The Comparison Between Standard Bayes Estimators of the Reliability Function of Exponential Distribution

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Abstract
In this paper, a Monte Carlo Simulation technique is used to compare the performance of the standard Bayes estimators of the reliability function of the one parameter exponential distribution .Three types of loss functions are adopted, namely, squared error loss function (SELF) ,Precautionary error loss function (PELF) and linear exponential error loss function (LINEX) with informative and non- informative prior .The criterion integrated mean square error (IMSE) is employed to assess the performance of such estimators
.

Key words: standard Bayes estimator, loss function, IMSE

1. Introduction

The reliability theory is associated with random occurrence of undesirable events or failure during the life of a physical or biological system [1]. Reliability is a substantial feature of a system. Basic concepts related with reliability has been recognized for a number of years, however, it has got greatest importance during the past decennium as a consequence of the use of highly complex systems. In reliability theory, the exponential distribution has a distinctive role in life testing experiments. Historically, it was the first life time model for which statistical procedures were widely developed. Many researchers gave numerous results and generalized the exponential distribution as a life time distribution, particularly, in the field of industrial life testing. The exponential distribution is desirable because of its simplicity and its own features such as lacks memory and self-producing property.

The probability density, cumulative distribution and reliability functions of one parameter exponential distribution are respectively defined as [2]:

$$f(t,\theta) = \theta e^{-\theta t}, \qquad t,\theta > 0 \tag{1}$$

The cumulative distribution function is given by

$$F(t) = pr(T \le t) = 1 - e^{-\theta t} \tag{2}$$

$$R(t) = 1 - F(t) = e^{-\theta t} \tag{3}$$

The one parameter exponential distribution is a member of exponential class of probability density functions which has the general form [3]

$$f(t,\theta) = \exp[p(\theta)k(t) + s(t) + q(\theta)] \tag{4}$$

then the exponential distribution the p.d.f. can be written as

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$f(t, \theta) = \exp[-\theta t + ln\theta]$

Hence $T = \sum_{i=1}^{n} k(t) = \sum_{i=1}^{n} t_i$ is the complete sufficient statistic for θ and it can be easily shown that T follows Gamma distribution with parameters n and $\frac{1}{\theta}$.

2. Standard Bayes Estimators

The researchers employed three types of loss functions, namely, the squared error loss function (SELF), precautionary error loss function (PELF) and linear exponential error loss function(LINEX) . The Bayes estimator of the parameter θ is the value of θ that minimize the risk function $R(\hat{\theta}, \theta)$ where [4]

$R(\hat{\theta}, \theta) = E[L(\hat{\theta}, \theta)]$

$$= \int_{\theta} L(\hat{\theta}, \theta) h(\theta | \underline{t})\, d\theta \qquad (5)$$

In the case of squared error loss function we have

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \qquad (6)$$

Then, the risk function will be

$$R(\hat{\theta}, \theta) = \int_{\theta} (\hat{\theta} - \theta)^2 h(\theta | \underline{t}) d\theta$$

$$= \int_{\theta} \hat{\theta}^2 h(\theta | \underline{t}) d\theta - 2\hat{\theta} \int_{\theta} \theta h(\theta | \underline{t}) d\theta + \int_{\theta} \theta^2 h(\theta | \underline{t}) d\theta$$

$$R(\hat{\theta}, \theta) = \hat{\theta}^2 - 2\hat{\theta} E(\theta | \underline{t}) + E(\theta^2 | \underline{t})$$

Differentiating $R(\hat{\theta}, \theta)$ with respect to $\hat{\theta}$ and setting the resultant derivative equal to zero, we get:

$$2\hat{\theta} - 2E(\theta | \underline{t}) = 0$$

Solving for $\hat{\theta}$ implies that

$$\hat{\theta}_{Sq} = E(\theta | \underline{t}) \qquad (7)$$

ThePrecautionary error loss function is defined as [5]:

$$L(\hat{\theta}, \theta) = \frac{(\theta - \hat{\theta})^2}{\hat{\theta}} \qquad (8)$$

