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Common Fixed Points in Modular Spaces

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Abstract

In this paper,there are new considerations about the dual of a modular spaces and weak convergence. Two common fixed point theorems for a P-non-expansive mapping defined on a star-shaped weakly compact subset are proved, Here the conditions of affineness, demi-closedness and Opial's property play an active role in the proving our results.

Keywords: Modular spaces, fixed points, best approximations.

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1. Introduction and Preliminaries

Dotson [1]proved existence of fixed points for non-expansive self-mappings of star-shaped subsets of Banach spaces(under appropriate conditions). Subrahmanyam[2] and Habinak [3] used the concept of Banach operator to generalize Dotson's theorem and its application to invariant approximation. Recently, Abed [4] introduced the notion of best approximation in modular spaces and gave conditions to existences of proximinal and Chebysev sets in finite dimension modular spaces. Also, Abed and Abdul Sada [5-7] proved a theorem of Brosowski-Meinaraus type on invariant approximation, proved that two fixed point theorems for compact set-valued mappings in modular spaces with an application on invariant best approximation. The object of the present paper is to extend and unified the above results [2], [3], [4] and others to modular spaces. For other results in this field see [8]- [10]

Definition (1.1)[5]: Let$M$ be a linear space over$F(R \text{ or } ₵)$. A function$\gamma: M \to [0, \infty]$ is called modular if

i. $\gamma\,(v) = 0$if and only if$v = 0$;

ii. $\gamma(\alpha v) = \alpha(v)$ for $\alpha \in F$ with $|\alpha| = 1$, for all $\alpha \in F$;

iii. $\gamma(\alpha v + \beta u) \leq \gamma(v) + \gamma(u)$iff $\alpha, \beta \geq 0$, for all ,$\in M$.

If (iii) replaced by

(iii′) $\gamma(\alpha v + \beta u) \leq \alpha\gamma(v) + \beta\gamma(u)$, for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, for all v,u $\in M$

Then M modular γ is called convex modular.

Definition 1.2 [6] A modular γ defines a corresponding modular space,*then*,the space$M_\gamma$given by

$$M_\gamma = \{v \in M : \gamma(\alpha v) \to 0 \text{ whenever } \alpha \to 0\}.$$

Remark 1.1[6] by condition (iii) above, if u= 0 then $\gamma(\alpha v) = \gamma\left(\frac{\alpha}{\beta}\, \beta v\right) \leq \gamma(\beta v)$, for all $\alpha, \beta \text{ in } F,\ 0 < \alpha < \beta$.this shows that γ is increasing function.

Definition 1.3[6] The γ-ball, $B_r(u)$centered at $u \in M_\gamma$with radius $r > 0$ as

$B_r(u) = \{\boldsymbol{v} \in M_\gamma; \gamma(u - v) < r\}$.

The class of all γ-balls in a modular spaceM_γ generates a topology which makes M_γ Hausdorff topological linear space. Every γ-ball is convex set, therefore every modular space locally convex Hausdorff topological vector space [4].

Definition 1.5[6] Let M_γ be a modular spase.

a) A sequence $\{v_n\} \subset M_\gamma$is said to be γ-convergent to $v \in M_\gamma$and write $v_n \to v$ if $\gamma(v_n - v) \to 0$ as n$\to \infty$.

b) A sequence $\{v_n\}$ is called γ– Cauchy whenever $\gamma(v_n$-$v_m) \to 0$ as $, m \to \infty$.

c) $M_\gamma$is called γ– complete if any γ– Cauchy sequence in M_γ is γ– convergent.

d) A subset $B \subset M_\gamma$ is called γ– closed if for any sequence $\{v_n\} \subset B$ γ– convergent to $\in M_\gamma$, we have $v \in B$.

e) A γ– closed subset $B \subset M_\gamma$ is called γ– compact if any sequence $\{v_n\} \subset B$ has a γ– convergent subsequence.

f) A subset $B \subset M_\gamma$ is said to be γ– bounded if $daim_\gamma(B) < \infty$, where $daim_\gamma(B)$ =sup$\{\gamma(v - u);\ v, u \in B\}$ is called the γ– diameter of B.

