On θ-Totally Disconnected and θ-Light Mappings

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Abstract

In our research, we introduced new concepts, namely $\theta, \theta^*$and θ^{**}-light mappings, after we knew $\theta, \theta^*$and θ^{**}-totally disconnected mappings through the use of θ-open sets.
Many examples, facts, relationships and results have been given to support our work.

Keywords: θ-open set, light mapping, θ-homeomorphism function, θ-totally disconnected set, θ-light mapping.

Introduction

Many researchers studied the light mappings such as the world's J.J.Charatonic and K.Omiljanowski[2]. In this paper, we provide other types of light mappings namely θ-light open mapping. Other scientists who studied the light mappings are the word M. Wldyslaw [5], M. K. Fort [3] and G. Sh. mohammed [1] and others.

In our work, we needed some basic definitions. Let (X, Ʈ) be topological space and A be a subset of X, a point x∈A is said to be θ-interior point to A if x∈ $\overline{U} \subseteq A$ for some U∈ τ containing x. The set of all θ-interior points are called θ-interior set and we denoted by $\theta - int$(A), a subset U of topological pace X is θ-open if and only if every point in U is a interior point [7]. Every θ-open set is an open set but the converse may not be true in general. A space X is said to be θ-Hausdorff if for every distinct point x, y∈X there exist θ-open sets U_x, V_y containing x and y respectively such that $U_x \cap V_y = \emptyset$[4]. A mapping f:X→Y is said to be θ-open(θ^*-open and θ**-open) if f(V) is θ-open(open and θ-open) in Y, whenever V is open (θ-open) in X [6]. Let X and Y be spaces and let f be a mapping from X into Y then f is said to be θ-homeomorphism if f is bijective, continuous and θ-closed (θ-open) [6]. A space X is said to be totally disconnected space if for every pair of distinct points, a, b ∈X has a disconnection A∪B to X such that a ∈ A and b ∈ B [8]. A surjective mapping f:X→Y is said to be totally disconnected mapping if and only if for every totally disconnected set U in X, f(U) is totally disconnected set in Y [1].

Definition(1): Let X be topological space, and let A and B are nonempty θ-open sets in X, then A∪B is said to be θ-disconnection in X if and only if A∪B=X and A∩B =∅.

Definition(2): Let X be topology space, G⊆X, let A, B are nonempty θ-open sets in X, then A∪B is said to be θ-disconnection in G if and only if satisfy the following:

1- G∩A≠ ∅.

2- G∩B≠ ∅.

3-(G∩A)∩(G∩B)=∅.

4-(G∩ A)∪(G∩B)=G.

Example (3): Let X={a, b, c} and let $Ʈ_D$ is discrete topology define to X. Then {a}, {b, c} are θ-disconnection to X and {a}, {b, c} are θ-disconnection to subset {a, b} to X.

*Its known that every θ −open set s is open but the converse may be not true.

Example (4):

(R, $Ʈ_{cof}$) the open subsets of R is open set but not θ-open.

Definition(5): A topology space X is said to be θ-totally disconnected if for every two distinct point p & q there exist θ-disconnection G∪H to X such that P∈G & q∈H.

Example (6): The rational numbers with relative usual topology is a θ-totally disconnected. Since if we take q_1&q_2 ∈Q where $q_1 < q_2$ there exist r∈Q^c such that $q_1 < r < q_2$

G={x∈Q:x< r} and H={ x∈Q:x>r}

Then G∪H is θ-disconnection to Q such that $q_1 \in G$ & q_2∈H

$\overline{G}_{inG}$=G & $\overline{H}_{inH}$=H

So Q is a θ-totally disconnected.

Proposition (7): Every θ-totally disconnected set is totally disconnected.

Proof:

Let X be θ-totally disconnected space to prove X is totally disconnected space.

Let x,y∈X with x≠y. So there exist a θ-totally disconnection to X (I mean there exist G and H which are θ-open sets and G, H≠ ∅ and G∪H=X , G∩H=∅ with x∈G, y∈H).

But every θ-open set is open set soX is totally disconnected space.

Remark (8):

The converse of above proposition is not true in general but in discrete space it is availed.

Definition (9): A surjective mapping f:X→Y is said to be θ-light mapping if for every y∈Y, $f^{-1}(y)$ is θ-totally disconnected set.

