Approximate Solution for Fuzzy Differential Algebraic Equations of Fractional Order Using Adomian Decomposition Method

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Abstract

In this paper we shall prepare an sacrificial solution for fuzzy differential algebraic equations of fractional order (FFDAEs) based on the Adomian decomposition method (ADM) which is proposed to solve (FFDAEs) . The blurriness will appear in the boundary conditions, to be fuzzy numbers. The solution of the proposed pattern of equations is studied in the form of a convergent series with readily computable components. Several examples are resolved as clarifications, the numerical outcomes are obvious that the followed approach is simple to perform and precise when utilized to (FFDAEs).

Keywords: fractional calculus, fuzzy set theory , fractional differential algebraic equation , Adomian decomposition method.

Introduction

The topic of fractional calculus (that is calculus of derivatives and integrals of any random complex or real order) has obtained large popularity and significance during the past three decades or so, back fundamentally to its pretended applications in abundant seemingly varied and widespread fields of engineering and science. It does de facto provide several potentially helpful tools for solving integral and differential equations, and different other problems including special functions of mathematical physics as well as generalizations of their expansions in one and more variables [1].

In this paper the approximate solution of (FFDAEs) will be investigated.The notion of fuzzy sets which was primarily introduced by Zadeh [2] leads to the definition of the fuzzy number and its application in approximate reasoning problems and fuzzy control. The basic arithmetic structure for fuzzy numbers was subsequently developed by Mizumoto and Tanaka [3], Nahmias and Ralescu [4,5] all of which observed the fuzzy number as a collection of α levels, $0 < \alpha \leq 1$.the basic notions on fuzzy sets, fuzzy differential equations and fuzzy differentials can be found in [6,7,8,9,10].

The study of fuzzy differential equations (FDEs) forms an appropriate setting for the mathematical modeling of real world problems in which suspicion or opacity pervades. The recently developed decomposition method proposed by American mathematician, Georg Adomian has been receiving much attention in recent years in applied mathematics

The ADM protrude as an alternative method for solving a spacious range of problems whose mathematical models involve integro-differential equations, algebraic , and partial differential equations.

Thus yields quickly convergent series solutions for both linear and nonlinear stochastic equations and deterministic; it has many features over the traditional techniques, namely it evades discretization and supply an efficient numerical solution with high precision, minimal calculations and avoidance of physically unrealistic assumptions, the theoretical treatment of convergence of the decomposition method has been considered in [11]and the obtained results about the speed of convergence of this method. The solution of the fractional differential equation have been obtained through the Adomian decomposition by [12].

However, El-Sayed and Kaya proposed ADM to approximate the numerical and analytical solution of system two-dimensional Burger sequations with initial conditions in [13], and the advantages of this work are that the decomposition method reduces the computational work and improves with regards to its accuracy and rapid convergence.

The convergence of decomposition method is proved as [14], in [15] applied ADM to obtain the approximate solution for the DAEs system and the result obtained by this method indicates a high degree of accuracy through the comparison with the analytic solutions. In [16], [17] standard and modified ADMs are applied to solve non-linear DAEs.

While, the error analysis of Adomian series solution to a class of nonlinear differential equation, where as numerical experiments show that Adomian solution using this formula converges faster is discussed in [18]. Also, a new discrete ADM to sacrificial is the theoretical solution of discrete nonlinear Schrodinger equations is presented in [19] where this tested for single solution waves and plane waves in case of continuous, semi discrete and fully discrete Schrodinger equations. Momani and Jafari, [20] presented numerical study of system of fractional differential equation by ADM.

This paper is orderly as follows: In section two, we retrieval the basic concept of fuzzy set theory, in section three, the fractional order definitions of derivatives and integration are considered. our approach is presented in section four, and two numerical examples are solved in section five to illustrate the ability and efficiency of the proposed method.

Some Basic Concept of Fuzzy Set Theory

In this part, we shall display some basic definitions of fuzzy set theory inclusive the definition of fuzzy functions and fuzzy numbers.

Definition (1), [21]:
Let Y be any set of elements, a fuzzy set $\tilde{B}$ is described by a membership function $\mu_{\tilde{B}}(Y): Y \to [0,1]$ and perhaps written as the set of points $\tilde{B} = \{(y, \mu_{\tilde{B}}) \mid y \in Y, 0 \leq \mu_{\tilde{B}}(y) \leq 1\}$.

Definition (2), [21]:
The fragile set of elements that belongs to the set B at minimal to the degree β is called the weak α-level set (or weak β -cut), and is defined by:
$B_\beta = \{y \in Y: \mu_{\tilde{B}}(y) \geq \beta\}$ While the strong β - level set (or strong β -cut) is defined by:
$B'_\beta = \{y \in Y: \mu_{\tilde{B}}(x) \geq \beta\}$

Definition (3), [2]
A fuzzy subset $\tilde{B}$ of a universal space Y is convex if and only if the sets B_β are convex, $\forall\, \beta \in [0,1]$. or equipollent, we can define convex fuzzy set immediately by using its membership function to offset:
$\mu_{\tilde{B}}[\lambda y_1 + \mu_{\tilde{B}}(1-\lambda)y_2] \geq Min\{\mu_{\tilde{B}}(y_1), \mu_{\tilde{B}}(y_2)\}$, $for\ all\ y_1, y_2 \in Y\ and\ \lambda \in [0,1]$.