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Definition (1.6) [7] Let M_γ be a modular space and $A \subseteq M_\gamma$ S:$A \to A$, S is called contraction mapping if ∃ h ∈ $(0\,,1)$ for all v ,u in M_γ. Such that

$$\gamma(Sv - Su) \leq h\,(v - u)$$

and if h = 1 then S is called a non –expansive mapping.

Definition (1.7): Let M_γ be a modular space and P, S: $M_\gamma \to M_\gamma$ be a mapping then S is said to be P **–** contraction if there exists h∈ $(0, 1)$ such that

$$\gamma(Sv - Su) \leq h\,\gamma(Pv - Pu)\ \forall\ v, u \text{ in } M_\gamma.$$

If h= 1 in (1.7), then S is called P**–** non– expansive mapping.

Definition (1.8)

a) A function S: $M_\gamma \to N_\delta$(whereM_γ, N_δ are modular spaces) is said to be continuous at a point $v \in M_\gamma$ if $\gamma(Sv_n - Sv) \to 0$ as n→ ∞ whenever $\delta(v_n - v) \to 0$ as n → ∞.

b) A mapping S: $M_\gamma \to N_\delta$ is said to be affine if $\forall v$, u in M_γ and $\forall \lambda$, $0 \leq \lambda \leq 1$, $S(\lambda v + (1-\lambda)u) = \lambda S(v) + (1-\lambda)S(u)$.

Definition (1.9): A two mappings S and P on M_γ are said to be commute if $SPv = PSv$ ∀ v∈ M_γ.

The purpose of this article is to prove the completeness of dual space of a modular space and to give some related concepts and properties, also, to prove the existence of common fixed points for pair mapping S, P where S is P − non − expansive.

2. Dual of a modular space

let P be a linear functional with domain in a modular space M_γ and range in the scalar field K P:$D(P) \to K$, P is bounded linear functional c such that for all $v \in D(P)$, $\gamma(Pv) \leq c\gamma(v)$. The set of all bounded linear functional on M_γ, $\boldsymbol{M'_\gamma}$ is linear space with point-wise operations. In the following, we reform some concepts about dual space in the setting of modular spaces, we begin with following:

Proposition (2.1): Let $P \in \boldsymbol{M'_\gamma}$, define $\boldsymbol{\gamma} : \boldsymbol{M'_\gamma} \to R^+$ ∋ $\boldsymbol{\gamma}(P) = \sup\{\gamma(Pv) : \gamma(v) = 1\}$ then

i. $\boldsymbol{\gamma}(\alpha P) = \boldsymbol{\gamma}(P)$, for $\alpha \in$ K with $|\alpha| = 1$
ii. $\boldsymbol{\gamma}(\alpha P + \beta Q) \leq \boldsymbol{\gamma}(P) + \boldsymbol{\gamma}(Q)$,
iii. $\boldsymbol{\gamma}(P) = 0$ iff $P = 0$.

Proof: For (i) $\boldsymbol{\gamma}(\alpha P) = \sup\{\gamma(\alpha Pv)\} = \sup\{\gamma(Pv)\} = \boldsymbol{\gamma}(P)$.

For (ii) $\boldsymbol{\gamma}(\alpha P + \beta Q) = \sup\{\gamma(\alpha Pv + \beta Qv)\}$

$$\leq \sup\{\gamma(Pv) + \gamma(Qv)\}$$
$$= \sup\{\gamma(Pv)\} + \sup\{\gamma(Qv)\}$$

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$$= \boldsymbol{\gamma}(P) + \boldsymbol{\gamma}(Q)$$

For (iii) $\boldsymbol{\gamma}(P) = 0$ iff $\sup\{\gamma(Pv) : \gamma(v) = 1\}$ iff $\gamma(Pv) = 0$ for all$v$ iff $P = 0$.

A modular $\boldsymbol{\gamma}$ defines a corresponding modular space,$i.e.$,the space$\boldsymbol{M}'_{\boldsymbol{\gamma}}$ given by

$$\boldsymbol{M}'_{\boldsymbol{\gamma}} = \{v \in M : \boldsymbol{\gamma}(\alpha P) \to 0 \ whenever \ \alpha \to 0\}$$

Theorem (2.2): $\boldsymbol{M}'_{\boldsymbol{\gamma}}$ is complete modular space.