Example(10): Let(Q, $Ʈ_D$) to topological space such that $Ʈ_D$ is the discrete topology define to the rational number Q and let (Q, $Ʈ_{ind}$) is the indiscrete topology such that k∈R.Let f:(Q, $Ʈ_D$)→(Q, $Ʈ_{ind}$) is a mapping define the following: f(x)=0.5 for each x∈Q note that $f^{-1}(x)$=Q if x =0.5 and $f^{-1}(x)$=∅ when x≠0.5 where ∅ and Q are θ-totally disconnected. Then f is θ-light mapping.

Remark (11): Every θ-totally disconnected is θ-hausdorff but the converse may be not true in general for example:

Example (12): (R, $Ʈ_u$) is θ-hausdorff but not θ-totally disconnected, where R is the set of real number .To show that (R, $Ʈ_u$) is not θ-totally disconnected.

Let x &y ∈Q⊆R such that x≠y, x<y.

Then ∃ p∈Q^c such that x<p<y, (p, ∞)&(-∞, p) are θ-open sets in R since P-1∈(-∞, p) there exist (-∞, p-1], p-1 ∈ (-∞, p-1] ⊆ (-∞, p) where $\overline{(-\infty, p]}$ =(-∞, p] the set (p, ∞) is similar.

(p, ∞)∩(-∞, p)=∅, but (p,∞)∪(-∞, p)≠R (R has no θ-disconnection)

So (R, $\mathcal{T}_U$) is not θ-totally disconnected.

Definition (13):_A surjective mapping f:X→Y is said to be θ-totally disconnected if and only if for every totally disconnected set U⊆X then f(U) is θ-totally disconnected in Y.

Definition (14):_A surjective mapping f:X→Y is said to be θ^*-totally disconnected mapping if and only if for every θ- totally disconnected set U⊆X then f(U) is totally disconnected

Examples (15):1-Let f: (R, $\mathcal{T}_u$)→(R, $\mathcal{T}_D$) such that f(x)=x for each x∈R .

Since (Q, $\mathcal{T}_u$) is totally disconnected set in (R, $\mathcal{T}_u$) and f(Q)=Q⊆(R,$\mathcal{T}_D$)

For each x, y∈Q there exist p∈Q^c such that x<p<y

G={x ∈Q:x<p} and H={x∈Q:x>p} are two open sets in (Q, $\mathcal{T}_u$) such that G∪H=Q, G∩H=∅

Now to prove (Q, $\mathcal{T}_D$) is θ-totally disconnected in (R, $\mathcal{T}_D$) where f(Q)=Q.

G={x∈Q:x≤0} is θ-open set in (Q, $\mathcal{T}_D$)

H={x∈Q:x>0} is θ-open set in (Q, $\mathcal{T}_D$)

H∪G=Q, H∩G=∅

So (Q, $\mathcal{T}_D$) is θ-totally disconnected in (R, $\mathcal{T}_D$) .

2- If we replace Q by (a, b] then the sets G={x ∈(a, b]:x<p} and H={x∈(a, b]:x>p} where p∈Q^c such that a<p<b then ((a, b], $\mathcal{T}_u$) is totally disconnected set in (R, $\mathcal{T}_u$)

Definition (16): A surjective mapping f:X→Y is said to be θ^{**}-totally disconnected mapping if and only if for every θ- totally disconnected set U⊆X then f(U) is θ-totally disconnected

Proposition (17): 1-Every θ-totally disconnected mapping is totally disconnected mapping.

2-Every θ-totally disconnected mapping is θ^{**}-totally disconnected mapping.

3-Every θ^{**}-totally disconnected mapping is θ^*-totally disconnected mapping.

Proof:

1-Let U be totally disconnected set in X, but f is θ-totally disconnected mapping then f(U) is θ-totally disconnection set in Y, but every θ-totally disconnected set is totally disconnected so f (U) is totally disconnected in Y, then f is totally disconnected mapping. The proof of 2 and 3 are similar.