Remark (1), [22]:
A fuzzy number $\tilde{N}$ *perhaps* singly represented in terms of its β -level sets, as the following closed intervals of the real line:

$$N_\beta = [n - \sqrt{1-\beta}, m + \sqrt{1-\beta}\,]$$

Or

$$N_\beta = [\beta n, \tfrac{1}{\alpha} n]$$

Where n is the mean value of $\tilde{N}$ and $\beta \in (0,1]$. This fuzzy number is perhaps written as $\tilde{N} = [\underline{N}, \overline{N}]$, where $\underline{N}$ refers to the greatest lower bound of $\tilde{N}$ and $\overline{N}$ to the least upper bound of $\tilde{N}$.

Remark (2), [22]:
like to the second approach given in remark (1), one can fuzzyfy any fragile or non fuzzy function f, by allowing:

$\underline{f}(Y) = \beta$ f(x), $\overline{f} = \frac{1}{\beta}$ f(y), y ∈ Y, $\beta \in (0,1]$, and hence the fuzzy function in terms of its β-levels is given by

$f_\beta = [\underline{f}, \overline{f}]$.

Fractional Order Derivative and Integration

In this section we shall present some definitions of fractional order derivatives and integral which will be used further in this paper .

Definition (4), [23]:

The Riemann-Liouville fractional integral operator of order q > 0 is specify as:

$I^q f(x) = \frac{1}{\Gamma(q)} \int_0^x (x-t)^{q-1} f(t)dt, q>0, x>0$ .....(1)

$I^0 f(x) = f(x)$

Definition (5)[23]:

The Riemann-Liouville fractional derivative operator of order q> 0 is specify as:

$$D_x^q f(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-q-1} f(t)dt \quad \ldots..(2)$$

Where m is an integer and m - 1 < q ≤ m

Definition (6)[23]:

Caputo fractional derivative operator of order q > 0 is defined as:

$$^cD_x^q f(x) = \frac{1}{\Gamma(m-q)} \int_0^x (x-t)^{m-q-1} \frac{d^m}{dx^m} f(t)dt \quad \ldots..(3)$$

Caputo fractional derivative has a useful property:

$$I^q\, {}^cD_x^q f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!} \quad \ldots.(4)$$

Where m is an integer and m - 1 < q ≤ m

And similar to integer order differentiation Caputo's fractional differentiation is a linear operation

i.e:

$$^cD_x^q f(x) \left[\lambda f(t) + \mu_{g(t)}\right] = \lambda\, {}^cD_x^q f(x) + \mu\, {}^cD_x^q g(x). \quad \ldots(5)$$

Where λ *and* μ are constants , for Caputo derivative also we have

$^cD_x^q c = 0$, c is constant

$$^cD_x^q f(x) \begin{cases} 0 & , for\ n \in N_0\ and\ n < \lceil q \rceil \\ \frac{\Gamma(n+1)}{\Gamma(n+1-q)} & , for\ n \in N_0\ and\ n \geq \lceil q \rceil \end{cases}$$

The Proposed Approach

In this section we shall employ the ADM to solve fuzzy differential algebraic equations of fractional order and , for this purpose let us consider the following system of fuzzy differential algebraic equation of fractional order with variable coefficients:

$$A(t)\,[{}^cD_{x_0}^q \tilde{x}(t)] + B(t)\tilde{x}(t) = f(t), 0 < q < 1 \quad \ldots(6)$$

With initial value

$$\tilde{x}(t_0) = \tilde{x}_{t_0} \quad \ldots(7)$$

Where ${}^cD_{x_0}^q$ represents the caputo fractional derivative of order q and with $A(t)$ is $n \times n$ singular matrix, $B(t)$ is $n \times n$ nonsingular matrix , f is $n \times 1$ vector function , $\tilde{x}_{t_0}$ is $n \times 1$ known fuzzy number vector.

The procedure of solution is given by writing system (6),equivalently as a system of fuzzy fractional DAEs, then by substituting the algebraic variable in the previous equations ,we get a system of ordinary differential ,equations which can be considered as

$$\left.\begin{aligned} {}^cD_{x_0}^q \tilde{x}_1 &= f_1(t, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n-1}) \\ {}^cD_{x_0}^q \tilde{x}_2 &= f_2(t, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n-1}) \\ {}^cD_{x_0}^q \tilde{x}_{n-1} &= f_{n-1}(t, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n-1}) \end{aligned}\right\} \quad \ldots(8)$$

Where each equation represents the fuzzy fractional of order q of one unknown functions as a mapping depending on the independent variable t and n-1 unknown functions $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{n-1}$.
We can present the system (8) by using the i^{th} equition as :
$L\tilde{x}_i = f_i(t, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{n-1})$, i=1,2,...,n-1 ...(9)

Where $L = {}^{c}D_{x_0}^{q}\tilde{x}(t)$ is the Caputo fractional derivative operator of order q with inverse J^q which is the Riemann-Liouville fractional integration of order q .Applying the inverse operator J^q on (9) we shall get the following form :

$$\tilde{x}_i = \sum_{k=0}^{n-1} \tilde{x}^{(k)}(0^+)\frac{t^k}{k!} + J^q f_i(t, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{n-1} \quad \dots (10)$$

As usual in ADM the solution of equation (5) and (6) is considered to be the infinite of a series

$$\tilde{x}_i = \sum_{j=0}^{\infty} \tilde{x}_{ij}(x) \text{ , i=1,2,...,n-1} \quad \dots (11)$$

Where

$$\tilde{x}_{i,0} = \tilde{x}_i(0)$$
$$\tilde{x}_{i,0} = J^q f_i \quad \text{,n=0,1,2,..}$$

And if the integrand f_i in eq.(10) is nonlinear , we define :

$$\left.\begin{array}{l} \tilde{x}_{i,0} = \tilde{x}_i(0) \\ \tilde{x}_{i,0} = J^q A_{i,n} \quad \text{,n=0,1,2,..} \end{array}\right\} \quad \dots (12)$$

Where $A_{i,n}$ are the Adomian polynomial .