Proof: We consider an arbitrary Cauchy sequence (S_n) in $\boldsymbol{M}'_{\boldsymbol{\gamma}}$ and show that (S_n) converges to a $S \in \boldsymbol{M}'_{\boldsymbol{\gamma}}$ Since (S_n) is Cauchy, for every $\epsilon > 0$ there is an L such that

$$\boldsymbol{\gamma}(S_n - S_m) < \in, \qquad (n, m > L),$$

For any $v \in M_\gamma$ and $n, m > L$, this implies that

$$|S_n v - S_m v| = |(S_n - S_m)v| \leq \gamma(S_n - S_m)\gamma(v) \leq \in \gamma(v). \qquad \dots (2.1)$$

Now, for any fixed point v and given $\in'$ we may choose $\in = \in_v$ so that $\in_v \ \gamma(v) < \in'$.

Then from (2.1), we have $|S_n v - S_m v| < \in'$ and $(S_n v)$ is Cauchy in K. By completeness of K, $(S_n v)$ converges, say, $S_n v \to r$. Clearly, the limit $r \in K$ depends on the choice of $v \in M_\gamma$.

This defines a functional S: $M_\gamma \to K$ where $r = Sv$ .The functional S is linear since $\lim_{n\to\infty} S_n(\alpha v + \beta z) = \lim_{n\to\infty}(\alpha S_n v - \beta S_n z) = \alpha \lim_{n\to\infty} S_n v + \beta \lim_{n\to\infty} S_n z$. We prove that S is bounded and $S_n \to S$, that is $\boldsymbol{\gamma}(S_n - S) \to 0$.

Since (2.1) holds for every $m > L$ and $S_m v \to S$, we may let $m \to \infty$. Using the continuity of the modular, then for every $n > L$ and all $v \in M_\gamma$.

$$|S_n v - Sv| = \left|S_n v - \lim_{m\to\infty} S_m v\right|$$

$$= \lim_{m\to\infty} |S_n v - S_m v|$$

$$\leq \epsilon\gamma(v) \qquad \dots (2.2)$$

This shows that $(S_n - S)$ with $n > L$ is a bounded linear functional. Since S_n is bounded, $S = S_n - (S_n - S)$ is bounded, that is, $S \in \boldsymbol{M}'_{\boldsymbol{\gamma}}$. Furthermore, if in (2.2) we take the supremum over all v of modular 1, we obtain

$$\gamma(S_n - S) \leq \epsilon, \ n > L.$$

Hence $\boldsymbol{\gamma}(S_n - S) \to 0$. This completes proof.

Definition (2.3): A sequence (v_n) in a modular space M_γ is said to be weakly convergent if there is an $v \in M_\gamma$ such that for every $P \in \boldsymbol{M}'_{\boldsymbol{\gamma}}$

$$\lim_{n\to\infty} \gamma(Pv_n - Pv) = 0 \qquad \text{This denoted by } v_n \xrightarrow{w} v.$$

Proposition (2.4): In a modular space M_γ, every convergent sequence is weakly convergent.

Proof: By definition, $v_n \to v$ means $\gamma(v_n - v) \to 0$ and implies that for every P∈ $\boldsymbol{M}'_{\gamma}$,

$$|P(v_n) - P(v)| = |P(v_n - v)| \leq \gamma(P)\gamma(v_n - v) \to 0.$$

This shows that $v_n \overset{w}{\to} v$.

Note that, the converse of proposition (2.4) is not necessary true. To show this recall the usual case is in a normed space.In the following some other needed properties of weak convergence are given:

Proposition (2.5): Let (v_n) be weakly convergent sequence in a modular space M_γ, say $v_n \overset{w}{\to} v$ Then:

i. The weak limit v of (v_n) is unique.
ii. Every subsequence of (v_n) converges weakly to v.