If (PELF)is adopted, it can be in the same manner show that the Bayes estimator of θ is

$$\hat{\theta}_P = \sqrt{E(\theta^2 | \underline{t})} \qquad (9)$$

Varian (1975) developed the following a symmetric linear exponential (LINEX) loss function

$$L(\Delta) = e^{\Delta} - \Delta - 1 \qquad (10)$$

Where $\Delta = (\hat{\theta} - \theta)$

And when the ((LINEX) is adopted, similarity the Bayes estimator of θ is

$$\hat{\theta}_{LI} = -ln \int_{\theta} e^{-\theta} h(\theta | \underline{t})\, d\theta \qquad (11)$$

4. Posterior Density Based on Jeffrey's Prior Information

Let us assume that θ has non informative prior density. Jeffrey's (1961) developed a general rule for obtaining the prior distribution of θ [6]. He established that the single unknown parameter θ which is regarded as a random variable follows such a distribution that is proportional to the square root of the fisher information on θ, that is [5]

$$g(\theta)\ \alpha \sqrt{I(\theta)} \qquad (12)$$

That is

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$g(\theta) = c\sqrt{I(\theta)}$

Where c is a constant of proportionality and I(ɵ) represent fisher information defined as follows:

$I(\theta) = -nE[\frac{\partial^2 lnf(t,\theta)}{\partial\theta^2}]$

If $g_1(\theta)$denote Jeffrey's prior information then

$$g_1(\theta) = c\sqrt{-nE\left(\frac{\partial^2 lnf(t;\theta)}{\partial\theta^2}\right)} \qquad (13)$$

For the exponential distribution we have

$lnf(t,\theta) = ln\theta - \theta t$

$- t\,\frac{\partial \ln f(t;\theta)}{\partial\theta} = \frac{1}{\theta}$

Hence,

$\frac{\partial^2 \ln f(t;\theta)}{\partial\theta^2} = \frac{-1}{\theta^2}$

Substituting in Equation (13) it follows that

$g_1(\theta) = \frac{c}{\theta}\sqrt{n}$

From Bayes theorem the posterior density function of ɵ denoted by $h_1(\theta|\underline{t})$ can be derived as [4]

$$h_1(\theta|t_1,\dots,t_n) = \frac{g_1(\theta)L(\theta;\ t_1,\dots,t_{n)}}{\int_0^\infty g_1(\theta)\,L(\theta;t_1,\dots,t_n)d\theta}$$

$$h_1(\theta|t_1,\dots,t_n) = \frac{\theta^{n-1}\,e^{-\theta T}}{\int_0^\infty \theta^{n-1}\,e^{-\theta T}d\theta}, T = \sum_{i=1}^{n} t_i$$

Hence, the posterior density function for ɵ based on Jeffery's prior information will be

$$h_1(\theta|t_1,\dots,t_n\,) = \frac{T^n\theta^{n-1}e^{-\theta T}}{\Gamma(n)} \qquad (14)$$

The posterior density in Equation (14) is defined identified as a density of the Gamma distribution, that is

Gamma (n,$\frac{1}{T}$) with E(ɵ) = $\frac{n}{T}$ and var(ɵ) =$\frac{n}{T^2}\theta|t_1,t_2,\dots,t_n\sim$

that is ɵ~Gamma (n ,$\frac{1}{T}$)

4. Posterior Density Based on Gamma Prior Distribution

Assuming that ɵ has informative prior as Gamma distribution which takes the following form:

$$g_2(\theta) = \frac{\beta^\alpha\theta^{\alpha-1}e^{-\theta\beta}}{\Gamma(\alpha)}; \theta > 0\ , \beta > 0\ , \alpha > 0 \qquad (15)$$

Where α,β are the shape parameter and scale parameter respectively

The posterior density function is

$$h_2(\theta|\underline{t}) = \frac{g_2(\theta)L(\theta;t_1,\dots,t_n)}{\int_0^\infty g_2(\theta)L(\theta;t_1,\dots,t_n)d\theta}$$

Thus

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$$h_2(\theta|\underline{t}) = \frac{P^{\alpha+n}\,\theta^{\alpha+n-1}e^{-\theta P}}{\Gamma(\alpha+n)} \qquad (16)$$

Where $P = (\beta + T)$

It can easily be noted that

$(\theta|\underline{t}) \sim Gamma\left(\alpha+n, \frac{1}{P}\right)$ with $E(\theta) = \frac{\alpha+n}{P}$, $Var(\theta) = (\frac{\alpha+n}{P^2})$

5. Bayes Estimator When (SELF)is Adopted

a: The case of Jeffrey's prior information.

