Torsion (Torsion Free) Fuzzy Modules Over Fuzzy Integral Domain

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Abstract

The study of torsion (torsion free) fuzzy modules over fuzzy integral domain as a generalization of torsion (torsion free) modules.

Introduction

A module M over an integral domain R is called a torsion (torsion free) module if T(M) = M (T(M) = (0)) where T(M)={ x: $x \in$ M and $\exists$ $r \neq 0$, $r\,x$ =0}.

These concepts has been fuzzified to torsion (torsion free) fuzzy μ-module where μ is a fuzzy integral domain.

In **Sections .1,** and 2, some of the known known definitions and results which are needed later.

In **S.2**, we give some basic properties of torsion (torsion free) fuzzy μ-modules.

In **S.3**, we study the behaviour of torsion (torsion free) fuzzy μ-modules under fuzzy homomorphism.

Finally, throughout this paper R denote commutative ring with unity, fuzzy μ-module means fuzzy module over fuzzy integral domain μ.

Section.1 Preliminaries

Definition 1.1 (1) Let S be a non-empty set and I be the closed interval [0,1] of the real line. A fuzzy set A in S is a function from S into [0,1].

Definition 1.2 (2) Let x_t : S $\rightarrow$ [0,1] be a fuzzy set in S, where $x \in$ S, $t \in$ [0,1], define by:

$$x_t(y) = \begin{cases} t & if \quad x = y \\ 0 & if \quad x \neq y \end{cases}$$

For all $y \in S$, x_t is called a fuzzy singleton.

Proposition 1.3 (3) Let a_k, b_t be two fuzzy singletons of a set S. If $a_k = b_t$, then $a = b$ and $k = t$.

Definition 1.4 (4) Let A be a fuzzy set in S, for all $t \in [0,1]$, the set $A_t = \{ x \in S, A(x) \geq t\}$ is called a level subset of A.

Note that, A_t is a subset of S in the ordinary sense.

Definition 1.5 (2) Let R be commutative ring, let X: R → [0,1], then X is called a fuzzy ring of R if:

- $X(a - b) \geq \min \{ X(a), X(b)\}$, for all $a, b \in R$.
- $X(ab) \geq \min \{ X(a), X(b)\}$, for all $a, b \in R$.
- $X(0) = 1$

Definition 1.6 (5) Let $\mu : R \to [0,1]$ be a fuzzy ring, let $\gamma : M \to [0,1]$ where M is an R-module. γ is called a left fuzzy μ-module if:

- $\gamma(a - b) \geq \min \{ \gamma(a), \gamma(b)\}$, for all $a, b \in M$.
- $\gamma(ra) \geq \min \{ \gamma(a), \mu(r)\}$, for all $r \in R$, $a \in M$.

If, moreover R is unitary and $1 \cdot x = x$, $\forall x \in M$, then γ is called unital. Similary, we defines a right fuzzy μ-module.

We shall deal only with left fuzzy μ-modules, and we shall call these simply fuzzy μ-modules.

Definition 1.7 (5) Let γ be a fuzzy μ-module, a fuzzy submodule of γ is a fuzzy μ-module. $\gamma' : M \to [0,1]$ such that $\gamma' \subseteq \gamma$, where $\gamma' \subseteq \gamma$ means $\gamma'(x) \leq \gamma(x)$, $\forall x \in M$.

Proposition 1.8. (5) Let γ be a fuzzy μ-module, a fuzzy submodule $\gamma' : M \to [0,1]$ where M ia an R-module and $\gamma' \neq 0$, is a fuzzy μ-module iff γ_t is a μ_t-submodule of M, $\forall\ t \in [0, \gamma(0)]$.

Proposition 1.9. Let $r_t \in \mu$ and $x_k \in \gamma$, then $r_t\ x_k = (r\ x)_\lambda$, where $\lambda = \min\{t,k\}$.

Proof: It's easiy. So we omitted.

Section.2 Torsion (Torsion Free)

Fuzzy Modules

In this section we fuzzify the concept of torsion (torsion free) modules into torsion (torsion free) fuzzy modules. Then we study some of their basic properties.