Since the initial value vector is fuzzy then the solution $\tilde{x}(t)$ will be a fuzzy vector solution and therefore $\tilde{x}(t)$will be written as $\tilde{x}(t)=[\underline{x}(t),\overline{x}(t)]$and to find $\overline{x}(t)$ we must solve the following problem

$$A(t)\,[{}^{c}D_{x_0}^{q}\overline{x}(t)] + B(t)\overline{x}(t) = f(t) \text{ ,} 0 < q < 1 \quad \dots (13)$$

With the initial value $\overline{x}(t_0) = \overline{x}_{t_0}$ ...(14)
Similarly in order to find $\underline{x}(t)$ we must solve

$$A(t)[{}^{c}D_{x_0}^{q}\,\underline{x}(t)] + B(t)\underline{x}(t) = f(t) \text{ ,} 0 < q < 1 \quad \dots (15)$$

With initial value $\underline{x}(t_0) = \underline{x}_{t_0}$ ... (16)

Illustrative Examples:

In this section, two examples are given in order to illustrate the proposed manner.

Example1:

Consider the following fuzzy differential algebraic equations of fractional order

$$\begin{pmatrix} 1 & -x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {}^{c}D_x^q \tilde{u}(x) \\ {}^{c}D_x^q \tilde{v}(x) \end{pmatrix} + \begin{pmatrix} 1 & -(1+x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}(x) \\ \tilde{v}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \sin x \end{pmatrix} \quad \dots (17)$$

With the following initial conditions :

$$\begin{pmatrix} \tilde{u}(0) \\ \tilde{v}(0) \end{pmatrix} = \begin{pmatrix} \tilde{1} \\ \tilde{0} \end{pmatrix} \quad \dots (18)$$

The solution $\tilde{u}(x)$can be written as $\tilde{u}(x) = [\underline{u}, \overline{u}]$,and to find $\underline{u}$ we must solve:

$$\begin{pmatrix} 1 & -x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {}^{c}D_x^q \underline{u}(x) \\ {}^{c}D_x^q \underline{v}(x) \end{pmatrix} + \begin{pmatrix} 1 & -(1+x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{u}(x) \\ \underline{v}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \sin x \end{pmatrix} \quad \dots (19)$$

With the following initial conditions

$$\begin{pmatrix} \underline{u}(0) \\ \underline{v}(0) \end{pmatrix} = \begin{pmatrix} 1 & -\sqrt{1-B} \\ - & \sqrt{1-B} \end{pmatrix} \quad \dots (20)$$

Equivalently eq. (15) can be written as

$${}^{c}D_x^q\,\underline{u}(x) - x\left[{}^{c}D_x^q\,\underline{v}(x)\right] + \underline{u}(x) - (1+x)\underline{v} = 0 \quad \dots (21)$$

$\underline{v}\ (x) = sin(x)$ … (22)

By using the Adomian decomposition manner on (21) , (22) we have

$\underline{u}\ (x) = \sum_{k=0}^{m-1} \underline{u}^{(k)}(0^+)\frac{x^k}{k!} + J^q\left\{x\,{}^cD_x^q\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!}\right]\right\}$ - $J^q[\underline{u}(x)] + J^q\left\{(1+x)\,{}^cD_x^q\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!}\right]\right\}$ ...(23)

Hence:

$\underline{u}\ (x) = \sum_{k=0}^{m-1} \underline{u}^{(k)}(0^+)\frac{x^k}{k!} + J^q\left[\frac{\Gamma(2)}{\Gamma(2-q)}x^{2-q} - \frac{\Gamma(4)}{3!\Gamma(4-q)}x^{4-q} + \frac{\Gamma(6)}{5!\Gamma(6-q)}x^{6-q} - \frac{\Gamma(8)}{7!\Gamma(8-q)}x^{8-q} + \frac{\Gamma(10)}{9!\Gamma(10-q)}x^{10-q}\right] + J^q\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!}\right] + J^q\left[x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^{10}}{9!}\right]$ ....(24)

$\underline{u_{k+1}}\ (x) = - J^q\left[\underline{u_k}(x)\right]$ ....(25)

Similarly in order to find $\overline{u}$ we must solve:

$\begin{pmatrix} 1 & -x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} {}^cD_x^q\,\overline{u}\,(x) \\ {}^cD_x^q\,\overline{v}\,(x) \end{pmatrix} + \begin{pmatrix} 1 & -(1+x) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \overline{u}\ (x) \\ \overline{v}\,(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \sin x \end{pmatrix}$ ...(26)

With the following initial conditions

$\begin{pmatrix} \overline{u}\ (0) \\ \overline{v}\ (0) \end{pmatrix} = \begin{pmatrix} 1 & +\sqrt{1-B} \\ & \sqrt{1-B} \end{pmatrix}$ ..... (27)

Equivalently eq. (26) can be written as

${}^cD_x^q\ \overline{u}\ (x) - x\left[{}^cD_x^q\ \ \overline{v}\ (x)\right] + \overline{u}\ (x) - (1+x)\,\overline{v} = 0$ ... (28)

$\overline{v}\ (x) = sin(x)$ …. (29)

By using the Adomian decomposition manner on (28) , (29) we have

$\overline{u}\ (x) = \sum_{k=0}^{m-1} \overline{u}^{(k)}(0^+)\frac{x^k}{k!} + J^q\left\{x\,{}^cD_x^q\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!}\right]\right\}$ - $J^q[\overline{u}(x)] + J^q\left\{(1+x)\,{}^cD_x^q\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!}\right]\right\}$ ...(30)

