$\|S[(1-\alpha_n)a_n + \alpha_n Ta_n] - a^*\| \leq (1-\alpha_n)\|Sa_n - a^*\| + \alpha_n\|STa_n - a^*\|$$

By applying Lemma (2.7), we get:

$$\lim_{n\to\infty}\|Sa_n - STa_n\| = 0$$

Now,

$$\|Sa_n - a_n\| \leq \|Sa_n - STa_n\| + \|STa_n - a_n\|$$

By using the hypothesis condition, we obtain:

$$\|Sa_n - a_n\| \leq 2\|Sa_n - STa_n\| \to 0 \ as\ n \to \infty.$$

Thus,

$$\lim_{n\to\infty}\|Sa_n - a_n\| = 0.$$

2-By Lemma (3.1) $\lim_{n\to\infty}\|w_n - a^*\|$ exists $\forall a^* \in F(T,S)$. Suppose that $\lim_{n\to\infty}\|w_n - a^*\| = c, \forall\, c \geq 0$.

If $c = 0$, no prove is needed.

Now, suppose $c > 0$,

Since $\|Tw_n - a^*\| \leq \|w_n - a^*\|$, and as proved by Lemma (3.1)

$$\|Su_n - a^*\| \leq \|u_n - a^*\| \text{ and } \|Tv_n - a^*\| \leq \|v_n - a^*\|.$$

Then, $\lim_{n\to\infty}\|Tw_n - a^*\| \leq c, \lim_{n\to\infty}\|Su_n - a^*\| \leq c$ and $\lim_{n\to\infty}\|Tv_n - a^*\| \leq c$

Moreover,

$$\lim_{n\to\infty}\|w_{n+1} - a^*\| = c$$

$c = \|w_{n+1} - a^*\| \leq (1-\alpha_n)\|w_n - a^*\| + \alpha_n\|Su_n - a^*\|$

By applying Lemma (2.7), we obtain:

$$\lim_{n\to\infty}\|w_n - Su_n\| = 0$$

Next,

$\|w_n - a^*\| \leq \|w_n - Su_n\| + \|Su_n - a^*\| \xrightarrow{yields} c \leq \lim_{n\to\infty} inf\|u_n - a^*\|$

Therefore, we get:

$$\lim_{n\to\infty}\|u_n - a^*\| = c$$

$c = \|u_n - a^*\| \le (1-\beta_n)\|w_n - a^*\| + \beta_n\|Tv_n - a^*\|$

$\le (1-\beta_n)\|w_n - a^*\| + \beta_n[(1-t)k + t]\|v_n - a^*\|$

$\le (1-\beta_n)\|w_n - a^*\| + \beta_n(1-\gamma_n)\|w_n - a^*\|$

$+\beta_n\gamma_n\|Tw_n - a^*\|$

$\le (1-\beta_n\gamma_n)\|w_n - a^*\| + \beta_n\gamma_n\|Tw_n - a^*\|$

So, by applying Lemma (2.7), we obtain:

$$\lim_{n\to\infty}\|w_n - Tw_n\| = 0.$$

Next,

$$\|w_n - Sw_n\| \le \|w_n - Su_n\| + \|Su_n - w_n\| + \|w_n - Sw_n\|$$

Letting n→∞, we obtain:

$$\|w_n - Sw_n\| \le \|w_n - Sw_n\|$$

That means $\lim_{n\to\infty}\|w_n - Sw_n\| = 0$.

Theorem (3.3): Let $T: B \to B$ be a Lipschitzain, firmly nonexpansive map, $S: B \to B$ be a Lipschitzain and generalized nonexpansive map, with $F(S,T) \neq \emptyset$ and,

1- (a_n) defined in (1) and $(\alpha_n) \in (0,1)$ satisfying $\sum_{i=0}^{\infty}\alpha_i = \infty$.

2- (w_n) defined in (2) and $(\alpha_n), (\beta_n)$ and $(\gamma_n) \in [0,1]$ satisfying $\sum_{i=0}^{\infty}\alpha_i\beta_i\gamma_i = \infty$.