Recall that an R-module M is called torsion (torsion free) R-module if $T(M) = \{x : x \in M; \text{ann } x \neq 0\} = M \quad (T(M) = \{0\})$.(6)

Firstly, we need the following definition:

Definition 2.1 (7) A fuzzy ring A is said to be an integral domain if $x\, y = 0$ and $\min\{A(x), A(y)\} > 0$ implies $x = 0$ or $y = 0$.

However we give another characterization of fuzzy integral domain.

Proposition 2.2 A fuzzy ring μ of a ring R is a fuzzy integral domain if and only if whenever $x_t \in \mu$, $y_k \in \mu$, $0 < t \leq 1$, $0 < k \leq 1$, such that $x_t\, y_k \subseteq 0_1$, then either $x_t \subseteq 0_1$ or $y_k \subseteq 0_1$.

Proof. ($\Rightarrow$) If $x_t\, y_k \subseteq 0_1$, then $(x\, y)_\lambda \subseteq 0_1$, where where $\lambda = \min\{t,k\}$

Thus $x\, y = 0$ by prop.(1.3.). On the other hand, since $\lambda = \min\{t,k\}$ and x_t, $y_k \in \mu$ then $\mu(x) \geq t$ and $\mu(y) \geq k$. Hence $\min\{\mu(x), \mu(k)\} \geq \min\{t,k\} = \lambda > 0$.

Thus $x = 0$ or $y = 0$ by def.(2.1.). So either $x_t = 0_t \subseteq 0_1$ or $y_k = 0_k \subseteq 0_1$.

($\Leftarrow$) To prove μ is a fuzzy integral domain. Suppose $x\, y = 0$ and $\min\{\mu(x), \mu(y)\} > 0$. We must prove $x = 0$ or $y = 0$.

Let $\mu(x) = t$ and $\mu(y) = k$. Since $\min\{\mu(x), \mu(y)\} > 0$. Hence $x_t \in \mu$ and $y_k \in \mu$. But $x_t\, y_k = (x\, y)_\lambda = 0_\lambda \subseteq 0_1$, where $\lambda = \min\{t,k\}$. Thus $x_t \subseteq 0_1$ or $y_k \subseteq 0_1$, which implies $x = 0$ or $y = 0$. Thus μ is a fuzzy integral domain.

Now, we shall fuzzify the concepts of torsion (torsion free) modules into torsion (torsion free) fuzzy modules.

Definition 2.3 Let μ be a fuzzy integral domain, let γ be a fuzzy μ-module. Let $T(\gamma) = \{x_t \in \gamma : \text{F-ann } x_t \not\subset 0_1\}$. Then

- γ is called a torsion fuzzy μ-module if $T(\gamma) = \gamma$.
- γ is called a torsion free fuzzy μ-module if $T(\gamma) = \{0_t : 0 \leq t \leq \gamma(0)\}$.

where $\text{F-ann } x_t = \{r_k : r_k \in \mu\, ; r_k\, x_t \subseteq 0_{\gamma(0)}\}$.

Remark We shall denote $T(\gamma)$ by T if there is no ambiguity.

Proposition 2.4 Let γ be a fuzzy μ-module over fuzzy integral domain μ, then T is a fuzzy submodule of γ.

