Some Results on Fuzzy Zariski Topology on Spec(μ)

R. N. Majeed
Department of Mathematics,College of Education
Ibn Al- Haitham, University of Baghdad

Abstract

The aim of this paper is to study some properties of fuzzy zariski topology on spec(μ) and find a subspace of it and defined a base for this subspace .Also, we prove the fuzzy zariski topology is T_1-space.

Preliminaries

Throughout this paper, R denotes a commutative ring with identity, and I is the unite interval [0,1], let $\mu:R\to I$ be a fuzzy subset of R (1). If $x\in R$ and $t\in[0,1]$, then the fuzzy subset x_t of R defined by $x_t(y)=t$ if $x=y$ and $x_t(y)=0$ if $x\neq y$ is called a fuzzy singleton (2). Let μ be a fuzzy set of R defined by $\mu(x)=0$ $\forall x\in R$, then μ is called empty fuzzy set denoted by Φ (1). Let μ and μ′ be two fuzzy subsets of R, we say that $\mu\subseteq\mu'$ if and only if $\mu(x)\leq\mu'(x)$ $\forall x\in R$ (3) , the intersection of μ and μ′ is defined by $(\mu\cap\mu')(x)=\min\{\mu(x),\mu'(x)\}\forall x\in R$, and the union is defined by $(\mu\cup\mu')(x)=\max\{\mu(x),\mu'(x)\}\forall x\in R$ (4). For $t\in[0,1]$ $\mu_t=\{x\in R, \mu(x)\geq t\}$is called a level subset of the fuzzy set μ and if $x\in\mu_t$ then $x_t\subseteq\mu$ (5). A fuzzy set μ of R is called a fuzzy ring of R if and only if $\forall x, y\in R$ $\mu(x-y)\geq\min\{\mu(x), \mu(y)\}$ and $\mu(xy)\geq\min\{\mu(x) , \mu(y)\}$ (6), if μ is a fuzzy ring of R then a fuzzy ideal of μ is a fuzzy set $\delta:R\to I$ such that the following properties hold: $\forall x, y\in R$ $\delta(x-y)\geq\min\{\delta(x), \delta(y)\}$, $\delta(xy)\}\geq\min\{\mu(x), \delta(y)\}$ and $\delta(x)\leq\mu(x)$ (7).

Definition 1.1 (7) :

A fuzzy ideal ρ of a fuzzy ring μ is said to be prime if $\rho \neq \lambda_R$ (where λ_R denotes the characteristic function of R, such that $\lambda_R(x)=1$ $\forall x \in R$) and it satisfies:
$\min\{\rho(xy),\mu(x),\mu(y)\} \leq \max\{\rho(x),\rho(y)\}$ $\forall x, y \in R$.

Proposition 1.2 (8) :

Given a fuzzy ring μ of R and a fuzzy set $\delta: R \rightarrow I$, the set δ is a fuzzy ideal of μ if and only if δ_t is an ideal of μ_t, for all $t \in [0, \min\{\delta(0),\mu(0)\}]$.

Proposition 1.3 (8) :

Given a fuzzy ring μ of R and a fuzzy set $\rho: R \rightarrow I$, the set ρ is prime fuzzy ideal of μ if and only if ρ_t is a prime ideal of μ_t, for all $t \in [0, \min\{\rho(0),\mu(0)\}]$.

Defenition 1.4 (9) :

Let $\mu : R \rightarrow I$ be a fuzzy ring , the set $X = \text{spec}(\mu) = \{\rho \mid \rho$ is a prime fuzzy ideal of $\mu\}$ is called the spectrum of μ .

Defenition 1.5 (9) :

For each proper fuzzy ideal δ of μ ,let

1. $V(\delta) = \{ \rho \in \text{spec}(\mu) \mid \delta \subseteq \rho \}$.
2. $X(\delta) = X \sim V(\delta)$, the complement of $V(\delta)$ in X .