Hence:

$\overline{u}\ (x) = \sum_{k=0}^{m-1} \overline{u}^{(k)}(0^+)\frac{x^k}{k!} + J^q\left[\frac{\Gamma(2)}{\Gamma(2-q)}x^{2-q} - \frac{\Gamma(4)}{3!\Gamma(4-q)}x^{4-q} + \frac{\Gamma(6)}{5!\Gamma(6-q)}x^{6-q} - \frac{\Gamma(8)}{7!\Gamma(8-q)}x^{8-q} + \frac{\Gamma(10)}{9!\Gamma(10-q)}x^{10-q}\right] + J^q\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!}\right] + J^q\left[x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^{10}}{9!}\right]$ ....(31)

$\overline{u_{k+1}}(x) = - J^q\left[\overline{u_k(x)}(x)\right]$ ....(32)

Tables (1) ,(2) and (3) represent the approximate values of $\underline{u}(x)$ $and\ \overline{u}\ (x)$ using ADM up to 10 terms for $q = 1$, **q=**$\frac{1}{4}$,$q = \frac{1}{2}$ and $B = 0.25,\ 0.5$, 0.75 and 1 respectively .

Example 2:

Consider the following fuzzy fractional differential algebraic equations

$$\begin{pmatrix} 1 & -x & x^2 \\ 0 & 1 & -x \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} {}^cD_x^q\,\tilde{u}\ (x) \\ {}^cD_x^q\,\tilde{v}\ (x) \\ {}^cD_x^q\,\tilde{z}\ (x) \end{pmatrix} + \begin{pmatrix} 1 & -(x+1) & -(x^2+2x) \\ 0 & 1 & -(x+1) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{u}\ (x) \\ \tilde{v}\ (x) \\ \tilde{z}\ (x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \text{sinx} \end{pmatrix} \quad \ldots(33)$$

With the following initial conditions:

$$\begin{pmatrix} \tilde{u}\ (x) \\ \tilde{v}\ (x) \\ \tilde{z}\ (x) \end{pmatrix} = \begin{pmatrix} \tilde{1} \\ \tilde{1} \\ \tilde{0} \end{pmatrix} \quad \ldots(34)$$

The solution $\tilde{u}\ (x)$can be written as $\tilde{u}\ \ (x)\ = [\underline{u}, \overline{u}]$,and to find $\underline{u}$ we must solve:

$$\begin{pmatrix}1 & -x & x^2\\ 0 & 1 & -x\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}{}^{c}D_x^q\,\underline{u}\,(x)\\ {}^{c}D_x^q\,\underline{v}\,(x)\\ {}^{c}D_x^q\,\underline{z}\,(x)\end{pmatrix}+\begin{pmatrix}1 & -(x+1) & -(x^2+2x)\\ 0 & 1 & -(x+1)\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\underline{u}\ (x)\\ \underline{v}\ (x)\\ \underline{z}\ (x)\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \sin x\end{pmatrix} \quad \ldots(35)$$

With following initial conditions

$$\begin{pmatrix}\underline{u}\ (0)\\ \underline{v}\ (0)\\ \underline{z}\ (0)\end{pmatrix}=\begin{pmatrix}1-\sqrt{1-B}\\ 1-\sqrt{1-B}\\ -\sqrt{1-B}\end{pmatrix} \quad \ldots(36)$$

Equivalently eq. (35) can be written as

$${}^{c}D_x^q\,\underline{\mathrm{u}}\,(x)-x[\,{}^{c}D_x^q\,\underline{\mathrm{v}}\,(x)]+x^2[\,{}^{c}D_x^q\,\underline{\mathrm{z}}\,(x)]+\underline{\mathrm{u}}\ (x)-x\underline{\mathrm{v}}\ (x)-\underline{\mathrm{v}}\ (x)+x^2\underline{\mathrm{z}}\ (x)+2\mathrm{x}\,\underline{\mathrm{z}}\,(\mathrm{x})=0 \quad ..(37)$$

$${}^{c}D_x^q\,\underline{\mathrm{v}}\,(x)-\ x[\,{}^{c}D_x^q\,\underline{\mathrm{z}}\,(x)]-\underline{\mathrm{v}}\,(x)+x\underline{\mathrm{z}}\,(x)-\underline{\mathrm{z}}\,(x)=0 \quad \ldots(38)$$

$$\underline{\mathrm{z}}\,(\mathrm{x})=\sin(\mathrm{x}) \quad \ldots(39)$$

By using the Adomian decomposition method on (38) , (39) we have

$$\underline{u}\,(x)=\ \sum_{k=0}^{m-1}\underline{u}^{(k)}(0^+)\frac{x^k}{k!}+J^q[2x\underline{v}(x)]+J^q[\underline{v}(x)]-J^q\left\{2\left[x^3-\frac{x^5}{3!}+\frac{x^7}{5!}-\frac{x^9}{7!}+\frac{x^{11}}{9!}\right]\right\}-J^q\left[x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-\frac{x^8}{7!}+\frac{x^{10}}{9!}\right]-\ J^q[\underline{u}(x)] \quad \ldots(40)$$

$$\underline{v}\,(x)=\ \sum_{k=0}^{m-1}\underline{v}^{(k)}(0^+)\frac{x^k}{k!}+J^q[\frac{\Gamma(2)}{\Gamma(2-q)}x^{2-q}-\frac{\Gamma(4)}{3!\Gamma(4-q)}x^{4-q}+\frac{\Gamma(6)}{5!\Gamma(6-q)}x^{6-q}-\frac{\Gamma(8)}{7!\Gamma(8-q)}x^{8-q}+\frac{\Gamma(10)}{9!\Gamma(10-q)}x^{10-q}]\quad +J^q\left[\underline{v}(x)-x^2+\frac{x^4}{3!}-\frac{x^6}{5!}+\frac{x^8}{7!}-\frac{x^{10}}{9!}+\frac{x^3}{3!}+\frac{x^5}{5!}-x-\frac{x^7}{7!}+\frac{x^9}{9!}\right] \quad \ldots(41)$$