Proof. (I) Let $a, b \in$ M. To prove that $T(a-b) \geq \min\{T(a), T(b)\}$
Let $T(a) = k$ and $T(b) = s$. Then $a_k \in$ T and $b_s \in$ T. Hence F-ann $a_k \neq 0_1$ and F-ann $b_s \neq 0_1$. So there exists $r_t \in \mu$ and $r_t \not\subset 0_1$ such that $r_t a_k \subseteq 0_{\gamma(0)}$ and there exisist $f_\ell \in \mu$ and $f_\ell \not\subset 0_1$ such that $f_\ell b_s \subseteq 0_{\gamma(0)}$. But $r_t \not\subset 0_1$ and $f_\ell \not\subset 0_1$ implies $r_t \cdot f_\ell \not\subset 0_1$ (see prop.(2.2)) it follows that $r_t f_\ell (a_k - b_s) \subseteq 0_{\gamma(0)}$. Hence $r_t f_\ell \in$ F-ann $(a_k - b_s)$. Thus F-ann $(a_k - b_s) \not\subset 0_1$.
Hence $a_k - b_s = (a-b)_\lambda \in$ T, $\lambda = \min\{k,s\}$.
Thus $T(a-b) \geq \min\{k,s\} = \min\{T(a), T(b)\}$.
(II) Let $r \in$ R, $a \in$ M. To prove $T(r\,a) \geq \min\{\mu(r), T(a)\}$. Assume that $\mu(r) = t$ and $T(a) = k$. Since $T(a) = k$, $a_k \in$ T and so F-ann $a_k \not\subset 0_1$. Hence there exist $c_\ell \not\subset 0_1$, $c_\ell \in \mu$ such that $c_\ell a_k \subseteq 0_{T(0)}$. Now, if $r_t \not\subset 0_1$, then $r_t c_\ell \not\subset 0_1$ since μ is a fuzzy integral domain (see prop.(2.2)). It follows that
$r_t c_\ell a_k = c_\ell r_t a_k \subseteq c_\ell 0_{T(0)} \subseteq 0_{T(0)}$. Thus $c_\ell \in$ F-ann $r_t a_k$; That is F-ann $(r_t a_k) \not\subset 0_1$
Thus $r_t a_k \in$ T, so $(r\,a)_\lambda \in$ T, where $\lambda = \min\{t,k\}$
Thus implies $T(r\,a) \geq \min\{\mu(r), T(a)\}$.
Therefore, T is a fuzzy submodule.
Recall that a fuzzy submodule A of fuzzy module X over fuzzy ring R is called a fuzzy prime submodule whenever $r_t a_k \in$ A for fuzzy singleton r_t of R and $a_k \subseteq$ X we have either $r_t \subseteq (A :_R X)$ or $a_k \in$ A,

where $(A :_R X) = \{ r_t : r_t X \subseteq A, r_t$ is a fuzzy singleton of R$\}$.[8]

We introduce the following:
A fuzzy submodule γ' of a fuzzy μ-module γ is called a prime fuzzy submodule, whenever $r_t a_k \in \gamma'$ for $r_t \in \mu$ and $a_k \subseteq \gamma$, we have either $a_k \subseteq \gamma'$ or $r_t \subseteq (\gamma : \gamma')$ where $(\gamma :_R \gamma') = \{ r_t : r_t \gamma \subseteq \gamma', r_t \in \mu\}$.

Recall that if M is a module over an integral domain R, then T(M) is a prime submodule of M (9, Rem.1.2 (d)).
We have the following:
Proposition 2.5 Let X be a fuzzy module over fuzzy integral domain μ, the fuzzy μ-submodule T(X) of X is prime.
Proof. Let $r_k \in \mu$, $x_t \in$ X, if $r_k x_t \in$ T(X). We must prove that either $x_t \in$ T(X) or $r_k \in$ (T(X):μ). Suppose $x_t \notin$ T(X), so F-ann $x_t \subseteq 0_1$. But $r_k x_t \in$ T(X), implies F-ann $(r_k x_t) \not\subset 0_1$. So there exists $c_s \in \mu$, $c_s \not\subset 0_1$ such that $c_s \in$F-ann $(r_k x_t)$

Hence $c_s\,(r_k\,x_t) \subseteq 0_{X(0)}$, and so $(c_s\,r_k)\,x_t \subseteq 0_{X(0)}$ which implies $c_s\,r_k \in$ F-ann $x_t \subseteq 0_1$ and so $c_s\,r_k \subseteq 0_1$. But μ is a fuzzy integral domain so either $c_s \subseteq 0_1$ or $r_k \subseteq 0_1$.
Hence $r_k \subseteq 0_1$ since $c_s \not\subset 0_1$. Therefore, $r_k \subseteq 0_1 \in (T(X)_\mu{:}X)$.
Thus T(X) is a prime fuzzy submodule.
Now, we shall study the relation between torsion (torsion free) fuzzy module and their levels.