Then (a_n) and (w_n) converge to a unique common fixed point $a^* \in F(S,T)$.

Proof:

1-$\|b_n - a^*\| \le (1-\alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\|$

$\le (1-\alpha_n)\|a_n - a^*\| + \alpha_n[(1-t)k + t]\|a_n - a^*\|$

Suppose $\xi = (1-t)k + t$

$\le (1-(1-\xi)\alpha_n)\|a_n - a^*\|$

$\|a_{n+1} - a^*\| = \|Sb_n - a^*\|$

$\le \delta\|b_n - a^*\| + \mu\{\|a^* - a^*\| + \|b_n - Sb_n\|\} + \omega\{\|a^* - Sb_n\| + \|b_n - a^*\|\}$

$\le (\delta + 2\mu + 2\omega)\|b_n - a^*\|$

$\le \|b_n - a^*\|$

$\le (1-(1-\xi)\alpha_n)\|a_n - a^*\|$

By induction:

$\|a_{n+1} - a^*\| \le \prod_{i=0}^{n}(1-(1-\xi)\alpha_i)\|a_0 - a^*\|$

$\le \|a_0 - a^*\|e^{-(1-\xi)\sum_{i=0}^{n}\alpha_i}$

Since $\sum_{i=0}^{\infty}\alpha_i = \infty, e^{-(1-\xi)\sum_{i=0}^{n}\alpha_i} \to 0\ as\ n \to \infty$.

Thus, $\lim_{n\to\infty}\|a_n - a^*\| = 0$.

2-$\|v_n - a^*\| \le (1-\gamma_n)\|w_n - a^*\| + \gamma_n\|Tw_n - a^*\|$

$\le (1-\gamma_n)\|w_n - a^*\| + \gamma_n[(1-t)k + t]\|w_n - a^*\|$

Setting $\xi = (1-t)k + t$

$\le (1-\gamma_n + \gamma_n\xi)\|w_n - a^*\|$

$\|u_n - a^*\| \le (1-\beta_n)\|w_n - a^*\| + \beta_n\|Tv_n - a^*\|$

$\le (1-\beta_n)\|w_n - a^*\| + \beta_n\xi\|v_n - a^*\|$

$\le (1-\beta_n)\|w_n - a^*\| + \beta_n\xi(1-\gamma_n + \gamma_n\xi)\|w_n - a^*\|$

$\le (1-\beta_n)\|w_n - a^*\| + \beta_n(1-\gamma_n + \gamma_n\xi)\|w_n - a^*\|$

Now,

$\|w_{n+1}-a^*\| \leq (1-\alpha_n)\|w_n-a^*\|+\alpha_n\|Su_n-a^*\|$
$\leq (1-\alpha_n)\|w_n-a^*\|+\alpha_n[\,\delta\|u_n-a^*\|+\mu\{\|a^*-a^*\|+\|u_n-Su_n\|\}$
$+\,\omega\{\|a^*-Su_n\|+\|u_n-a^*\|\}]$
$\leq (1-\alpha_n)\|w_n-a^*\|+\alpha_n(\delta+2\mu+2\omega)\|u_n-a^*\|$
$\leq (1-\alpha_n)\|w_n-a^*\|+\alpha_n(1-\beta_n)\|w_n-a^*\|$
$+\alpha_n\beta_n(1-\gamma_n+\gamma_n\xi)\|w_n-a^*\|$
$\leq [1-\alpha_n\beta_n\gamma_n+\alpha_n\beta_n\gamma_n\xi]\|w_n-a^*\|$
$\leq [1-\alpha_n\beta_n\gamma_n]\|w_n-a^*\|$

By induction:

$\|w_{n+1}-a^*\| \leq \prod_{i=0}^{n}[1-\alpha_i\beta_i\gamma_i]\|w_0-a^*\|$
$\leq \|w_0-a^*\|e^{-\sum_{i=0}^{n}\alpha_i\beta_i\gamma_i}$

Since $\sum_{i=0}^{\infty}\alpha_i\,\beta_i\gamma_i=\infty, e^{-\sum_{i=0}^{n}\alpha_i\beta_i\gamma_i}\to 0\ as\ n\to\infty$. Thus, $\lim_{n\to\infty}\|w_n-a^*\|=0$.