Proposition 1.6 (8):

Let $\mu : R \rightarrow I$ be a fuzzy ring , then :

1. (i) $V(\Phi) = \text{spec}(\mu)$
   (ii) $V(\mu) = \varnothing$, where $\varnothing$ is empty set .

2. If $\delta_1 \subseteq \delta_2$, then $V(\delta_2) \subseteq V(\delta_1)$,$\forall$ δ_1 ,δ_2 fuzzy ideals of μ.
3. $V(\cup\delta_i \mid i \in \Lambda) = \cap\{V(\delta_i) \mid i \in \Lambda\}$,for any collection$\{ \delta_i \mid i \in \Lambda \}$ of fuzzy ideals of μ.
4. $V(\delta_1 \cap \delta_2) = V(\delta_1) \cup V(\delta_2)$ for any fuzzy ideals $\delta_1$and δ_2 of μ.

Definition 1.7 (9) :

Let $\mu : R \rightarrow I$ be a fuzzy ring, let δ be a fuzzy subset of μ and $\langle\delta\rangle$ the intersection of all fuzzy ideals δ' of μ , such that $\delta \subseteq \delta'$. then $\langle\delta\rangle$ is called fuzzy ideal of μ generated by δ.

i.e. $\langle\delta\rangle = \cap\{\delta' : \delta'$ is a fuzzy ideal of μ and $\delta \subseteq \delta'\}$

Proposition 1.8 (9):

Let $\mu : R \rightarrow I$ be a fuzzy ring, let δ be a fuzzy subset of μ then: $V(\delta)=V(\langle\delta\rangle)$.

Theorem 1.9 (8):

Let $\mu : R \rightarrow I$ be a fuzzy ring, let $X = \text{spec}(\mu)$. Let $T=\{X(\delta) : X(\delta) = X\sim V(\delta), \delta$ is a fuzzy ideal of $\mu\}$ then the pair $(\text{spec}(\mu), T)$ is a topological space, which is called fuzzy zariski topology on $\text{spec}(\mu)$.

Lemma 1.10 (8):

$V(\delta)$ is closed subset in the topological space $(\text{spec}(\mu), T)$ for any fuzzy ideal δ of μ.

Theorem 1.11 (8):

The subfamily $\{ X(x_t) \mid x \in R, t \in (0,1] \}$ of T is a base for T. (where x_t is a fuzzy singleton of μ).

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Proposition 2.1 :

$V(x_t) \cap V(y_t) = V((x+y)_t)$, where $x, y \in R$ and $t \in (0,1]$.

Proof :

Let $\rho \in V(x_t) \cap V(y_t) \Leftrightarrow \rho \in V(x_t)$ and $\rho \in V(y_t)$

$\Leftrightarrow x_t \subseteq \rho$ and $y_t \subseteq \rho$ (by defenition 1.5)

$\Leftrightarrow x \in \rho_t$ and $y \in \rho_t$

and since ρ_t is a prime ideal of R, and $x, y \in R$ we have

$\Leftrightarrow x + y \in \rho_t$

$\Leftrightarrow (x+y)_t \subseteq \rho$

$\Leftrightarrow \rho \in V((x+y)_t)$

Hence, $V(x_t) \cap V(y_t) = V((x+y)_t)$. ■

Proposition 2.2 :

$V(x_t) \cup V(y_t) = V((xy)_t)$, where $x,y \in R$, $t \in (0,1]$.