Then:

$$\underline{u}_0(x)=\ 1-\frac{2\Gamma(4)}{\Gamma(4+q)}x^{3+q}+\frac{2\Gamma(6)}{3!\Gamma(6+q)}x^{5+q}-\frac{2\Gamma(8)}{5!\Gamma(8+q)}x^{7+q}+\frac{2\Gamma(10)}{7!\Gamma(10+q)}x^{9+q}-\frac{2\Gamma(12)}{9!\Gamma(12+q)}x^{11+q}-\frac{\Gamma(3)}{\Gamma(3+q)}x^{2+q}\quad +\frac{\Gamma(5)}{3!\Gamma(5+q)}x^{4+q}-\frac{\Gamma(7)}{5!\Gamma(7+q)}x^{6+q}+\frac{\Gamma(9)}{7!\Gamma(9+q)}x^{8+q}-\frac{\Gamma(11)}{9!\Gamma(11+q)}x^{10+q} \quad \ldots(42)$$

$$\underline{u}_{k+1}\,(x)=-\ J^q\left[\underline{u_k}(x)\right]+J^q\left[\underline{v_k}(x)\right]+J^q\left[2x\underline{v_k}(x)\right] \quad \ldots(43)$$

$$\underline{v}_0\,(x)=\ \sum_{k=0}^{m-1}\underline{v}^{(k)}(0^+)\frac{x^k}{k!}+\frac{\Gamma(3-q)}{\Gamma(2-q)\Gamma(3)}x^2-\frac{\Gamma(5-q)}{\Gamma(4-q)\Gamma(5)}x^4+\frac{\Gamma(7-q)}{\Gamma(6-q)\Gamma(7)}x^6-\frac{\Gamma(9-q)}{\Gamma(8-q)\Gamma(9)}x^8+\frac{\Gamma(11-q)}{\Gamma(10-q)\Gamma(11)}x^{10}-\frac{\Gamma(3)}{\Gamma(3+q))}x^{2+q}+\frac{\Gamma(5)}{3!\Gamma(5+q))}x^{4+q}-\frac{\Gamma(7)}{5!\Gamma(5+q))}x^{6+q}+\frac{\Gamma(9)}{7!\Gamma(9+q))}x^{8+q}-\frac{\Gamma(11)}{9!\Gamma(11+q))}x^{10+q}+\frac{\Gamma(2)}{\Gamma(2+q))}x^{1+q}-\frac{\Gamma(4)}{3!\Gamma(4+q))}x^{3+q}+\frac{\Gamma(6)}{5!\Gamma(6+q))}x^{5+q}-\frac{\Gamma(8)}{7!\Gamma(8+q))}x^{7+q}+\frac{\Gamma(10)}{9!\Gamma(10+q))}x^{9+q} \quad \ldots(44)$$

$$\underline{v}_{k+1}\,(x)=J^q\left[\underline{v_k}(x)\right] \quad \ldots(45)$$

Similarly in order to find $\overline{u}$ we must solve:

$$\begin{pmatrix}1 & -x & x^2\\ 0 & 1 & -x\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}{}^{c}D_x^q\,\overline{u}\ \ (x)\\ {}^{c}D_x^q\,\overline{v}\ \ (x)\\ {}^{c}D_x^q\,\overline{z}\ \ (x)\end{pmatrix}+\begin{pmatrix}1 & -(x+1) & -(x^2+2x)\\ 0 & 1 & -(x+1)\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\overline{u}\,(x)\\ \overline{v}\,(x)\\ \overline{z}\ (x)\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \sin x\end{pmatrix} \quad \ldots(46)$$

With following initial conditions

$$\begin{pmatrix}\overline{u}\,(0)\\ \overline{v}\,(0)\\ \overline{v}\,(0)\end{pmatrix}=\begin{pmatrix}1+\sqrt{1-B}\\ 1+\sqrt{1-B}\\ \sqrt{1-B}\end{pmatrix} \quad \ldots(47)$$

Equivalently eq. (46) can be written as

$${}^{c}D_x^q\,\overline{u}\,(x) - x\left[{}^{c}D_x^q\,\overline{v}\,(x)\right] + x^2\left[{}^{c}D_x^q\,\overline{z}\,(x)\right] + \overline{u}\,(x) - x\overline{u}\,(x) - \overline{v}\,(x) + x^2\overline{z}\,(x) + 2x\,\overline{z_3}\,(x) = 0 \quad \ldots(48)$$

$${}^{c}D_x^q\,\overline{v}\,(x) - x\left[{}^{c}D_x^q\,\overline{z}\,(x)\right] - \overline{v}\,(x) + x\overline{z}\,(x) - \overline{z}\,(x) = 0 \quad \ldots(49)$$

$$\overline{z}\,(x) = \sin(x) \quad \ldots(50)$$

By using the Adomian decomposition method on (48) , (49) and (50) we have

$$\overline{u}\,(x) = \sum_{k=0}^{m-1}\overline{u}^{\,(k)}(0^+)\frac{x^k}{k!} + J^q[2x\overline{v}(x)] + J^q[\overline{v}(x)] - J^q\left\{2\left[x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \frac{x^{11}}{9!}\right]\right\} - J^q\left[x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^{10}}{9!}\right] - J^q[\overline{u}(x)] \quad \ldots(51)$$