Proposition (18): Let f:X→Y be bijective θ-open mapping. Then Y is θ-totally disconnected set whenever X is totally disconnected

Proof:

Let y_1, y_2 ∈Y with $y_1 \neq y_2$ since f is bijective, then there exist two distinct points x_1, x_2 ∈ X such that $f(x_1)=y_1$, $f(x_2)=y_2$. But X is totally disconnected space, then there exist disconnection G∪H to X such that x_1∈G & x_2∈H. also f is θ- open mapping and G, H are open sets in X. So f(G) and f(H) are θ- open sets in Y. But f(G)∪f(H)=f(G∪H)=f(X)=Y and f is one to one mapping. So f(G)∩f(H)=f(G∩H)=f(∅)=∅ Such that y_1∈f(G), y_2∈f(H) So f(G)∪f(H) is θ-disconnection to Y. therefor Y is θ-totally disconnected set.

Corollary (19): A property of space being θ-totally disconnected a topological property.

Proposition (20): Let X and Y be topological space, let f:X→Y be homeomorphism. So if X is θ- totally disconnected then Y is totally disconnected set.

Proof:

Let y_1, y_2 ∈Y with $y_1 \neq y_2$.since f is bijective, then there exist two distinct points x_1, x_2 ∈Xsuch that $f(x_1)=y_1$, $f(x_2)=y_2$. But X is θ-totally disconnected set, then there exist θ-disconnection G∪H to X such that x_1∈G & x_2∈H. also f is homeomorphism, so f is open mapping. Since G and H are θ-open sets in X. So f(G) and f(H) are open sets in Y. But f(G)∪f(H)=f(G∪H)=f(X)=Y. Since f is bijective mapping.

So f(G)∩f(H)=f(G∩H)=f(∅)=∅ Such that y_1∈f(G), y_2∈f(H) which implies f(G)∪f(H) is disconnection to Y. Therefor Y is totally disconnected set.

Proposition (21): Let f:X→Y be bijective θ^{**}-open mapping. Then Y is θ-totally disconnected set whenever X is θ-totally disconnected

Proof:

Let y_1, y_2 ∈Y with $y_1 \neq y_2$.since f is bijective, then there exist two distinct points x_1, x_2 ∈Xsuch that $f(x_1)=y_1$, $f(x_2)=y_2$. But X is θ-totally disconnected set, then there exist θ-disconnection G∪H to X such that x_1∈G & x_2∈H. also f is θ- homeomorphism, so f is θ-open mapping. Since G and H are θ- open sets in X. So f(G) and f(H) are θ- open sets in Y. But f(G)∪f(H)=f(G∪H)=f(X)=Y. Since f is bijective mapping.

So f(G)∩f(H)=f(G∩H)=f(∅)=∅ such that f(G)∪f(H) is θ-disconnection to Y. therefore Y is also θ-totally disconnected.

Corollary (22): Let X and Y be topological space, let f:X→Y be θ-homeomorphism. So if X is θ- totally disconnected then Y is θ-totally disconnected set again.

Definition (23): A surjective mapping f:X→Y is $\theta(\theta *, \theta **)$-Inversely totally disconnected, if $f^{-1}(U)$ is θ-totally disconnected (totally disconnected, θ-totally disconnected)set for every totally disconnected (θ-totally disconnected) set U in Y.

Proposition (24): 1-Every θ^{**}-Inversely totally disconnected mapping is θ^{*}-Inversely totally disconnected mapping.

2-Every θ-Inversely totally disconnected mapping is θ^{**}-Inversely totally disconnected mapping.

3-Every θ-Inversely totally disconnected mapping is θ^{*}-Inversely totally disconnected mapping.

Proof:

1-Let U is θ-totally disconnected set in Y. Since f is θ^{**}-Inversely totally disconnected mapping. $f^{-1}(U)$ is θ-totally disconnected in X (proposition 7) so $f^{-1}(U)$ is totally disconnected in X, then f is θ^{**}-Inversely totally disconnected mapping.

2-Let U is θ-totally disconnected set in Y. Then U is totally disconnected set in Y. To prove $f^{-1}(U)$ is θ-totally disconnected set in Y. Since f is θ-Inversely totally disconnected mapping, $f^{-1}(U)$ is θ-totally disconnected in X (proposition 7). Then f is θ^{**}-Inversely totally disconnected mapping

3-Let U is θ-totally disconnected set in Y. Then U is totally disconnected set in Y(proposition 7). Since f is θ^{*}-Inversely totally disconnected mapping. But $f^{-1}(U)$ is θ-totally disconnected in X so $f^{-1}(U)$ is totally disconnected in X. Then f is θ^{*}-Inversely totally disconnected mapping

Theorem (25): If f:X→Y is θ-Inversely totally disconnected mapping then f is θ-light mapping .

