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Mathematical Model for One Year Planning of a Manufactory

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Abstract

This paper is an attempt to help the manager of a manufactory to plan for the next year by a scientific approach, to maximize the profit and provide optimal monthly quantities of production, inventory, work-force, prices and sales. The computer programming helps us to execute that huge number of calculations.

Introduction

Holt, Modigliani, Muth and Simon (H.M.M.S.) developed a dynamic model to plan aggregate control of production, inventory and work-force which fully reported in their text (1). It was developed under the assumption that the receipt of orders would be erratic and fluctuating and it would therefore need to eradicate any excessive movements in the rates of production, inventory and work-force, in order to cut down the costs of running a manufactory in mathematical terms and, to that end, H.M.M.S subdivided the total cost as follows:

a. Regular payroll costs = $c_1 W_t + c_{13}$.
b. Hiring and Layoff costs = $c_2(W_t - W_{t-1} - c_{11})^2$
c. Over time and Idle time costs = $c_3 (P_t - c_4 W_t)^2 + c_5 P_t - c_6 W_t + c_{12} P_t W_t$.
d. Inventory related costs = $c_7 [I_t - (c_8 + c_9 S_t)]^2$.

The total cost function to be minimized was obtained as the sum of the foregoing components of cost added for values of t from t = 1 to t = T represented the planning horizon involved in any particular application.

$$C_T = \sum_{t=1}^{T} \{c_1 W_t + c_{13} + c_2(W_t - W_{t-1} - c_{11})^2 + c_3 (P_t - c_4 W_t)^2 + c_5 P_t - c_6 W_t +$$
$$c_{12} P_t W_t + c_7 [I_t - (c_8 + c_9 S_t)]^2 \} \quad ...[1.1]$$

the function 1.1 subject to the following restriction

$I_t \equiv I_{t-1} + P_t - S_t$ ...[1.2]

where

P_t = production rate required in period t.

I_t = level of inventory at the end of period t.

W_t = level of work-force required during period t.

$S_t \equiv O_t$ shipment in month t must equal the order level for that month.

c_1 to c_{13} numerical constants which must be evaluated from historical costs in any particular application.

To minimize the quadratic function above by differentiate it with respect to the independent decision variables yield to obtain a linear decision rules,

$$\frac{\partial C_T}{\partial W_r} = c_1 + 2c_2(W_r - W_{r-1} - c_{11}) - 2c_2(W_{r+1} - W_r - c_{11}) - 2c_3c_4(P_r - c_4 W_r) - c_6 + c_{12} P_r$$

i.e.

$$\frac{\partial C_T}{\partial W_r} = c_1 - c_6 + 2c_2 (\Delta W_{r-1} - c_{11}) - 2c_2 (\Delta W_r - c_{11}) - 2c_3 c_4 (P_r - c_4 W_r) + c_{12} P_r = 0 \quad ...[1.3]$$

for $r = 1, 2, \ldots, T-1$.

where

$\Delta W_r = W_{r+1} - W_r$

$$\frac{\partial W_{t-1}}{\partial W_r} = \begin{cases} 1 & \text{if } t = r+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial W_t}{\partial W_r} = \begin{cases} 1 & \text{if } t = r \\ 0 & \text{otherwise} \end{cases}$$

Solving equation 1.3 for P_t we obtain:

$$P_t = \frac{c_{10}}{c_{14}} - c_{15}\,\Delta^2 W_{t-1} + c_{16} W_t$$

$$= \frac{c_{10}}{c_{14}} - c_{15} W_{t+1} + c_{23} W_t - c_{15} W_{t-1} \quad ...[1.4]$$

for $t = 1, 2, ..., T-1$.

And also differentiate C_T with respect to I_t and setting the derivatives equal to zero, we obtain

$$I_t = \frac{c_3}{c_7}\,\Delta P_t - \frac{c_{14}}{2c_7}\,\Delta W_t + c_8 + c_9 S_t \quad ...[1.5]$$

for $t = 1, 2, ..., T-1$.

Prediction Model

H.M.M.S. did not take in consideration price variable in his model and treated as an exogenous variable while in this model it was treated as endogenous, allowing the manufacturer to vary his prices in such a way to influence the ordering pattern, hopefully to move heavy demand away from peak periods and smoothing production, inventory and work force levels and reducing overall costs.

The following is an inverse Price-demand relationship of classical economic theory used, namely

$$O_t = a - b_t\, p_t \quad ...[2.1]$$

where

O_t = the forecasted order quantity for time t.

a = maximum productive capacity.

b_t = the measure of change in demand per unit change in price.