Proposition 2.6 Let X be a fuzzy module over fuzzy integral domain μ, then X is a torsion μ-module iff X_t is a torsion μ_t-module, $\forall\, t \in (0,X(0)]$.
Proof. (⇒) If X is a torsion fuzzy μ-module, we must prove that $T(X_t) = X_t$, $\forall\, t \in (0,X(0)]$. It's clear that $T(X_t) \subseteq X_t$. To prove $X_t \subseteq T(X_t)$. Let $y \in X_t$, hence $y_t \in X$, so $y_t \in T(X)$; That is F-ann $y_t \not\subset 0_1$
This implies that there exists $r_s \in \mu$, $r_s \not\subset 0_1$ such that $r_s\,y_t \subseteq 0_{X(0)}$. It follows that; $(r\,y)_\lambda \subseteq 0_{X(0)}$, where $\lambda = \min\{s,t\}$.
Hence $r\,y = 0$, that $r \in$ ann y . But $r \neq 0$, therefore ann $y \neq 0$ and $y \in T(X_t)$
Thus $T(X_t) = X_t$.

Conversely; If $T(X_t) = X_t$. To prove T(X) = X. It's clear that T(X) ⊆ X. Now, to prove X ⊆ T(X). Let $y_t \in X$, then $y \in X_t$ which implies $y \in T(X_t)$. Hence ann $y \neq 0$, and so F-ann $y_t \not\subset 0_1$. Because ann $y \neq 0$ implies there exists $r \in R$ such that $r \neq 0$, $r \in$ ann y and so $r\,y = 0$. It follows that $r_t\,y_k = (r\,y)_\lambda = 0_\lambda \subseteq 0_{X(0)}$, where $\lambda = \min\{t,k\}$. Hence, $r_k \in$ F-ann $y_t \not\subset 0_1$. Hence X ⊆ T(X). Thus T(X) = X.
Proposition 2.7 Let μ be a fuzzy integral domain. Let X be a fuzzy μ-module. Then X is a torsion free fuzzy μ-module iff X_t is a torsion free μ_t-submodule, $\forall\, t \in (0,X(0)]$.
Proof. If X is a torsion free fuzzy μ- module. To prove X_t is a torsion free μ_t-submodule, $\forall\, t \in (0,X(0)]$. Suppose there exists $r \in \mu_t$, $r \neq 0$ such that $r\,y = 0$. But $y \in T(X_t)$ implies $y \in X_t$, hence $X(y) \geq t$ and so $y_t \in X$, this implies $0 < t \leq X(0)$. On the other hand, $r \in \mu_t$ implies $r_t \in \mu$. It follows that $r_t\,y_t = (r\,y)_t = 0_t \subseteq 0_{X(0)}$. But $r_t \not\subset 0_1$ and $y_t \not\subset 0_{X(0)}$. Thus F-ann $y_t \not\subset 0_1$, so $y_t \in T(X)$. But X is torsion free, so $y_t = 0_t$ (see def.(2.3). Hence $y = 0$ which is a contradiction.

Conversely; If X_t is a torsion free μ_t-submodule, $\forall\, t \in (0,X(0)]$. To prove X is a torsion free μ-fuzzy μ-module. Suppose X is not a