Proof: For (i), suppose that $v_n \overset{w}{\to} v$ as well as $v_n \overset{w}{\to} u$. Then $P(v_n) \to P(v)$ as well as $P(v_n) \to P(u)$**.** Since $\big(P(v_n)\big)$ is a sequence of numbers, its limit is unique. Hence $P(v) = P(u)$, that is, for every P∈ $\boldsymbol{M}'_{\gamma}$.We have $P(v) - P(u) = P(v-u) = 0$.This implies $v-u = 0$ and shows that the weak limit is unique. Part (ii) follows from the fact that $\big(P(v_n)\big)$ is convergent sequence of numbers. So that every subsequence of $\big(P(v_n)\big)$ converges and has same limit as the sequence.

Definition (2.6): A a subset of a modular space M_γ is said to be weakly compact if every sequence in M_γ has a weak convergent subsequence.

Definition (2.7): Let M_γ, N_ρ be two modular spaces and $S : M_\gamma \longrightarrow N_\rho$ be mappings then:

i. S is continuous if $v_n \longrightarrow v \Rightarrow S(v_n) \longrightarrow S(v)$.
ii. S is weakly continuous if $v_n \overset{w}{\to} v \Rightarrow S(v_n) \overset{w}{\to} S(v)$.

Definition (2.8): Let M_γ be a modular space, $A \subseteq M$ and S: $A \to M_\gamma$ be a mapping, S is called demi-closed of $v \in A$, if for every sequence (v_n) in A such that $v_n \overset{w}{\to} v$ and $v_n \to u \in M_\gamma$ then $u = Sv$ and S is demi closed on A if it is demi-closed of each v in A.

Definition (2.9): Let M_γ be a modular space, M_γ is said to be Opial if for every sequence (v_n) in M_γ weakly convergent to $v \in M_\gamma$ the inequality

$$\lim_{n\to\infty} inf\, \gamma(v_n - v) < \lim_{n\to\infty} inf \gamma(v_n - u)$$

holds for all $u \neq v$.

3. Common fixed point for commuting mappings

Mongkolkeha, Sintunavarat and Kumamstudy[11]and [12] proved the existence theorems of fixed points for contraction mappings in modular metric spaces with condition $\gamma(P(v)) < \infty$ to guarantee the existence and uniqueness of the fixed points. We start with following

Proposition (3.1): Let P be a continuous self-mapping of a complete modular space (M_γ, γ) if S: $M_\gamma \to M_\gamma$is P- contraction mapping which commutes with P and $S(M) \subseteq P(M)$ and $\exists\, v \in M_\gamma$ such that $\gamma(P(v)) < \infty$ then $F(P) \cap F(S) =$ singleton.

Proof: Suppose $p(a) = a$ for some $a \in M_\gamma$, define S: $M_\gamma \to M_\gamma$ by $S(v) = a\ \forall\, v \in M_\gamma$ then $S(P(v)) = a$ and $P(S(v)) = P(a)$ for all $v \in M_\gamma$ so $S(P(v)) = P(S(v)), \forall\, v \in M_\gamma$ and S commutes with P moreover $S(v) = a = P(a)\ \forall\ v \in M_\gamma$ so that $S(M) \subseteq P(M)$. Finally, $\forall a \in (0,1)$, $\forall\ v,u$ in $M_\gamma$we have

$$\gamma(S(v), S(u)) = \gamma(a, a) = 0 \ \leq a\, \gamma(P(v), P(u)).$$

This completes the proof.

Now, it is easy to show that the following needed lemma.

Lemma (3.2): Let $M_\gamma$be a modular space, S: $M_\gamma \to M_\gamma$ be mapping, and $u \in M$. If

$S(hu + (1-h)v) = hSu + (1-h)v, \forall v \in M_\gamma$ and $h \in (0,1)$, then u is a fixed point.

Theorem (3.3): Let $\emptyset \neq A$ weakly compact subset of a complete modular space M_γ . Let p be a continuous and affine mapping on M_γ with $p(A) = A$, S: $A \to A$ be an P- non – expansive mapping commutes with P. If A is star-shaped with respect to S,and there is some $v \in A$ $\gamma(S(v)) < \infty$ and $(P - S)$ is demi-closed on M_γ, then $F(S) \cap F(P) \neq \emptyset$.