Proof:

Since f is θ-Inversely totally disconnected mapping to prove f is θ-light mapping. Let y∈Y to prove $f^{-1}(y)$ is θ-totally disconnected set. Since f is θ-Inversely totally disconnected mapping, and {y} is totally disconnected in Y, then $f^{-1}(\{y\})$ is θ-totally disconnected set in X so f is θ-light mapping.

Proposition (26): let f:X→Z and g:Z→Y be surjective mapping if f is θ**-inversely totally disconnected and g is θ-light mappings, then h:X→Y is θ-light mapping

Proof:

Let c∈Y so $h^{-1}(c)=(g\circ f)^{-1}(c)=(f^{-1}\circ g^{-1})(c)= f^{-1}(g^{-1}(c))$. As g is θ-light mapping so $g^{-1}(c)$ is θ-totally disconnected. Also As f is θ^{**}-Inversely totally disconnected mapping so $f^{-1}(g^{-1}(c))$ is θ-totally disconnected. $h^{-1}(c)$ is θ-totally disconnected then h is θ-light mapping.

Theorem (27): Let h:X→Y be a suriective mapping and h=g∘f such that for every f:X→Z , g:Z→Y be a surjective mappings then:

1-If h is θ-light mapping and f is θ^{**}-totally disconnected mapping then g is θ-light mapping.
2-If g is injective mapping and h is θ-light mapping then f is θ-light mapping.
3-If g be a surjective mapping and f is θ-light mapping then h is also θ-light mapping.

Proof:

1-Let y∈Y, so $h^{-1}(y)$ is θ-totally disconnected set in X as f is θ^{**}-totally disconnected mapping then $f(h^{-1}(y))$ is θ-totally disconnected set to Z, Let $f(h^{-1}(y)) = f((g\circ f)^{-1}(y)) = f((f^{-1}\circ g^{-1})(y)) = f((f^{-1}(g^{-1}(y))) = g^{-1}(y)$. So $g^{-1}(y)$ is θ-totally disconnected set to Z. In other words g is θ-light mapping.

2-Let z∈Z so g(z)∈Y since h is θ-light mapping , $h^{-1}(g(z))$ is θ-totally disconnected set to X. But $h^{-1}(g(z))=(g\circ f)^{-1}(g(z))=(f^{-1}\circ g^{-1})(g(z))=f^{-1}(z)$, So $f^{-1}(z)$ is θ-totally disconnected set in X. In other words f is θ-light mapping.

3-Let y∈Y as g is bijective mapping, then there exist only one point z∈Z such that g(z)=y. As f is θ-light mapping, then $f^{-1}(z)$ is θ-totally disconnected set to X. As $f^{-1}(z)=h^{-1}(y)$, then $h^{-1}(y)$ is also θ-totally disconnected set to X. So h is θ-light mapping.

References

1. Mohammed, G. Sh., (2001). “Open Light Mappings” M.Sc. thesis, college of Education, The University of Al- Mustansiriyah
2. Charatonic, J.J and omiljanowski ,K. (1989)., "On Light Open Mappings", Baku international Topology conference proceed, ELM, Baku
3. Fort ,M. K. (1951). "A characterization of plane light open Mappings, Amer. Math. SOC. 3
4 Saleh, M. (2004), “ On θ-closed sets and some forms of continuity”, Archivum Mathematicum (Brno), 40, 383-393.
5. Wladyslaw, M. (1994). "On Open Light Mappings", comment. M ath., university Carulinae.35
6. Mohammed, N. S. (2015). “Certain Types of Perfect Mappings", M.SC. thesis, college of Since, Al-mustansiriyah University.
7. Veliko, N. V. (1968), "H-closed topological spaces, Amer. Math. Soc. Transl. AMS.78103-118.
8. Lipschutz, S. (1965). "General Topology", Professor of Mathematics Temple University