torsion free fuzzy μ-module. Then there exists $x_t \in$ T(X) such that F-ann $x_t \not\subset 0_1$. Hence there exists $r_k \in \mu$ (i.e. $r \in \mu_k$) such that $r_k \not\subset 0_1$ and $r_k\, x_t \subseteq 0_{X(0)}$ so $(r\,x)_\lambda \subseteq 0_{X(0)}$, where $\lambda = \min\{k,t\}$. Thus $r\,x = 0$. But $r_k \not\subset 0_1$, so $r \neq 0$. Also, $x_t \in$T(X) implies $x_t \in$ X and so $x \in X_t$.
Now, if $k \geq t$, then $\mu_k \subseteq \mu_t$. But $r \in \mu_k$, so $r \in \mu_t$
Thus $r\,x = 0$, $r \in \mu_t$ and $x \in X_t$, which implies $x \in$T(X). But $T(X_t) = (0)$ since X_t is a torsion free μ_t-submodule, so ann $x = (0)$.
Hence $r = 0$ which is a contradiction.
If $t \geq k$, then $X_t \subseteq X_k$, hence $x \in X_t$. Thus $r \in \mu_k$, $x \in X_k$ and $r\,x = 0$, which implies $x \in T(X_k)$. But $T(X_k) = (0)$ since X_k is a torsion free, so ann $x = (0)$.
Hence $r = 0$ which is a contradiction. Thus X is a torsion free fuzzy μ-module.
Now, we study the direct sum of torsion (torsion free) fuzzy modules. But first we have the following result:
Lemma 2.8 If X and Y are modules over integral domain R, then T(X⊕Y) = T(X) ⊕ T(Y). In particular, if X and Y are torsion (torsion free) modules over integral domain R, then so is X⊕Y.
Proof. Let $(a,b) \in$ T(X⊕Y) then $\underset{R}{ann}\,(a,b) = (0,0)$.

Hence there exists $r \in$ R such that $r \neq 0$ and $r\,(a,b) = (0,0)$, which implies that $r\,a = 0$ and $r\,b = 0$. Then $r \in$ ann a and $r \in$ ann b; That is $(a,b) \in$ T(X) ⊕ T(Y)
Thus T(X⊕Y) $\subseteq$ T(X) ⊕ T(Y)
Now, let $(a,b) \in$ T(X) ⊕ T(Y)
Hence $a \in$ T(X) and $b \in$ T(Y). This implies there exists $r_1, r_2 \in$ R such that $r_1 \neq 0$, $r_2 \neq 0$. $r_1\,a = 0$ and $r_2\,b = 0$
It follows that $(r_1\,r_2)\,(a,b) = (r_1\,r_2\,a, r_1\,r_2\,b) = (0,0)$. But $r_1\,r_2 \neq 0$ since R is an integral domain. Hence $(a,b) \in$ T(X⊕Y). Thus T(X) ⊕ T(Y) $\subseteq$ T(X⊕Y).
Therefore, T(X⊕Y) = T(X) ⊕ T(Y).
Now, we shall define the direct sum of fuzzy modules over fuzzy ings, as a generalization of the concept of direct sum of fuzzy modules over rings (see (8)).
Definition 2.9 Let X : $M_1 \rightarrow [0,1]$ and Y : $M_2 \rightarrow [0,1]$ be μ-fuzzy μ-modules (where M_1 and M_2 are R-modules). Let X⊕Y : $M_1 \oplus M_2 \rightarrow [0,1]$ defined by: $(X\oplus Y)(a,b) = \min\{X(a), Y(b)\}$, for all $(a,b) \in M_1 \oplus M_2$.

Then $X\oplus Y$ is called the direct sum of X and Y.

Lemma 2.10 Let $X : M_1 \to [0,1]$, $Y : M_2 \to [0,1]$ be fuzzy μ-modules where M_1 and M_2 are R-modules, then $X\oplus Y$ is a fuzzy μ-modules.

Proof. For all $(a,b), (c,d) \in M_1 \oplus M_2$

(1) $$\begin{aligned}(X\oplus Y)((a,b)-(c,d)) &= X\oplus Y(a-c, b-d)\\ &= \min\{X(a-c), Y(b-d)\}\\ &\geq \min\{\min\{X(a), X(c)\}, \min\{Y(b), Y(d)\}\}\\ &= \min\{\min\{X(a), Y(b)\}, \min\{X(c), Y(d)\}\}\\ &= \min\{(X\oplus Y)(a,b), (X\oplus Y)(c,d)\}.\end{aligned}$$

(2) Let $r \in R$, $(a,b) \in X\oplus Y$.

$$\begin{aligned}(X\oplus Y)(r(a,b)) &= (X\oplus Y)(ra, rb)\\ &= \min\{X(ra), Y(rb)\}\\ &\geq \min\{\min\{\mu(r), X(a)\}, \min\{\mu(r), Y(b)\}\}\\ &= \min\{\mu(r), X(a), Y(b)\}\\ &= \min\{\mu(r), \min\{X(a), Y(b)\}\}\\ &= \min\{\mu(r), (X\oplus Y)(a,b)\}.\end{aligned}$$

Thus $X\oplus Y$ is a fuzzy μ-module.