Proof: Since A is star-shaped with respect to $u \in A$, then S: $A \to A$, we define S_n on A for any v in A by, $S_n(v) = h_n Sv + (1 - h_n)u$ and there is $u \in A$, and the sequence $h_n \to 1$ as $n \to \infty$, $0 < h_n < 1$ such that $(1 - h_n)u + h_n Sv \in A\ \forall\ v,u \ \in A$. It is clear that $S_n : A \to A$.

Note that $S(A) \subseteq A$ and $S_n(A) \subseteq p(A)$. Since S commutes with P and P is affine mapping, for each $v \in A$.

$$\begin{aligned} S_n Pv &= h_n Spv + (1 - h_n)Pu \\ &= h_n PSv + (1 - h_n)Pu \\ &= P(h_n Sv + (1 - h_n)) \\ &= PS_n v \end{aligned}$$

$\ni S_n$commutes with P. Further, we observe that for each $n \geq 1$, S is P- non-expansive mapping,

$\gamma(S_n v - S_n u) = \gamma(h_n S v + (1-h_n)u - h_n S u - (1-h_n)u)$

$= h_n \gamma(Sv - Su)$

$\leq h_n \gamma(Pv - Pu)$

$\forall\, v, u \in A$ hence $S_n$is P- contraction. Thus by proposition (3.1),

there is a unique $v_n \in A$ such that $v_n = S_n = Pv_n$ for all $n \geq 1$.

Since A is weakly compact, there is a subsequence (v_{n_i}) of sequence (v_n) which converges weakly to some $v_0 \in A$.

Since P is a continuous affine mapping then P is weakly continuous and so, since $Sv_{ni} = \frac{S_{ni}\, v_{ni}\ +(1-h_{ni})u}{h_{ni}}$ and $Pv_{ni} = v_{ni}$.

Now, $(P-S)v_{ni} = Pv_{ni} - Sv_{ni}$

$= v_{ni} - \left(\frac{S_{ni} v_{ni} - (1-h_{ni})u}{h_{ni}}\right)$

$= \frac{h_{ni} v_{ni} - S_{ni} v_{ni} + (1-h_{ni})u}{h_{ni}}$

$= \frac{-v_{ni}(1-h_{ni}) + (1-h_{ni})u}{h_{ni}}$

$= \frac{(1-h_{ni})(u - v_{ni})}{h_{ni}}$

$= \frac{(1-h_{ni})}{h_{ni}}(u - v_{ni})$

$= \left(\frac{1}{h_{ni}} - 1\right)(u - v_{ni})$

Therefore $(P-S)v_{ni} = \left(\frac{1}{h_{ni}} - 1\right)(u - v_{ni})$

Thus $(P-S)v_{ni} = \left|\frac{1}{h_{ni}} - 1\right| \gamma(u - v_{ni}) \leq \left|\frac{1}{h_{ni}} - 1\right| [\gamma(v_{ni}) + \gamma(u)]$.

Since A is bounded, $v_{ni} \in A$ implies $(\gamma(v_{ni}))$ is bounded and so by the fact that $h_{ni} \to 1$,

We have $\gamma(P-S)v_{ni} \to 0$

Now, since $P{-}S$ is demi-closed then $(P-S)v_0 = 0$ and thus $Pv_0 = v_0 = Sv_0$. Hence, $F(S) \cap F(P) \neq \emptyset$.

Another common fixed point theorem will be given for Opial's space.

Theorem (3.4): Let$\emptyset \neq A$ weakly compact subset of Opia's complete modular space M_γ . Let P be a continuous and affine mapping on M_γ with $P(A) = A$, S: $A \to A$ be P- non-

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expansive mapping commutes with P. If A has star-shaped with respect to S, then $F(S) \cap F(P) \neq \emptyset$.