Proof :

Let $\rho \in V(x_t) \cup V(y_t) \Leftrightarrow \rho \in V(x_t)$ or $\rho \in V(y_t)$

$\Leftrightarrow x_t \subseteq \rho$ or $y_t \subseteq \rho$ (by definition 1.5)

$\Leftrightarrow x \in \rho_t$ and $y \in \rho_t$

Since ρ_t is a prime ideal of R ,and x, y $\in$R .Thus

$\Leftrightarrow x\,y \in \rho_t$

$\Leftrightarrow (xy)_t \subseteq \rho$

$\Leftrightarrow \rho \in V((xy)_t)$

Hence, $V(x_t) \cup V(y_t) = V((xy)_t)$. ■

Theorem 2.3 :

$X(x_t)=\varnothing$ if and only if x is nilpotent element, where $x \in R$ and $t\in(0,1]$

Proof :

Suppose $X(x_t)=\varnothing$, this means $X\sim V(x_t)=\varnothing$ (by definition 1.5) then $V(x_t)=X$ which implies $x_t \subseteq \rho$ for all $\rho\in X$, and therefore $x\in\rho_t$,which is a prime ideal of μ_t (by proposition 1.3) .
Hence $x\in \cap\{\rho_t | \rho_t$ is a prime ideal of $\mu_t\}$, and since
$\cap\{\rho_t | \rho_t$ is a prime ideal of $\mu_t\} = r(\mu_t)$ = the set of all nilpotent element
Hence x is nilpotent element.

Conversely, assume that x is a nilpotent element.
Let $\rho\in spec(\mu)$, then ρ_t is a prime ideal of μ_t (by proposition 1.3), and so $x\in\rho_t$. Therefore $x_t\subseteq\rho$ for all $\rho\in X$, thus $V(x_t)=X$ which implies $X(x_t)=\varnothing$. ■

Theorem 2.4 :

If $X(x_t)=X$ then x is a unit, where $x\in R$ and $t\in(0,1]$.

Proof :

Since $X(x_t)=X=spec(\mu)$ then $V(x_t)=\varnothing$ which implies $x_t \not\subset \rho$ for all $\rho\in X$, that is mean $\exists$ y $\in$R such that $\rho(y) < x_t(y)$,and by definition of fuzzy singleton we have $\rho(y) < t$ if x =y or $\rho(y) < 0$ if $x \neq y$ (which is not posipole), thus $\rho(x) < t$ so that $x\notin\rho_t$
Hence $x\notin\cup\{\rho_t | \rho_t$ is a prime ideal of μ_t },consequently, x is a unit.
■

Now, we define a subspace of X=spec(μ) and prove it is a compact space.

Proposition 2.5 :

Let X=spec(μ),T={X(δ) =X~V(δ)| δ is a fuzzy ideal of μ},let A={$\rho\in$X | Imρ ={α,1},$\alpha\in$(0,1]}, and let T'={X(δ)$\cap$A | X(δ)$\in T$} then T' is a topology on A and the pair (A, T') is a subspace of (X, T).

Proof:

To prove (A, T') is a topological space, we must prove T' is a topology on A, that's mean we must satisfy the following conditions:

1. $\varnothing$, $A\in T'$.(where $\varnothing$ is the empty set).
2. {X(δ_1) $\cap A$}$\cap${X(δ_2) $\cap A$}$\in T'$ for each X(δ_1) $\cap A$, X(δ_2)$\cap A\in T'$.
3. $\cup${X(δ_i) $\cap A$ | i$\in\Lambda$}$\in T'$, for any family {X(δ_i) $\cap A$ | i$\in\Lambda$} in T'.

Now, **1**. Consider the fuzzy ideals Φ and μ of μ . Since X(Φ)=$\varnothing\in T$(by theorem 1.9), then X(Φ) $\cap$A$\in$T′, implies $\varnothing\in$T′.
Since X(μ)=X$\in T$ (by theorem 1.9), then X(μ)$\cap A\in T'$, implies X$\cap A\in T'$, thus $\varnothing$,$A\in T'$.

2.LetX(δ_1)$\cap A$, X(δ_2)$\cap A\in T'$,we must prove{X(δ_1)$\cap$A}$\cap${X(δ_2)$\cap$A}$\in T'$.
Now,
{X(δ_1)$\cap A$} $\cap${X(δ_2)$\cap A$} ={X(δ_1) $\cap$ X(δ_2)}$\cap A$,
Since X(δ_1)$\cap$X(δ_2)=X($\delta_1\cap\delta_2$)$\in T$ (by theorem1.9),then X($\delta_1\cap\delta_2$)$\cap A$ $\in T'$. Thus {X(δ_1) $\cap A$}$\cap${X(δ_2) $\cap A$}$\in T'$.