Unfortunately there is no information about parameter 'a' which might be evaluated directly. We would suggest that 'a' may be determined from a formula such as:

a = optimal value of labour productivity x initial level of work-force x possible maximum shift ratio x v

i.e.

$$a = c_4\, W_0 \times N \times v \quad ...[2.2]$$

where

$$N = \frac{\text{number of shifts possible Per day}}{\text{number of shifts worked Per day}}$$

v = a factor to compensate for unknown components in the productive capacity and for any large forecasted demands in the interval t = 1 to t = 12.

W_0 = initial work-force for that year.

By substituting equation 2.2 into equation 2.1, we obtain:

$O_t = c_4\, W_0 \times N \times v - b_t\, p_t$ ...[2.3]

The expected inventory connected costs (d in 1) above would now have to be modified to:

Inventory connected costs = $c_7\, [\, I_t - c_8 - c_9\, (c_4 W_0 \times N \times v - b_t\, p_t)]^2$ ...[2.4]

Also, as a result of using price variable (p_t) the manufacturer will bear a new cost namely

Opportunity cost = $Q{\cdot}P_c - \sum_{t=1}^{T} p_t\ (c_4\ W_0 \times N \times v - b_t\ p_t)$ ...[2.5]

where

P_c = the (constant) selling price.

Q = the total quantity that would have been sold during the period t = 1 to t = T.

From equations 2.4 and 2.5, we obtain total cost function instead function 1.1 above

$$c_T = \sum_{t=1}^{T} \{(c_1 - c_6)W_t + c_{13} + c_2(W_t - W_{t-1} - c_{11})^2 + c_3\,(P_t - c_4\,W_t)^2 + c_5\,P_t + c_{12}\,P_t\,W_t + c_7\,[I_t - c_8 - c_9\,(c_4\,W_0 \times N \times v - b_t\,p_t)]^2 - p_t\,(c_4\,W_0 \times N \times v - b_t\,p_t)\} + Q{\cdot}P_c \quad ...[2.6]$$

which has to be minimized subject to the restriction

$I_t = I_{t-1} + P_t - c_4 W_0 \times N \times v + b_t\ p_t$ ...[2.7]

Linear Decision Rules

By differentiating C_T in 2.6 with respect to W_r, we obtain the production rate in terms of work-force level, as follows:

$P_t = g_1 - g_2 W_{t-1} + g_3 W_t - g_2 W_{t+1}$ ...[2.8]

where

$$g_1 = \frac{c_1 - c_6}{2c_3 c_4},\ g_2 = \frac{c_2}{c_3 c_4} \text{ and } g_3 = 2g_2 + c_4$$

Differentiating c_T in 2.6 w.r.t. I_r yield the result

$I_t = g_4 \Delta P_t - g_5 \Delta W_t + g_6 - c_9 b_t p_t$ ...[2.9]

Differentiating C_T in 2.6 w.r.t. p_t, we obtain

$p_t = C_{6(t)} P_t - C_{7(t)} W_t - C_{8(t)} I_t + C_{10(t)}$ ...[2.10]

The expression 2.8 for P_t above is given entirely in terms of the W_t but those for I_t and p_t are not. By substituting amongst equations 2.8, 2.9 and 2.10 obtain the necessary expression in the forms:

$I_t = C_{26(t)} + C_{27(t)} W_{t-1} - C_{28(t)} W_t + C_{29(t)} W_{t+1} - C_{30(t)} W_{t+2}$ ...[2.11]

$p_t = C_{36(t)} - C_{37(t)} W_{t-1} + C_{38(t)} W_t - C_{39(t)} W_{t+1} + C_{40(t)} W_{t+2}$ ...[2.12]

where

$C_{26(t)} = c_8 + c_9 (c_4 (c_4 W_0 \text{ x } N \text{ x } v - c_5 b_t) - b_t c_{10}) / 2 c_4$

$C_{27(t)} = c_2 (1 + c_7 c_9 b_t (c_9 + 1)) / (c_4 c_7)$

$C_{28(t)} = c_2 (3 + c_7 c_9 b_t (3c_9 + 2)) / (c_4 c_7)$

$C_{29(t)} = c_2 (3 + c_7 c_9 b_t (3c_9 + 1)) / (c_4 c_7)$

$C_{30(t)} = c_2 (1 + c_7 c_9^2 b_t) / (c_4 c_7)$

$C_{36(t)} = (c_4^2 W_0 \text{ x } N \text{ x } v + b_t(c_{10} + c_4 c_5)) / (2 c_4 b_t)$

$C_{37(t)} = c_2 (c_9 + 1) / c_4$

$C_{38(t)} = c_2 (3c_9 + 2) / c_4$

$C_{39(t)} = c_2 (3c_9 + 1) / c_4$

$C_{40(t)} = c_2 c_9 / c_4$

The decision variables in equations 2.8, 2.11 and 2.12 in terms of the W_{t+i} $(i = -1, 0, 1, 2)$.