Theorem(3.4): Let $B,S,T,(a_n)$ and (w_n) be as in Lemma (3.2) and S,T satisfying condition (A). If $F(S,T)\neq\emptyset$, then (a_n) and (w_n) converge strongly to a common fixed point of S and T.

Proof: Now, we will show that (a_n) strong convergence. By Lemma (3.1), $\lim_{n\to\infty}\|a_n-a^*\|$ exists. Suppose that $\lim_{n\to\infty}\|a_n-a^*\|=c, c\geq 0$. From Lemma (3.1), we have,

$$\|a_{n+1}-a^*\|\leq\|a_n-a^*\|$$

That gives:

$$inf_{a*\in F}\|a_{n+1}-a^*\|\leq inf_{a*\in F}\|a_n-a^*\|$$

Which means, $D(a_{n+1},F)\leq D(a_n,F)\xrightarrow{yields}\lim_{n\to\infty}D(a_n,F)$ exists.

By using condition (A), we have:

$$\lim_{n\to\infty}g(D(a_n,F))\leq\lim_{n\to\infty}\|a_n-Ta_n\|=0.$$

Or,

$$\lim_{n\to\infty}g(D(a_n,F))\leq\lim_{n\to\infty}\|a_n-Sa_n\|=0.$$

In both situation, we obtain

$$\lim_{n\to\infty}g(D(a_n,F))=0.$$

Since g is a non-decreasing function and $g(0)=0$, it follows that:
$\lim_{n\to\infty}D(a_n\,,F)=0.$ Now to show that (a_n) is a Cauchy sequence in A. Let $\epsilon>0$, $\lim_{n\to\infty}D(a_n\,,F)=0,\exists$ a positive integer n_0, such that:

$$D(a_n,F)<\frac{\epsilon}{4},\qquad\forall n\geq n_0$$

In particular,

$$\inf\{\|a_n-a^*\|,a^*\in F\}<\frac{\epsilon}{2}$$

Thus must exist $a^{**}\in F$ such that $\|a_n-a^{**}\|<\frac{\epsilon}{2}$.

Now, $\forall\, n,w\geq n_0$, we obtain:

$$\|a_{n+w}-a_n\|\leq\|a_{n+w}-a^{**}\|+\|a_n-a^{**}\|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

Hence, (a_n) is Cauchy sequence in B of M. Then (a_n) converges to a point $p\in B$.

$$\operatorname*{Lim}_{n\to\infty}D(a_n\,,F)=0\xrightarrow{yields}D(p,F)=0.$$

Since F is closed, hence $p \in F$.
By utilizing the same procedure, we can prove (w_n) convergence strongly.
Theorem (3.5): Let M be a Banach space, $\emptyset \neq B \subseteq M$. Let $T: B \to B$ be lipschitzain, firmly nonexpansive maps, $S: B \to B$ be Lipschitzain, generalized nonexpansive map and $a^* \in B$ be a common fixed point of S and T. Let (a_n) and (w_n) be the Picard-Mann and Noor iterations defined in (1) and (2). Suppose $(\alpha_n), (\beta_n)$ and (γ_n)satisfied the following conditions:
1- $(\alpha_n), (\beta_n)$ and $(\gamma_n) \in (0,1), \forall\, n \geq 0$.
2- $\sum \alpha_n = \infty$.
3- $\sum \alpha_n \beta_n < \infty$.
If $w_0 = a_0$ and $R(T), R(S)$ are bounded, then the Picard-Mann iteration sequence $a_n \to a^*$ and The Noor iteration sequence $w_n \to a^*$.
Proof: Since the range of T and S are bounded, let

$$M_1 = sup_{a\in B}\{\|Ta\|\} + \|a_0\| < \infty$$

And,

$$M_2 = sup_{a\in B}\{\|Sa\|\} + \|a_0\| < \infty$$

Let $M = max\{M_1, M_2\}$
Then,

$$\|a_n\| \leq M, \|b_n\| \leq M, \|w_n\| \leq M, \|u_n\| \leq M, \|v_n\| \leq M$$