3.We must prove $\cup${X(δ_i) $\cap A$| i$\in\Lambda$}$\in T'$, for any family {X(δ_i)$\cap A$| i$\in\Lambda$}$\in T'$.
Since $\cup${X(δ_i)| i$\in\Lambda$}=X(< $\cup\delta_i$ >| i$\in\Lambda$)$\in T$ (by theorem 1.9), then $\cup${X(δ_i) $\cap A$| i$\in\Lambda$} = X(< $\cup\delta_i$ >| i$\in\Lambda$) $\cap A\in T'$
Therefore T' define a topology on A and the pair (A, T') is a topological space which is subspace of (X, T). ■

roposition 2.6 :

Let X=spec(μ),T={X(δ) =X~V(δ)| δ is a fuzzy ideal of μ},
let A={$\rho\in$X| Imρ ={α,1},$\alpha\in$(0,1]} be a subspace of X, then the subfamily {X(x_β) $\cap A$ | x$\in$R, $\beta\in$(α,1]} of T' is a base for A.

Proof :

Let $X(\delta)\cap A\in T'$, for some $X(\delta)\in T$, since the subfamily $\{X(x_t)\,|\,x\in R, t\in(0,1]\}$ is a base for T (by theorem 1.11), then

$X(\delta)\cap A=(\cup X(x_\beta)\,|\,x_\beta\subseteq\delta)\cap A$, where $\beta\in(0,1]$

$=\cup(X(x_\beta)\cap A\,|\,x_\beta\subseteq\delta)$

now, if $\beta>\alpha$, then $X(x_\beta)\cap A\neq\varnothing$

if $\beta<\alpha$, then $X(x_\beta)\cap A=\varnothing$

Thus $X(\delta)\cap A=\cup(X(x_\beta)\cap A\,|\,x_\beta\subseteq\delta$ and $\beta>\alpha)$

Hence, the subfamily $\{X(x_\beta)\cap A\,|\,x\in R, \beta\in(\alpha,1]\}$ is a base for A. ■

Theorem 2.7 :

Let $\alpha\in[0,1)$ and let $A=\{\rho\in X\,|\,\text{Im}\rho=\{\alpha,1\}\}$, then the subspace A is compact.

Proof :

To prove A is compact, we must prove any open cover of A is reducible to a finite sub cover of A.

(By proposition 2.6) we show that the family $\{X(x_\beta)\cap A\,|\,x\in R, \beta\in(\alpha,1]\}$ constitutes a base for A.

Now, let $\{X((x_i)_t)\cap A\,|\,i\in\Lambda, t\in k\subset(\alpha,1]\}$ be any covering of A

Let $\beta=\sup\{t\,|\,t\in k\}$.Then$\{X((x_i)_\beta)\cap A\,|\,i\in\Lambda\}$is also a cover of A.

Now, $A=\cup\{X((x_i)_\beta)\cap A\,|\,i\in\Lambda\}$

$=(\cup\{X((x_i)_\beta)\,|\,i\in\Lambda\}\cap A$

$=(X\sim V(\cup\{(x_i)_\beta\,|\,i\in\Lambda\})\cap A$

$=A\sim(V(\{\cup(x_i)_\beta\,|\,i\in\Lambda\})\cap A$

this shows that $V(\{\cup(x_i)_\beta\,|\,i\in\Lambda\})\cap A=\varnothing$

Now, $V(\{\cup(x_i)_\beta\,|\,i=1,2,\ldots,n\})\cap A=\varnothing$ because if

$\exists\rho\in V(\{\cup(xi)_\beta\,|\,i=1,2,\ldots,n\})\cap A$ hold, then

$\cup\{(x_i)_\beta\,|\,i=1,2,\ldots,n\}\subseteq\rho$ and $\text{Im}\,\rho=\{\alpha,1\}$ which imply,