From Equation (7) we found that:

$$\hat{\theta}_{JSq} = E(\theta|\underline{t}) = \int_0^\infty \theta h_1(\theta|\underline{t})\, d\theta$$

$$\hat{\theta}_{JSq} = \frac{n}{T} \qquad (17)$$

Similarly, the Bayes estimator of the reliability function can be obtained as follows:

$$\hat{R}(t)_{JSq} = E(R(t)|\underline{t}) = \int_0^\infty R(t)h_1(\theta|\underline{t})\, d\theta$$

$$\hat{R}(t)_{JSq} = (\frac{T}{T+t})^n \qquad (18)$$

b: The case of Gamma prior distribution.

In this case we have

$$\hat{\theta}_{GSq} = E[h_2(\theta|\underline{t})] = \frac{\alpha+n}{P} \qquad (19)$$

The estimator of the reliability function can be obtained as

$$\hat{R}(t)_{GSq} = E(R(t)|\underline{t}) =$$

$$\hat{R}(t)_{GSq} = \int_0^\infty e^{-\theta t}\frac{P^{\alpha+n}\theta^{\alpha+n-1}e^{-\theta P}}{\Gamma(\alpha+n)}\, d\theta$$

Which, implies that

$$\hat{R}(t)_{GSq} = \int_0^\infty R(t)h_2(\theta|\underline{t})\, d\theta = (\frac{P}{P+t})^{\alpha+n} \qquad (20)$$

6. Bayes Estimator When (PELF) is Adopted

a: The case of Jeffrey's prior information

From Equation (9) we have

$$\hat{\theta}_{JP} = \sqrt{E(\theta^2|\underline{x})}$$

The r^{th} moment of $\theta|\underline{t}$ can be evaluated as follows:

$E(\theta^r|\underline{t}) = \int_0^\infty \theta^r h_1(\theta|\underline{t})d\theta$

Hence,

$$E(\theta^r|\underline{t}) = \frac{\Gamma(n+r)}{\Gamma(n)T^r} \qquad (21)$$

When r=2, we get

$$E(\theta^2|\underline{t}) = \frac{\Gamma(n+2)}{\Gamma(n)T^2} = \frac{n(n+1)}{T}$$

Hence,

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$$\hat{\theta}_{JP} = \sqrt{\frac{n(n+1)}{T}} \tag{22}$$

Similarly, the Bayes estimators of the reliability function can be obtained as follows

$$\hat{R}(t)_{JP} = \sqrt{\mathrm{E}[\left(\left(R(t)\right)^2 \mid \underline{t}\right)]}$$

Now, we have to determine $\mathrm{E}[((R(t))^2 \mid \underline{t})]$

$$\mathrm{E}\left((R(t))^2 \mid \underline{t}\right) = \int_0^\infty \left(R(t)\right)^2 h_1(\theta \mid \underline{t}) d\theta$$

$$=\left(\frac{T}{T+2t}\right)^n$$

Hence,

$$\hat{R}(t)_{JP} = \sqrt{\left(\frac{T}{T+2t}\right)^n} \tag{23}$$

b: The case of Gamma prior distribution

From Equation (9) we have

$$\hat{\theta}_P = \sqrt{E(\theta^2 \mid \underline{x})}$$

The r^{th} moment of $\theta \mid \underline{t}$ can be evaluated as follows:

$$E\left(\theta^r \mid \underline{t}\right) = \int_0^\infty \theta^r\, h_2\left(\theta \mid \underline{t}\right) d\theta$$

$$E\left(\theta^r \mid \underline{t}\right) = \frac{\Gamma(\alpha+n+r)}{\Gamma(\alpha+n)P^r} \tag{24}$$