$$\overline{v}\,(x) = \sum_{k=0}^{m-1}\overline{v}^{(k)}(0^+)\frac{x^k}{k!} + J^q\Big[\frac{\Gamma(2)}{\Gamma(2-q)}x^{2-q} - \frac{\Gamma(4)}{3!\Gamma(4-q)}x^{4-q} + \frac{\Gamma(6)}{5!\Gamma(6-q)}x^{6-q} - \frac{\Gamma(8)}{7!\Gamma(8-q)}x^{8-q} + \frac{\Gamma(10)}{9!\Gamma(10-q)}x^{10-q}\Big] + J^q\left[\overline{v}(x) - x^2 + \frac{x^4}{3!} - \frac{x^6}{5!} + \frac{x^8}{7!} - \frac{x^{10}}{9!} + \frac{x^3}{3!} + \frac{x^5}{5!} - x - \frac{x^7}{7!} + \frac{x^9}{9!}\right] \quad \ldots.(52)$$

Then:

$$\overline{u_0}(x) = 1 - \frac{2\Gamma(4)}{\Gamma(4+q)}x^{3+q} + \frac{2\Gamma(6)}{3!\Gamma(6+q)}x^{5+q} - \frac{2\Gamma(8)}{5!\Gamma(8+q)}x^{7+q} + \frac{2\Gamma(10)}{7!\Gamma(10+q)}x^{9+q} - \frac{2\Gamma(12)}{9!\Gamma(12+q)}x^{11+q} - \frac{\Gamma(3)}{\Gamma(3+q)}x^{2+q} + \frac{\Gamma(5)}{3!\Gamma(5+q)}x^{4+q} - \frac{\Gamma(7)}{5!\Gamma(7+q)}x^{6+q} + \frac{\Gamma(9)}{7!\Gamma(9+q)}x^{8+q} - \frac{\Gamma(11)}{9!\Gamma(11+q)}x^{10+q} \quad \ldots.(53)$$

$$\overline{u}_{k+1}(x) = -\,J^q[\overline{u}_k(x)] + J^q[\overline{v}_k(x)] + J^q[2x\overline{v}_k(x)] \quad \ldots.(54)$$

$$\overline{v}_0\,(x) = \sum_{k=0}^{m-1}\overline{v}(0^+)\frac{x^k}{k!} + \frac{\Gamma(3-q)}{\Gamma(2-q)\Gamma(3)}x^2 - \frac{\Gamma(5-q)}{\Gamma(4-q)\Gamma(5)}x^4 + \frac{\Gamma(7-q)}{\Gamma(6-q)\Gamma(7)}x^6 - \frac{\Gamma(9-q)}{\Gamma(8-q)\Gamma(9)}x^8 + \frac{\Gamma(11-q)}{\Gamma(10-q)\Gamma(11)}x^{10} - \frac{\Gamma(3)}{\Gamma(3+q))}x^{2+q} + \frac{\Gamma(5)}{3!\Gamma(5+q))}x^{4+q} - \frac{\Gamma(7)}{5!\Gamma(5+q))}x^{6+q} + \frac{\Gamma(9)}{7!\Gamma(9+q))}x^{8+q} - \frac{\Gamma(11)}{9!\Gamma(11+q))}x^{10+q} + \frac{\Gamma(2)}{\Gamma(2+q))}x^{1+q} - \frac{\Gamma(4)}{3!\Gamma(4+q))}x^{3+q} + \frac{\Gamma(6)}{5!\Gamma(6+q))}x^{5+q} - \frac{\Gamma(8)}{7!\Gamma(8+q))}x^{7+q} + \frac{\Gamma(10)}{9!\Gamma(10+q))}x^{9+q} \quad \ldots.(55)$$

$$\overline{v}_{k+1}\,(x) = J^q[\overline{v}_k(x)] \quad \ldots.(56)$$

Tables (4) ,(5) ,(6) and (7) represent the approximate values of $\underline{u}(x)$,$\overline{u}\,(x)$, $\underline{v}(x)$ and $\overline{v}\,(x)$ using ADM up to 10 terms for $q = 1$ and $q = \frac{1}{2}$ and $B = 0.25,\ 0.5$ and 1 respectively.

Conclusions

In this paper, the approximate solution of the fuzzy differential algebraic equations of fractional order was found by using the Adomian decomposition Method.It seems from the results of the illustrative examples that the proposed method gave an accurate result and the method followed to find the approximate solution is efficient and can be considered for such types of problems.

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Table(1) : The approximate values of $\underline{u}(x)$ and $\overline{u}\ (x)$ for q=1 and B $= 0.25$, 0.5 , 0.75 and 1

X	$B=0.25$		$B=0.5$		$B=0.75$		$B=1$		
	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	Exact solution of $\underline{u}(x)$
0	0.1339746	1.8660254	0.2928932	1.7071068	0.5	1.5	1	1	1
0.1	0.131208573	1.698432947	0.275004069	1.554637735	0.462402051	1.367239469	0.914821265	0.914821265	0.91482076
0.2	0.149422991	1.567506247	0.279534536	1.437403817	0.449099243	1.267829996	0.858480964	0.858480964	0.858464619
0.3	0.187906887	1.471041679	0.305636681	1.353381202	0.459065172	1.199883393	0.829599653	0.829599653	0.829474283
0.4	0.245573197	1.406601569	0.35209952	1.300367951	0.49092736	1.161247406	0.826620776	0.826620776	0.826087383
0.5	0.320972472	1.371514386	0.417361475	1.276021044	0.542978099	1.149508759	0.847886169	0.847886169	0.846243429
0.6	0.412312304	1.362881937	0.49952868	1.27790162	0.613191302	1.162002938	0.8891720554	0.8891720554	0.88759712
0.7	0.517482199	1.377593172	0.59639884	1.303528556	0.699245033	1.195830338	0.956524185	0.956524185	0.947537684
0.8	0.634083542	1.412344138	0.705490273	1.350440403	0.798549357	1.247878323	1.0408727	1.0408727	1.023213837
0.9	0.759464232	1.463663549	0.824075715	1.416265655	0.908279057	1.314848724	1.143623156	1.143623156	1.111563878

Table(2): The approximate values of $\underline{u}(x)$ and $\overline{u}\ (x)$ for q=1/4 and B $=$

$0.25, 0\ 0.5$, 0.75 and 1 .