$(x_i)_\beta(x)\leq\rho(x)$ for all $x\in R$ and for all i= 1,2,…,n .And by definition of fuzzy singleton we have: $\beta<\rho(x)$ if $x=xi$ and $0<\rho(x)$ if $x\neq xi$ for all i =1,2,..,n .Thus $\beta<\rho(xi)$ for all i= 1,2,…,n ,and since $\beta>\alpha$, then $\rho(x_i)=1$, for all i=1,2,..,n . But then $\rho=\lambda_R$ which is contradiction (by definition 1.1).

Thus the family $\{X((x_i)_\beta)\cap A\,|\,i=1,2,..n\}$covers A. Hence A is compact. ■

Theorem 2.8 :

The space X=spec(μ) is a T_1-space.

Proof :

To prove X is T_1-space, that's mean we must prove for any distinct points ρ_1,ρ_2 of X, there exists an open set in X containing ρ_1 but not ρ_2, and an open set in X containing ρ_2 but not ρ_1.

Let $\rho_1,\rho_2 \in X$ such that $\rho_1 \neq \rho_2$, then either $\rho_1 \not\subset \rho_2$ or $\rho_2 \not\subset \rho_1$.

Let $\rho_1 \not\subset \rho_2$, then $\rho_2 \notin V(\rho_1)$ (by definition 1.5) implies $\rho_2 \in X(\rho_1)$ and $\rho_1 \notin X(\rho_1)$ but $X(\rho_1)$ is open in X, thus there exists an open set $X(\rho_1)$ containing ρ_2 but not ρ_1.

And similarly if $\rho_2 \not\subset \rho_1$. ■

Theorem 2.9:

The space(X= Sepc(μ), T) is a T_1-space if and only if each $\rho \in X$, ρ is closed subset of X.

Proof:

Let (X, T) be a T_1-space and consider any prime fuzzy ideal $\rho \in X$. We show that $\{\rho\}$ is closed by showing $X \sim \{\rho\}$ is open .

Let $\rho' \neq \rho$ be any prime fuzzy ideal in X, since X is T_1-space then (by theorem 2.8) there exists two open sets $X(\rho')$, $X(\rho)$ such that $\rho' \in X(\rho)$ and $\rho \notin X(\rho)$, and $\rho \in X(\rho')$ and $\rho' \notin X(\rho')$.

Since $\rho \notin X(\rho)$ then $X(\rho) \subseteq X \sim \{\rho\}$, therefore (by theorem (10) : the set U is open in (X,τ) ⇔ for each x∈U there exists an open set V(x)⊂ U) thus $X \sim \{\rho\}$ open and, consequently, $\{\rho\}$ is a closed subset of X.

For the converse, let each $\rho \subseteq X$ be closed subset of X and show that (X, T) is T_1-space . Thus, let ρ_1, $\rho_2 \in X$ such that $\rho_1 \neq \rho_2$. Then since $\{\rho_2\}$ is closed , $X \sim \{\rho_2\}$ is open and contains ρ_1 but not ρ_2,and since $\{\rho_1\}$ is closed , $X \sim \{\rho_1\}$ is open and contains ρ_2 but not ρ_1.

It follows that (X= Sepc(μ), T) is a T_1-space.

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بعض النتائج حول فضاء زارسكي الضبابي التبولوجي

رشا ناصر مجيد
قسم الرياضيات ، كلية التربية- أبن الهيثم، جامعة بغداد

الخلاصة

يهدف البحث الى دراسة بعض خواص فضاء زارسكي الضبابي التبولوجي $spec(\mu)$ وتعريف الفضاء التبولوجي الجزئي منه وتعريف اساس لذلك الفضاءالجزئي، وكذلك اثبتنا

ان الفضاء الجزئي هو T_1.