If r=2 then

$$E\left(\theta^2 \mid \underline{t}\right) = \frac{(\alpha+n)(\alpha+n+1)}{P^2}$$

Hence,

$$\hat{\theta}_{GP} = \sqrt{\frac{(\alpha+n)(\alpha+n+1)}{P^2}} \tag{25}$$

Similarly, the r^{th}Bayes estimators of the reliability function can be obtained as follows

$$\hat{R}(t)_{GP} = \sqrt{\mathrm{E}[((R(t))^2 \mid \underline{t})]}$$

Now, we have to determine $\mathrm{E}[((R(t))^2 \mid \underline{t})]$

$$\mathrm{E}((R(t))^2 \mid \underline{t}) = \int_0^\infty \left(R(t)\right)^2 h_2(\theta \mid \underline{t}) d\theta$$

$$= \left(\frac{P}{P+2t}\right)^{\alpha+n}$$

$$\hat{R}(t)_{GP} = \sqrt{\left(\frac{P}{P+2t}\right)^{\alpha+n}} \tag{26}$$

7. Bayes Estimator When (LINEX) Is Adopted

a: The case of Jeffrey's prior information

From Equation (11) we have

$$\hat{\theta}_{LI} = -ln \int_\theta e^{-\theta} h(\theta \mid \underline{t})\, d\theta$$

Hence,

$$\hat{\theta}_{JLI} = -ln\int_0^{\infty} e^{-\theta} h_1(\theta|\underline{t})d\theta$$

By evaluating the integral we get

$$\hat{\theta}_{JLI} = -ln\left(\frac{T}{1+T}\right)^n \tag{27}$$

The estimator of the reliability function can be obtained as

$$\hat{R}(t)_{JLI} = e^{-\hat{\theta}_{JLI}t}$$

$$\hat{R}(t)_{JLI} = \left(\frac{T}{1+T}\right)^n t \tag{28}$$

b: The case of Gamma prior distribution

From Equation (11) we have

$$\hat{\theta}_{LI} = -ln\int_{\theta} e^{-\theta} h(\theta|\underline{t})\, d\theta$$

Hence,

$$\hat{\theta}_{GLI} = -ln\int_0^{\infty} e^{-\theta} h_2(\theta|\underline{t})d\theta$$

By evaluating the integral we get

$$\hat{\theta}_{GLI} = -ln\left(\frac{P}{1+P}\right)^{\alpha+n} \tag{29}$$

The estimator of the reliability function can be obtained as

$$\hat{R}(t)_{GLI} = e^{-\hat{\theta}_{GLI}t}$$

$$\hat{R}(t)_{GLI} = \left(\frac{P}{1+P}\right)^{\alpha+n} t \tag{30}$$

8. Simulation Study

The simulation study was conducted in order to compare the performance of the Bayesian estimators of the reliability function R(t)of one parameter exponential distribution.

The integrated mean squared error (IMSE) as a criterion of comparison where

$$IMSE\left[\hat{R}(ti)\right] = \tfrac{1}{L}\textstyle\sum_{i=1}^{L}[\tfrac{1}{n_t}\sum_{r=1}^{n_t}(\hat{R}_i(t_r) - R(t_r))^2]$$

$$= \frac{1}{n_t}\sum_{r=1}^{n_t} MSE(\hat{R}\ \ (t_r))$$

Where $n_t$is the random limitsof t_r, using t=(0.1,0.2,0.3,0.4, 0.5,0.6,0.7,0.8,0.9,1)

L is the number of replications which we assumed that L=1000 in our study, $\hat{R}(t_r)$ is the estimator of R(t) at the L^{th}replication.