By substituting these expressions into identity 2.7 obtain a general equation expressed in terms of W_{t+i} $(i = 0, 1, 2)$, but it will not hold for initial period $t = 1$.

$C_{27(t)} W_{t-2} - C_{41(t)} W_{t-1} + C_{42(t)} W_t - C_{43(t)} W_{t+1} + C_{44(t)} W_{t+2} = c_4 W_0 \text{ x } N \text{ x } v - C_{45(t)}$ ...[2.13]

Because of presence of the variable I_{t-1} in identity 2.7, it is necessary to use a slightly modified form of equation for period t = 1 as follow

$C_{47(1)}\ W_1 - C_{48(1)}\ W_2 + C_{49(1)}\ W_3 = c_4W_0 \times N \times v - I_0 + C_{46(1)}\ W_0 - C_{50(1)}$ ...[2.14]

where I_0 = initial inventory

$C_{27} = c_2\ (1 + c_7\ c_9\ b_t\ (c_9 + 1))\ /\ (c_4\ c_7)$

$C_{46} = c_2\ (1/c_3 + 1/c_7 + b_t\ (c_9 + 1)^2)\ /\ c_4$

$C_{47} = c_2\ (2/c_3 + c_4^2/c_2 + 3/c_7 + b_t\ (3c_9 + 2)(c_9 + 1))\ /\ c_4$

$C_{48} = c_2\ (1/c_3 + 3/c_7 + b_t\ (3c_9 + 1)(c_9 + 1))\ /\ c_4$

$C_{49} = c_2\ (1/c_7 + b_t\ c_9\ (c_9 + 1))\ /\ c_4$

$C_{50} = c_4W_0 \times N \times v - c_8 + c_{10}/c_7 - (c_9 + 1)\ (c_4(c_4W_0 \times N \times v - b_t\ c_5) - b_t\ c_{10})\ /\ 2c_4$

$C_{44} = c_2\ (1/c_7 + b_t\ c_9\ (c_9 + 1))\ /\ c_4$

$C_{45} = c_4W_0 \times N \times v + c_{10}/(2c_3\ c_4) - (c_9 + 1)\ (c_4(c_4W_0 \times N \times v - c_5\ b_t) - b_t\ c_{10})\ /\ 2c_4 +$

$c_9(c_4(c_4W_0 \times N \times v - c_5\ b_{t-1}) - b_{t-1}\ c_{10})\ /\ 2c_4$

$C_{43} = c_2\ (1/c_3 + 4/c_4 + b_t\ (3c_9 + 1)(c_9 + 1) + b_{t-1}\ c_9^2)\ /\ c_4$

$C_{42} = c_2\ (2/c_3 + c_4^2/c_2 + 6/c_7 + b_t\ (3c_9 + 2)(c_9 + 1) + b_{t-1}c_9\ (3c_9 + 1))\ /\ c_4$

$C_{41} = c_2\ (1/c_3 + 4/c_7 + b_t\ (c_9 + 1)^2 + b_{t-1}c_9\ (3c_9 + 2))\ /\ c_4$

From equations 2.13 and 2.14, we have got 12-period of simultaneous linear equations to be solved for optimizing values of W_r.

The system would always contain two more unknowns than the are equations by imposing two end conditions $W_{10} = W_{11} = W_{12}$.

By applying the Gauss-Jordan method to the system above, we have got the optimal values of W_t, t = 1 to 14.

Forecasting Future Demands

When analyzing the customer demand per unit time, main factors should be known is the average (or mean) demand per unit time. An estimate of the mean demand per unit time will give an indication of what demand would be expected in a typical time period. It must be realized that such a mean value can only be calculated from past data, and to use such a value to predict what will happen in the future implies that one assumes that what has happened in the past will necessarily happen again and, of course, this is rarely so. See (2, p.13).

It seems to us that some form of forecasting would be an essential part of the running of the system in any real-life situation and, in consequence, I computerized a 12-month weighted moving average and exponentially-weighted moving average. The user is at free to choose which method to use.