Therefore,

$$\|Ta_n\| \leq M, \|Tw_n\| \leq M\ .$$

$$\begin{aligned}
\|a_{n+1} - w_{n+1}\| &= \|Sb_n - (1-\alpha_n)w_n - \alpha_n Su_n\| \\
&\leq \|Sb_n - w_n\| + \alpha_n\|Su_n - w_n\| \\
&\leq (\delta + 2\mu + 2\omega)\|b_n - a^*\| + \alpha_n(\delta + 2\mu + 2\omega)\|u_n - a^*\| + (1+\alpha_n) \\
&\quad (\delta + 2\mu + 2\omega)\|w_n - a^*\| \\
&\leq \|b_n - a^*\| + \alpha_n\|u_n - a^*\| + (1+\alpha_n)\|w_n - a^*\|
\end{aligned}$$

$$\|b_n - a^*\| \leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n(M + \|a^*\|)$$

$$\|v_n - a^*\| \leq (1-\gamma_n)\|w_n - a^*\| + \gamma_n\|Tw_n - a^*\|$$

Since T is lipschitzain and firmly nonexpansive, setting $\xi = k - kt + t$

$$\begin{aligned}
&\leq (1-\gamma_n)\|w_n - a^*\| + \gamma_n\xi\|w_n - a^*\| \\
&\leq \|w_n - a^*\|
\end{aligned}$$

$$\begin{aligned}
\|u_n - a^*\| &\leq (1-\beta_n)\|w_n - a^*\| + \beta_n\|Tv_n - a^*\| \\
&\leq (1-\beta_n)\|w_n - a^*\| + \beta_n\xi\|v_n - a^*\| \\
&\leq \|w_n - a^*\| \\
&\leq M + \|a^*\|
\end{aligned}$$

Then,

$$\begin{aligned}
\|a_{n+1} - w_{n+1}\| &\leq \|b_n - a^*\| + \alpha_n\|u_n - a^*\| + (1+\alpha_n)\|w_n - a^*\| \\
&\leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n(M + \|a^*\|) + \\
&\quad \alpha_n(M + \|a^*\|) + (1+\alpha_n)(M + \|a^*\|) \\
&\leq (1-\alpha_n)\|a_n - w_n\| + (1-\alpha_n)(M + \|a^*\|) \\
&\quad + 2\alpha_n(M + \|a^*\|) + (1+\alpha_n)(M + \|a^*\|) \\
&\leq (1-\alpha_n)\|a_n - w_n\| + 2\,(1+\alpha_n)(M + \|a^*\|)
\end{aligned}$$

Let $\Upsilon_n = \|a_n - w_n\|, \varkappa_n = (2+2\alpha_n)(M + \|a^*\|)$ and $\frac{\varkappa_n}{\tau_n} \to 0\ as\ n \to \infty$. By applying Lemma (2.6), we get

$$\lim_{n\to\infty}\|a_n - w_n\| = 0$$

If $a_n \to a^* \in F(S,T)$, then

$$\|w_n - a^*\| \le \|w_n - a_n\| + \|a_n - a^*\| \to 0 \; as \; n \to \infty$$

If $w_n \to a^* \in F(S,T)$, then

$$\|a_n - a^*\| \le \|a_n - w_n\| + \|w_n - a^*\| \to 0 \; as \; n \to \infty.$$

Theorem (3.6): Let $T: B \to B$ be a Lipschitzain, firmly nonexpansive map with $kt < 1$, $S: B \to B$ be a Lipschitzain and generalized nonexpansive map.Suppose that the Picard-Mann and Noor iterations converge to the same common fixed point a^*.Then picard-Mann iteration converges faster than Noor iteration.