The Bayesian estimators of R(t) are derived with respect to three loss function which are the square error loss function (SELF),precautionary error loss function (PSELF) and linear exponential error loss function(LINEX) , moreover, the informative and non-informative prior were postulated .The sample sizes n=10,50, 100 and 200 were chosen to represent small, moderate, large and very large sample sizes from the one parameter exponential distribution .The postulated values of the unique parameter θ were θ=0.5,1.5 and the values of the parameters for Gamma prior were α=0.3,1 and β=1.2,3 .

The results are presented in Tables below:

Table 1. (IMSE's) values of the reliability function estimators by using Jeffrey's prior information at θ=0.5

Estimator \ n	10	50	100	200
JSqu	0.000524	0.004635	0.004064	0.002635
JPre	0.000484	0.004524	0.004041	0.002632
JLIN	0.027725	0.010603	0.005875	0.002981

Table 2. (IMSE) values of the reliability function estimators by using Jeffrey's prior information at θ=1.5

Estimator \ n	10	50	100	200
JSqu	0.000963	0.003425	0.002309	0.001260
JPre	0.000932	0.003376	0.002301	0.001259
JLIN	0.019773	0.005140	0.002565	0.001279

Table 3. (IMSE's) of the reliability function estimators by using Gamma prior information at θ=0.5

Estimator \ n			10	50	100	200
GSqu	α=0.3	β=1.2	0.000416	0.004250	0.003941	0.002618
		β=3	0.000312	0.003806	0.003743	0.002543
	α=1	β=1.2	0.000528	0.004457	0.003954	0.002609
		β=3	0.000335	0.003899	0.003749	0.002556
GPre	α=0.3	β=1.2	0.000389	0.004178	0.003918	0.002614
		β=3	0.000308	0.003739	0.003719	0.002538
	α=1	β=1.2	0.000478	0.004384	0.003931	0.002605
		β=3	0.000320	0.003832	0.003725	0.002551
GLIN	α=0.3	β=1.2	0.027407	0.010439	0.005859	0.002981
		β=3	0.026386	0.010204	0.005817	0.002980
	α=1	β=1.2	0.028030	0.010539	0.005859	0.002981
		β=3	0.026935	0.010251	0.005830	0.002980

Table 4. (IMSE's) of the reliability function estimators by using Gamma prior information at θ=1.5.

Estimator \ n			10	50	100	200
GSqu	α=0.3	β=1.2	0.000838	0.003000	0.002209	0.001250
		β=3	0.001465	0.002462	0.002038	0.001227
	α=1	β=1.2	0.000713	0.003055	0.002217	0.001250
		β=3	0.001173	0.002497	0.002063	0.001227
GPre	α=0.3	β=1.2	0.000909	0.002949	0.002200	0.001249
		β=3	0.001665	0.002410	0.002027	0.001226
	α=1	β=1.2	0.000760	0.003004	0.002208	0.001249
		β=3	0.001336	0.002446	0.002053	0.001225
	α=0.3	β=1.2	0.018831	0.005123	0.002564	0.001279

GLIN		β=3	0.017556	0.005081	0.002564	0.001279
	α=1	β=1.2	0.019206	0.005127	0.002565	0.001279
		β=3	0.017887	0.005087	0.002564	0.001279

9. Simulation Results and Conclusions

From our simulation study, the following results are clear

- From **Table 1.** and **Table 2.**: The Bayes estimator under
- Precautionary error loss function with Jeffrey's prior is the best comparing to other estimators for all sample sizes.
- From **Table 3.**: The Bayes estimator under precautionary error loss function with Gamma prior (α=0.3, β=3) is the best comparing to other estimators for all sample sizes.
- From **Table 4.**: for (n=10) the performance of Bayes estimator under squared error loss function with Gamma prior (α=1, β=1.2) is the best, and for (n=50,100) the performance of Bayes estimator under precautionary error loss function with Gamma prior (α=0.3, β=3) is the best and for (n=200) the performance of Bayes estimator under precautionary error loss function with Gamma prior (α=1, β=3) is the best.
- That is the performance of Bayes estimator under precautionary error loss function is superior to the performance of other estimators in almost cases that are studied in this paper, where the integrated mean squared error (IMSE) is employed as a criterion to assess the performance of such estimators.

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