X	$B=0.25$		$B=0.5$		$B=0.75$		$B=1$	
	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$
0	0.133974596	1.866025404	0.2928932	1.7071068	0.5	1.5	1	1
0.1	0.095426051	1.015639719	0.179913959	0.931208552	0.28989008	0.82117569	0.555532885	0.555532885
0.2	0.128128109	0.836262752	0.19314433	0.771290196	0.277774567	0.686616294	0.482195431	0.482195431
0.3	0.186594594	0.765776439	0.239771225	0.71263552	0.308990119	0.643380913	0.476185516	0.476185516
0.4	0.26514489	0.75404169	0.310032148	0.709184578	0.36846094	0.650725639	0.50959329	0.50959329
0.5	0.360783267	0.781875558	0.399445165	0.743239625	0.449770538	0.692888286	0.571329412	0.571329412
0.6	0.471227589	0.839153267	0.505008082	0.805395461	0.548979433	0.761401423	0.655190428	0.655190428
0.7	0.59439238	0.919376096	0.624230227	0.889558288	0.66306952	0.850698956	0.756884238	0.756884238
0.8	0.72820384	1.017741457	0.754787261	0.991175888	0.789390338	0.956554959	0.872972648	0.872972648
0.9	0.870526445	1.130312923	0.894378314	1.106477073	0.925425787	1.075413581	1.000419684	1.000419684

Table(3):The approximate values of $\underline{u}(x)$ and $\overline{u}\ (x)$for q=1/2

and B $= 0.25, 0\ 0.5$, 0.75 and 1

X	$B=0.25$		$B=0.5$		$B=0.75$		$B=1$	
	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$
0	0.133974596	1.866025404	0.292893219	1.707106781	0.5	1.5	1	1
0.1	-0.146022093	-2.520043007	-0.363842571	-2.302222529	-0.647711743	-2.018353357	0.14323225	0.14323225
0.2	-0.110009819	-2.725867835	-0.350019264	-2.48585839	-0.662805662	-2.173071992	0.162262125	0.162262125
0.3	-0.039305401	-2.574450255	-0.271909276	-2.341846381	-0.575044547	-2.03871111	0.211655623	0.211655623
0.4	0.048256041	-2.260467298	-0.163573273	-2.048637984	-0.439634608	-1.772576649	0.275037413	0.275037413
0.5	0.143670526	-1.853853425	-0.039605711	-1.670577187	-0.278455954	-1.431726945	0.344942937	0.344942937
0.6	0.239521719	-1.390544496	0.089960357	-1.240983134	-0.104951805	-1.046070973	0.415049138	0.415049138
0.7	0.327710101	-0.894254744	0.215592737	-0.78213738	0.069478545	-0.636023188	0.478055111	0.478055111
0.8	0.397569739	-0.385061423	0.325761991	-0.313253676	0.232180314	-0.219671998	0.523909414	0.523909414
0.9	0.432382935	0.114269464	0.40319548	0.143456919	0.365157648	0.181494751	0.53638979	0.53638979

Table(4): The approximate values of $\underline{u}(x)$ and $\overline{u}\ (x)$ for q=1 and $\mathrm{B}=0.25$, 0.5 and 1

X	$B=0.25$		$B=0.5$		$B=0.75$		$B=1$		
	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	Exact solution of $\underline{u}(x)$
0	**0.133974596**	**1.866025404**	**0.292893219**	**1.707106781**	**0.5**	**1.5**	**1**	**1**	**1**
0.1	**0.136767408**	**1.904716153**	**0.298979728**	**1.742503833**	**0.510378939**	**1.531104622**	**1.020741781**	**1.020741781**	**1.01535451**
0.2	**0.145795016**	**2.02730293**	**0.318426584**	**1.854671362**	**0.543404423**	**1.629693523**	**1.086548973**	**1.086548973**	**1.063011305**
0.3	**0.162473097**	**2.24595656**	**0.353636267**	**2.05479339**	**0.60276496**	**1.805664698**	**1.204214829**	**1.204214829**	**1.145775863**
0.4	**0.188782322**	**2.576163909**	**0.407828665**	**2.357117567**	**0.693295415**	**2.071650816**	**1.382473116**	**1.382473116**	**1.267049925**
0.5	**0.227201133**	**3.036859531**	**0.484992095**	**2.779068569**	**0.820951816**	**2.443108848**	**1.632030332**	**1.632030332**	**1.430891295**
0.6	**0.280601148**	**3.650548822**	**0.589799609**	**3.341350362**	**0.992754887**	**2.938395083**	**1.965574985**	**1.965574985**	**1.642082916**
0.7	**0.352114402**	**4.4434352**	**0.72750008**	**4.068049522**	**1.216712219**	**3.578837383**	**2.397774801**	**2.397774801**	**1.906212199**
0.8	**0.444982619**	**5.445565242**	**0.903794624**	**4.986753236**	**1.501730069**	**4.388817792**	**2.94527393**	**2.94527393**	**2.229761706**
0.9	**0.56239945**	**6.691006841**	**1.124709657**	**6.128696634**	**1.857526583**	**5.395879709**	**3.626703146**	**3.626703146**	**2.62021246**