Proof: Let $a^* \in F(T,S)$. Then for Picard-Mann iteration.

$$\|b_n - a^*\| \le (1-\alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\|$$

Setting $\xi = (1-t)k + t$, then we have

$$\le (1-(1-\xi)\alpha_n)\|a_n - a^*\|$$

Next,

$$\|a_{n+1} - a^*\| = \|Sb_n - a^*\|$$
$$\le \delta\|b_n - a^*\| + \mu\{\|a^* - a^*\| + \|b_n - Sb_n\|\} + \omega\{\|a^* - Sb_n\| + \|b_n - a^*\|\}$$
$$\le (\delta + 2k\mu + 2k\omega)\|b_n - a^*\|$$
$$\le (\delta + 2\mu + 2\omega)\|b_n - a^*\|$$
$$\le \|b_n - a^*\|$$
$$\le (1-(1-\xi)\alpha_n)\|a_n - a^*\|$$

.

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$$\le (1-(1-\xi)\alpha)^n\|a_1 - a^*\|$$

Let $f_n = (1-(1-\xi)\alpha)^n\|a_1 - a^*\|$

Now, Noor iteration,

$$\|v_n - a^*\| \le (1-\gamma_n)\|w_n - a^*\| + \gamma_n\|Tw_n - a^*\|$$
$$\le (1-\gamma_n)\|w_n - a^*\| + \gamma_n\xi\|w_n - a^*\|$$
$$= \|w_n - a^*\|$$
$$\|u_n - a^*\| \le (1-\beta_n)\|w_n - a^*\| + \beta_n\|Tv_n - a^*\|$$

.

$$\le (1-(1-\xi)\beta_n)\|w_n - a^*\|$$

Then,

$$\|w_{n+1} - a^*\| \le (1-\alpha_n)\|w_n - a^*\| + \alpha_n\|Su_n - a^*\|$$
$$\le (1-\alpha_n)\|w_n - a^*\| + \alpha_n(\delta + 2\mu + 2\omega)\|u_n - a^*\|$$
$$\le (1-\alpha_n + \alpha_n(1-(1-\xi)\beta_n)\|w_n - a^*\|$$

Assume that $\alpha_n \le (1-\alpha_n + \alpha_n(1-(1-\xi)\beta_n)$

$$\le \alpha_n\|w_n - a^*\|$$

.

.

$$\le \alpha^n\|w_1 - a^*\|$$

Let $g_n = \alpha^n\|w_1 - a^*\|$

Now,

$$\frac{f_n}{g_n} = \frac{(1-(1-\xi)\alpha)^n \|a_1 - a^*\|}{\alpha^n \|w_1 - a^*\|} \leq \left(1-(1-\xi)\right)^n \frac{\|a_1 - a^*\|}{\|w_1 - a^*\|} \to 0 \ as\ n \to \infty.$$

Then, (a_n) converges faster than (w_n) to a^*.

Example (3.7): Let $T, S: R \to R$ (where R is the set of all real numbers) be two maps defined by $Ta = \frac{2a}{3}$ and $Sa = \frac{a}{2}$ $\forall a \in R$. Choose $\alpha_n = \frac{3}{7}, \beta_n = \frac{1}{7}, \gamma_n = \frac{3}{7}$ $, \forall n$ with initial value $a_1 = 20$. The Picard-Mann iteration converges faster than Noor iteration, it is clear from **Table 1.** and **Figure 1.**

Table 1. Numerical results corresponding to $a_1 = 20$ for 50 steps.

n	Picard-Mann	n	Noor Iteration	n	Picard-Mann	n	Noor iteration
0	20.0000	0	20.0000	26	-	26	0.0244
1	8.5714	1	15.4519	27	-	27	0.0189
2	3.6735	2	11.9381	28	-	28	0.0146
3	1.5747	3	9.2233	29	-	29	0.0113
4	0.6747	4	7.1259	30	-	30	0.0087
5	0.2892	5	5.5054	31	-	31	0.0067
6	0.1239	6	4.2534	32	-	32	0.0052
7	0.0531	7	3.2862	33	-	33	0.0040
8	0.0228	8	2.5389	34	-	34	0.0031
9	0.0098	9	1.9615	35	-	35	0.0024
10	0.0042	10	1.5155	36	-	36	0.0019
11	0.0018	11	1.1708	37	-	37	0.0014
12	0.0008	12	0.9046	38	-	38	0.0011
13	0.0003	13	0.6989	39	-	39	0.0009
14	0.0001	14	0.5400	40	-	40	0.0007
15	0.0001	15	0.4172	41	-	41	0.0005
16	0.0000	16	0.3223	42	-	42	0.0004
17	0.0000	17	0.2490	43	-	43	0.0003
18	-	18	0.1924	44	-	44	0.0002
19	-	19	0.1486	45	-	45	0.0002
20	-	20	0.1148	46	-	46	0.0001
21	-	21	0.0887	47	-	47	0.0001
22	-	22	0.0685	48	-	48	0.0001
23	-	23	0.0530	49	-	49	0.0001
24	-	24	0.0409	50	-	50	0.0000
25	-	25	0.0319				