Results Obtained From New Model

Here we report on performance of this model when working on data of the paint factory originally presented by H.M.M.S. and compare the results with their model. But before presenting these comparisons, we should make some general observations on the effect on the decision variables resulting from change in the value of productive capacity (equation 2.2 above) by giving different values to the parameter v in running the computer program many times. From table 3-1 below, we note the following:

a. The maximum and minimum of work-force, production and sales increase with increasing the parameter v and that is normal because of positive relationship between the productive capacity and work-force, production and sales levels.

b. The variation of work-force, production, price and sales increase with increasing v. This means a negative relationship between v and smoothing of these variables.

c. For the inventory rate which is more stable than the others because H.M.M.S. consider $c_9 = 0$ which eliminates the time dependent terms in $c_{26} - c_{30}$ above so the change in I_t depends only on the variation of W_t (see equation 2.11).

d. The increasing in the v yields increasing in the revenue and profit because a positive relationship between the v and sales and also a negative relationship between v and basic costs which consist opportunity cost which remarkably increased in negative value with increasing v.

In the real practice, the decision maker can do the same and then choose the reasonable production policy by specify the value of v. But he must have a good knowledge about

a. Market needs from his product.

b. Size of capital.

c. His ability on hiring or firing work-force.

d. Machine productivity.

e. Raw material requirement.

f. Store capacity and others.

One of the main purposes of H.M.M.S. and this model is to smooth out the time-series representing fluctuations in work-force, production, inventory levels. In table 3-1 below the smallest value of deviation for W_t, P_t and I_t is when $v = 1$, it is realistic results and optimal, or near-optimal solution.

Comparison With H.M.MS. in Terms of Smoothing

In table 3-2 shows the results of prediction model when $v = 1$ as well as H.M.M.S. results. We compare in terms of smoothing time-series for each variable in this table.

The maximum of production, work-force and sales are considerably larger than H.M.M.S. as well as the minimum. Also the total of production and sales are larger than H.M.M.S. model. Prediction model shows considerably less variation in all variables in this table and this smoothing is effective in increasing the profit of the factory. In terms of inventory level, the model predicts less amount than H.M.M.S., this wovld reduce the inventory holding cost.

Comarison With H.M.MS. in Terms of Profit

Table 3-3 below contains the relative costs and profits for our model comparable with results of H.M.M.S. model, It shows a great reduction in the basic costs. This reduction being a direct result of the superiority of our model in its capacity to smooth the relevant time-series, as noted in the previous subsection.

The other cost = production rate x O_c

where

O_c = the other cost per unit of production.

Prediction model suggests a greater quantity of production than H.M.M.S. model. This yields to an other larger cost than H.M.M.S. model.

In table 3-2 below, we note the total sale in our model is larger than H.M.M.S. and this case yields to a greater revenue than in the H.M.M.S. model.

Because of that reduction in the cost and increasing revenue yields to a maximization of profits.

The check column obtained from the equation 2.7 above and must equal zero otherwise means there is an error in mathematical operations of this model or an error in programming the model.

Simple Description of Computer Programme

In this section we outline, main steps, the construction of the prediction computer program and its name "Pred". Execution time is 1 to 2 seconds and consist of 355 programming instructions or statements.

1. Declarations of variables and dimensions.
2. Read c_1 to c_{13} of H.M.M.S., no of months, historical demand, W_0 and I_0.
3. Compute initial coefficients and common terms.
4. Test for forecasting method to be used:

$$\text{FORCA}\begin{cases} =1 & \text{Moving average forecasting subroutine} \\ =2 & \text{Exponential weighted average subroutine} \\ =3 & \text{Forecasted sales equal to actual demands} \end{cases}$$

See (2), (3), (4) and (5).

5. Compute the value of b_t by equation 2.3 above.
6. Evaluate the coefficients c_{41} to c_{45} and c_{27}, c_{46} to c_{50}.
7. Build up the matrix by using equations 2.13 and 2.14.
8. Solve the system of equations by the Gauss-Jordan method to obtain values of W_t ($t=1, 2, \ldots, 14$), see (6), (7) and (8).
9. Computation of the values of decision variables P_t, I_t, p_t equations 2.8, 2.11 and 2.12 after the computation of c_{26} to c_{40} above.
10. Computation the various costs for period t from equations a, b, c and d in section 1 and equation 2.5 above.
11. Prediction of sales, revenue and profit for period t and checking the consistency of predicted values of the decision variables.
12. Print out monthly P_t, I_t, W_t, p_t and S_t and yearly totals. Also components of costs as mentioned in 10 above and yearly totals. The same thing to revenue and profit.