Table(5) : The approximate values of $\underline{v}(x)$ and $\overline{v}\ (x)$ for q=1 and $\mathrm{B}=0.25$, $0.$

X	$B=0.25$		$B=0.5$		$B=0.75$		$B=1$		
	$\underline{v}(x)$	$\overline{v}\ (x)$	$\underline{v}(x)$	$\overline{v}\ (x)$	$\underline{v}(x)$	$\overline{v}\ (x)$	$\underline{v}(x)$	$\overline{v}\ (x)$	Exact solution of $\underline{v}(x)$
0	0.133974596	1.866025404	0.292893219	1.707106781	0.5	1.5	1	1	1
0.1	0.158055629	2.07226781	0.333687869	1.89663557	0.562576261	1.667747179	1.020741781	1.11516172	1.11515426
0.2	0.203490098	2.319021732	0.397593743	2.124918088	0.650554536	1.871957294	1.086548973	1.261255915	1.261136624
0.3	0.2701062	2.608130238	0.484623903	2.393612535	0.764188815	2.114047623	1.204214829	1.439118219	1.43851487
0.4	0.357538354	2.941454527	0.594617081	2.7043758	0.903584092	2.395408789	1.382473116	1.64949644	1.647592035
0.5	0.465241212	3.320910221	0.727253726	3.058897707	1.068715081	2.717436352	1.632030332	1.893075717	1.88843404
0.6	0.592508524	3.748510864	0.882077135	3.458942254	1.259450294	3.081569094	1.965574985	2.170509694	2.160904284
0.7	0.738496635	4.226418638	1.058519442	3.906395831	1.475581283	3.48933399	2.397774801	2.482457636	2.464705088
0.8	0.902252333	4.757002293	1.255932233	4.403322393	1.71685685	3.942397776	2.94527393	2.829627313	2.799423801
0.9	1.082744686	5.342902227	1.473621424	4.952025489	1.983021905	4.442625008	3.626703146	3.212823456	3.16459733

Table(6) :The approximate values of $\underline{u}(x)$ and $\overline{u}\ (x)$ for q=0.5 and $\mathrm{B}=0.25$, 0.5 and 1

X	$B=0.25$		$B=0.5$		$B=0.75$		$B=1$	
	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$	$\underline{u}(x)$	$\overline{u}\ (x)$
0	0.133974596	1.866025404	0.292893219	1.707106781	0.5	1.5	1	1
0.1	0.203151908	2.765538086	0.438255221	2.530434773	0.744647822	2.224042172	1.484344997	1.484344997
0.2	0.301733687	3.901403252	0.632009524	3.571127415	1.062433374	3.140703565	2.101568469	2.101568469
0.3	0.434374951	5.280743725	0.879037572	4.836081104	1.458533181	4.256585496	2.857559338	2.857559338
0.4	0.606107664	6.924096442	1.185793937	6.34441017	1.941255793	5.588948314	3.765102053	3.765102053
0.5	0.822070637	8.855550327	1.559156133	8.118464831	2.519744652	7.157876312	4.838810482	4.838810482
0.6	1.088184717	11.101915959	2.006961679	10.183138996	3.204335124	8.985765551	6.095050338	6.095050338
0.7	1.41362934	13.694697983	2.540438386	12.567888938	4.008924519	11.099402804	7.554163662	7.554163662
0.8	1.821207649	16.68024835	3.184550039	15.316905961	4.961292426	13.540163573	9.250728	9.250728
0.9	2.383433053	20.15575553	4.014074026	18.525114556	6.13916671	16.400021873	11.269594291	11.269594291

Table(7):The approximate values $\underline{v}(x)$ and $\overline{v}(x)$ for q=0.5 and $B = 0.25$, 0.5 and 1

X	$B = 0.25$		$B = 0.5$		$B = 0.75$		$B = 1$	
	$\underline{v}(x)$	$\overline{v}(x)$	$\underline{v}(x)$	$\overline{v}(x)$	$\underline{v}(x)$	$\overline{v}(x)$	$\underline{v}(x)$	$\overline{v}(x)$
0	0.134	1.866025404	0.292893219	1.707106781	0.5	1.5	1	1
0.1	0.287677641	3.547790052	0.586798529	3.248669164	0.976620458	2.858847235	1.917733847	1.917733847
0.2	0.442038115	4.678520642	0.83074263	4.289816127	1.337312215	3.783246542	2.560279378	2.560279378
0.3	0.628764605	5.808485816	1.104012882	5.333237538	1.723368493	4.713881927	3.21862521	3.21862521
0.4	0.845417279	6.981687514	1.408430565	6.418674228	2.142163761	5.684941032	3.913552396	3.913552396
0.5	1.086791209	8.211147628	1.740463091	7.557475746	2.592344871	6.705593966	4.648969418	4.648969418
0.6	1.346550152	9.500469641	2.094686206	8.752333586	3.069676091	7.777343701	5.423509896	5.423509896
0.7	1.619538072	10.851066494	2.466546588	10.004057978	3.570389574	8.900214992	6.235302283	6.235302283
0.8	1.910853017	12.272980584	2.861595938	11.322237663	4.100628231	10.083205369	7.0919168	7.0919168
0.9	2.266861039	13.816655418	3.32657442	12.756942037	4.707619782	11.375896675	8.041758229	8.041758229

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