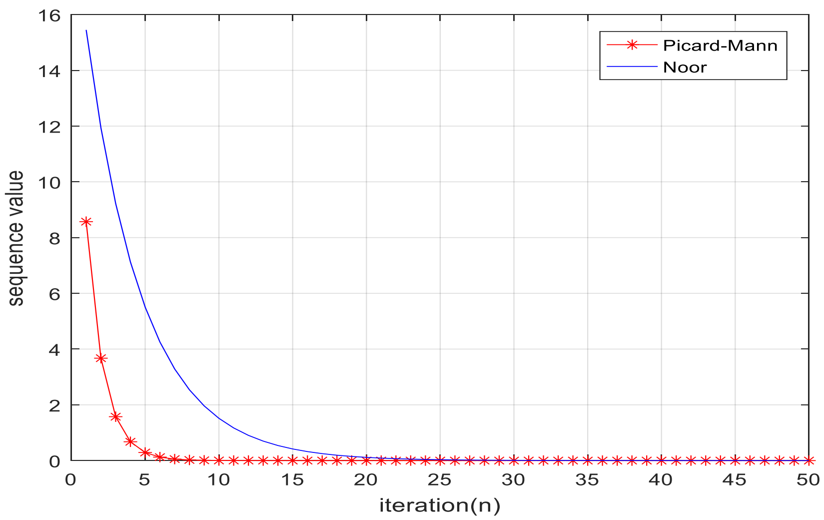

Figure 1. Convergence behavior corresponding to $a_1 = 20$ for 50 steps.

4. Applications

Let the space C([f, h]) of all continuous real valued functions on a closed interval [f, h] be endowed with the chebyshev norm $\|a-b\|_\infty = \max_{t\in[f,h]}|a(t)-b(t)|$.

$(C[f,h], \|.\|_\infty)$ be a Banach space. The following delay differential equation:

$w'(t) = g\big(t, w(t), w(t-\tau)\big), t \in [t_0, h]$ with initial condition $w(t) = \vartheta(t), t \in [t_0-\tau, t_0]$ (3)

Assume the conditions are satisfied:

i- $t_0, h \in R, \tau > 0$.

ii- $g \in C([t_0,h]\times R^2, R)$.

iii- $\vartheta \in C([t_0-\tau, h], R)$.

iv- There is $L_g > 0$ such that

$$|g(t,x_1,x_2) - g(t,y_1,y_2)| \le L_g \sum_{i=1}^{2} |x_i - y_i| \ \forall\, x_i, y_i \in R, i = 1,2, t \in [t_0, h].$$

v- $2L_g(h - t_0) < 1$.

Now, let us consider the following integral equation:

$$w(t) = \begin{cases} \vartheta(t) & t \in [t_0-\tau, t_0] \\ \vartheta(t) + \int_{t_0}^{t} g\big(r, w(r), w(r-\tau)\big)dr & t \in [t_0, h] \end{cases}$$

This is the solution of the above delay differential equation [15].

Theorem (4.1): Suppose the conditions (i-v) are accomplished the problem (3) has a unique solution a^* in $C([t_0-\tau,h],R) \cap C^{-1}([t_0,h,R])$ and the Noor iteration converges to a^*.