Proof: Since A has star-shaped then S:$A \rightarrow A$ and there is $u \in A$ and the sequence $h_n \rightarrow 1$, as $n \rightarrow \infty$, $(0 < h_n < 1) \ni (1 - h_n)u + h_n Sv \in A$ for all $v \in A$. Now, define $S_n$on A for any v in A by, $S_n(v) = h_n Sv + (1 - h_n)u$ and there is $u \in A$, it is clear that $S_n: A \rightarrow A$. Note that $S(A) \subseteq A$ and $S_n(A) \subseteq p(A)$. Since S commutes with p and p is affine mapping, for each $v \in A$.

$$S_n Pv = h_n SPv + (1 - h_n)Pu$$
$$= h_n PSv + (1 - h_n)Pu$$
$$= P(h_n Sv + (1 - h_n)u)$$
$$= PS_n v$$

Thus each h_n commutes with P. Further observe that for each $n \geq 1$, S is P – non-expansive mapping.

$$\gamma(S_n v - S_n u) = \gamma(h_n Sv + (1 - h_n)u - h_n Su - (1 - h_n)u)$$
$$= h_n \gamma(Sv - Su)$$
$$\leq h_n \gamma(Pv - Pu)$$

$\forall\, u \in A$, hence S_n is P- contraction.

Thus by proposition (3.1), there is a unique $v_n \in A$ such that $v_n = S_n v_n = Pv_n$ for all $n \geq 1$. Since A is weakly compact, there is a subsequence (v_{ni}) of sequence (v_n) which converges weakly to some $v_0 \in A$. Since P is a continuous affine mapping then P is weakly continuous and so we have:

$$Pv_0 = \lim_{i \to \infty} Pv_{ni} = \lim_{i \to \infty} v_{ni} = v_0$$

Since $Sv_{ni} = \frac{S_{ni} v_{ni} + (1 - h_{ni})u}{h_{ni}}$ and $Pv_{ni} = v_{ni}$, we have:

$$(P - S)v_{ni} = Pv_{ni} - Sv_{ni}$$
$$= v_{ni} - \left(\frac{S_{ni} v_{ni} + (1 - h_{ni})u}{h_{ni}}\right)$$
$$= \frac{h_{ni} v_{ni} \; - v_{ni} + (1 - h_{ni})u}{h_{ni}}$$
$$= \frac{-v_{ni}(1 - h_{ni}) + (1 - h_{ni})u}{h_{ni}}$$
$$= \frac{(1 - h_{ni})(u - v_{ni})}{h_{ni}}$$
$$= \frac{(1 - h_{ni})}{h_{ni}}(u - v_{ni})$$

$(P-S)v_{ni} = \left(\frac{1}{h_{ni}} - 1\right)(u - v_{ni})$

Therefore $(P-S)v_{ni} = \left(\frac{1}{h_{ni}} - 1\right)(u - v_{ni})$.

Thus $\gamma(P-S)v_{ni} = \left|\frac{1}{h_{ni}} - 1\right| \gamma(u - v_{ni}) \le \left|\frac{1}{h_{ni}} - 1\right| [\gamma(v_{ni}) + \gamma(u)]$.

Since A is bounded by A is weakly compact, $v_{ni} \in A$ implies $\left(\gamma(v_{ni})\right)$ is bounded and so by the fact that $h_{ni} \to 1$, we have $\gamma(P-S)v_{ni} \to 0$

Now, since M_γ is Opial space and suppose that, $Sv_0 \neq v_0$ we have:

$\lim_{i\to\infty} inf\gamma(v_{ni} - v_0) < \lim_{i\to\infty} \inf\gamma(v_{ni} - Sv_0)$

$$= \lim_{i\to\infty} \inf\gamma(Sv_{ni} + (P-S)v_{ni} - Sv_0)$$

$$\le \lim_{i\to\infty} \inf\gamma(Sv_{ni} - Sv_0) + \lim_{i\to\infty} inf\gamma(P-S)v_{ni},$$ since $v_{ni} = (P-S)v_{ni} + Sv_{ni}$. And thus

$$\lim_{i\to\infty} inf\gamma(v_{ni} - v_0) < \lim_{i\to\infty} \inf\gamma(Sv_{ni} - Sv_0)$$

But on the other hand, we have

$\lim_{i\to\infty} \inf\gamma(Sv_{ni} - Sv_0) \le \lim_{i\to\infty} \inf\gamma(Pv_{ni} - Pv_0) = \lim_{i\to\infty} \inf\gamma(v_{ni} - v_0)$

This is a contradiction. Hence $v_0 \in F(S) \cap F(P) \Rightarrow F(S) \cap F(P) \neq \emptyset$.

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