This program is loaded on one of the personal computers of the computers sciences department of this college.

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Table (3-1): Results obtained with the pred program using various values v with H.M.M.5 Data

Values of v	Work-force			Production			Inventory			Price			Sales			Total Cost	Total	Total
	Max.	Min.	Var.	Max.	Min.	Var.	Max.	Min.	Var.	Max.	Min.	Var.	Max.	Min.	Var.	Other+basic	Revenue	Profit
1	85	82	3	467	463	4	320	299	21	95.2	90.6	4.6	467	431	36	233793.4	511957	278163.7
1.1	90	83	7	498	481	17	319	298	21	94.5	88.2	6.3	499	446	53	229507.6	532813	303305.5
1.2	95	84	11	527	493	34	319	297	22	94	86.9	7.1	531	459	72	224679.5	553684	329004.5
1.3	100	85	15	556	504	52	318	296	22	93.8	86	7.8	563	471	92	219555.6	574297.7	354742
1.4	104	86	18	583	514	69	318	296	22	93.7	85.3	8.4	593	482	111	214308.9	594501.1	380192.3
1.6	112	87	25	633	531	102	317	289	28	93.6	84.3	9.3	651	500	151	203830.9	633377.6	429546.7
1.8	119	88	31	680	546	134	317	280	37	93.7	83.9	9.8	704	515	189	193699.4	670028.2	476328.7

$C_1 = 340.0$, $C_2 = 64.3$, $C_3 = 0.20$, $C_4 = 5.67$, $C_5 = 51.2$, $C_6 = 281.0$, $C_7 = 0.0825$, $C_8 = 32.0$, $C_9 = C_{11} = C_{12} = 0$

$N = 3$, $W_0 = 81$ men, $I_0 = 275$ units, $P_c = 100$

Table (3-2): Comparison between prediction model and H.M.M.S Model whem v=1

Month	Production		Work-force		Inventory		Price		Sales	
	H.M.M.S.	Pred.	H.M.M.S.	Pred.	H.M.M.S.	Pred.	H.M.M.S.	Pred.	H.M.M.S.	Pred.
1	446	467	79	82	303	299		95.28	397	431
2	416	463	75	83	282	312		94.13	437	450
3	382	463	71	84	342	316		94.22	322	459
4	376	464	69	84	322	316		94.41	396	464
5	367	465	67	84	314	318		92.81	348	463
6	359	465	66	85	381	318		92.68	319	465
7	382	465	67	85	308	318		92.14	455	466
8	379	465	67	85	287	320		90.64	409	463
9	371	466	68	85	302	320		91.85	347	467
10	374	466	70	85	387	319		92.04	289	467
11	418	466	74	85	375	318		91.46	430	467
12	459	465	77	85	339	319		90.02	459	465
Max.	459	467	79	85	387	320		95.28	459	467
Min.	359	463	66	82	282	299		90.02	289	431
Var.	100	4	13	3	105	21		5.26	170	36
Total	4729	5580			3942	3793			4608	5527

Table (3-3): The Relative costs and profits for prediction Model Componable with reseluts of H.M.M.S model

M	T. Basic Cost	Other Cost	Revenue	Profit	Check
1	18031.23	2972.14	41092.85	20089.48	0
2	17234.53	2945.8	42392.05	22211.72	0
3	17366.6	2946.23	43278.39	22965.57	0
4	17554.96	2952.21	43816.82	23309.65	0
5	16620.73	2954.63	42977.55	23402.19	0
6	16593.06	2958.23	43101.01	23549.73	0
7	16285.5	2959.89	42897.55	23652.16	0
8	15292.39	2959.51	41970.12	23718.22	0
9	16155.89	2964.18	42865.9	23745.83	0
10	16291.83	2966.16	43025.3	23767.31	0
11	15917.68	2964.17	42713.08	23831.22	0
12	14946.98	2958.95	41826.58	23920.66	0
Total	198291.4	35502.07	511957.2	278163.7	
H.M.M.S.	298910.6	30076.4	469800	140813	

أنموذج رياضي لتخطيط مصنع لسنة واحدة

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الخلاصة

ان هذا البحث محاولة لمساعدة مدير المصنع أو المعمل للتخطيط لسنة قادمة باستعمال الطريقة العلمية لتعظيم الارباح ويوفر الكميات المثالية شهرياً من الانتاج، الخزين، القوى العاملة، الاسعار والمبيعات. برامجيات الحاسوب ساعدتنا لتنفيذ الاعداد الضخمة أو الكبيرة من الحسابات.