Proof: Let (w_n) be an iterative sequence generated by Noor for an map defined by

$$Tw(t) = \begin{cases} \vartheta(t) & t \in [t_0-\tau, t_0] \\ \vartheta(t) + \int_{t_0}^{t} g\big(r, w(r), w(r-\tau)\big)dr & t \in [t_0, h] \end{cases}$$

Let (a^*) be a fixed point. Now, it is easy to see $w_n \to a^*$ for each $t \in [t_0-\tau, t_0]$.

Next, for $t \in [t_0, h]$, we get:

$$\|v_n - a^*\|_\infty = \|(1-\gamma_n)w_n + \gamma_n T w_n - a^*\|_\infty$$

$$\le (1-\gamma_n)\|w_n - a^*\|_\infty + \gamma_n \max_{t\in[t_0-\tau,t_0]}|Tw_n(t) - Ta^*(t)|$$

$$\le (1-\gamma_n)\|w_n - a^*\|_\infty$$

$$+\gamma_n \max_{t\in[t_0-\tau,t_0]} \left|\vartheta(t) + \int_{t_0}^{t} g\big(r, w(r), w(r-\tau)\big)dr - \vartheta(t) - \int_{t_0}^{t} g\big(r, a^*(r), a^*(r-\tau)\big)dr\right|$$

$$\le (1-\gamma_n)\|w_n - a^*\|_\infty$$

$$+\gamma_n \max_{t\in[t_0-\tau,t_0]} \int_{t_0}^{t} \left(\left|g\big(r, w(r), w(r-\tau)\big)\right| + \left|g\big(r, a^*(r), a^*(r-\tau)\big)\right|\right)dr$$

$$\leq (1-\gamma_n)\|w_n - a^*\|_\infty + \gamma_n \int_{t_0}^{t} (L_g(\max_{t\in[t_0-\tau,t_0]} |w_n(t) - a^*(t)| + \max_{t\in[t_0-\tau,t_0]} |w_n(t-\tau) - a^*(t-\tau)|) dr$$

$$\leq (1-\gamma_n)\|w_n - a^*\|_\infty + \gamma_n \int_{t_0}^{t} L_g(\|w_n - a^*\|_\infty + \|w_n - a^*\|_\infty) dr$$

$$\leq (1-\gamma_n)\|w_n - a^*\|_\infty + 2\gamma_n L_g(t-t_0)\|w_n - a^*\|_\infty$$

$$\leq [1-(1-2L_g(h-t_0)\gamma_n]\|w_n - a^*\|_\infty$$

Next,

$$\|u_n - a^*\|_\infty = \|(1-\beta_n)w_n + \beta_n T v_n - a^*\|_\infty$$

$$\leq (1-\beta_n)\|w_n - a^*\|_\infty + \beta_n \max_{t\in[t_0-\tau,t_0]} |Tv_n(t) - Ta^*(t)|$$

$$\leq [1-(1-2L_g(h-t_0)\beta_n]\|v_n - a^*\|_\infty$$

Therefore,

$$\|w_{n+1} - a^*\|_\infty = \|(1-\alpha_n)w_n + \alpha_n T u_n - a^*\|_\infty$$

$$\leq (1-\alpha_n)\|w_n - a^*\|_\infty + \alpha_n \max_{t\in[t_0-\tau,t_0]} |Tu_n(t) - Ta^*(t)|$$

$$\leq [1-(1-2L_g(h-t_0)\alpha_n]\|u_n - a^*\|_\infty$$

$$\leq [1-(1-2L_g(h-t_0)\alpha_n][1-(1-2L_g(h-t_0)\beta_n][1-(1 - 2L_g(h-t_0)\gamma_n]\|w_n - a^*\|_\infty$$

$$\leq [1-(\alpha_n+\beta_n+\gamma_n)[1-(1-2L_g(h-t_0)]]\|w_n - a^*\|_\infty$$

setting $\lambda_n = \alpha_n + \beta_n + \gamma_n$ and by condition (v) $2L_g(h-t_0) < 1$.

Now, under the conditions (i-v) and using theorem (3.3) , therefore the delay differential equation has a unique solution $a^*\ in\ C([t_0-\tau,h],R) \cap C^{-1}([t_0,h,R])$ and the Noor iteration converges to a^*.

In support of this work, we would like to refer to our other results in this field in [16].

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