Theorem 2.11 Let X and Y be torsion (torsion free) fuzzy μ-modules, then $X\oplus Y$ is a torsion (torsion free) fuzzy μ-modules.

Proof. If X and Y are torsion fuzzy modules.

X_t, $\forall\, t \in (0, X(0)]$ and Y_t, $\forall\, t \in (0, Y(0)]$ are torsion μ_t-modules by prop.(2.6)

Hence $X_t \oplus Y_t$ is a torsion module by lemma.(2.8)

Hence $(X\oplus Y)_t$ is a torsion module by ([3], lemma (2.2.4))

Thus $X\oplus Y$ is a torsion fuzzy μ-modules by (prop.(2.6))

In a Similar way we can prove the case of torsion free.

Section.3 The Image and Inverse Image of Torsion
(Torsion Free) Fuzzy Modules

In this section, we shall indicate the behaviour of torsion (torsion free) fuzzy μ-modules under fuzzy homomorphisms.

To do this we need some definitions and lemmas:

Definition 3.1 (1) Let f be a mapping from a set M into a set N, let A be a fuzzy set in M and B be a fuzzy set in N. The image of A denoted by $f(A)$ is the set in N defined by:

$$f(A)(y)=\begin{cases}\sup\{A(z)/z\in f^{-1}(y)\} & \text{if } f^{-1}(y)\neq\phi,\ \forall\ y\in N\\ 0 & \text{otherwise}\end{cases}$$

And the inverse image of f denoted by $f^{-1}(B)$, where $f^{-1}(B)(x) = B(f(x))$, for all $x \in M$.

Definition 3.2 (5) If γ, γ' are fuzzy μ-modules, a homomorphism from γ to γ' is a R-module homomorphism $f : M \rightarrow M'$ such that $\gamma(x) = \gamma'(f(x))$, $\forall\ x \in M$.

Proposition 3.3 ((5) prop.(3.1)) Let γ, γ' be fuzzy μ-modules, and $f : M \rightarrow M'$ be a homomorphism between them. Let γ_1' be a fuzzy submodule of γ'. Then $f^{-1}(\gamma_1')$ is a fuzzy submodule of γ.

Proposition 3.4 ((5) prop.(3.2)) Let γ, γ' be fuzzy μ-modules, and $f : M \rightarrow M'$ be a homomorphism between them. Let γ_1' be a fuzzy submodule of γ, then $f(\gamma_1)$ is a fuzzy submodule of γ'.

Now we give the following:

Proposition 3.5 Let X and Y be fuzzy μ-modules, let f be fuzzy epimorphism. If X is a torsion fuzzy μ-module, then Y is torsion fuzzy μ-module.

Proof. By definition (3.2) $Y(f(x)) = X(x)$, for all $x \in M$

We must prove that $T(Y) = Y$. It's clear that $T(Y) \subseteq Y$

let $y_t \in Y$, hence $Y(y) \geq t$. Since f is an epimorphism, $y = f(x)$, for some $x \in M$

Hence $Y(f(x)) \geq t$ which implies that $X(x) \geq t$. It follows that $x_t \in X = T(X)$, and so F-ann $x_t \neq 0_1$. Hence there exists $r_k \in \mu$, $r_k \neq 0_1$ such that $r_k x_t = 0_{X(0)}$. This implies $f(r\,x) = 0$ and so $r\,f(x) = 0$; That is $r\,y = 0$.

Hence $r_k y_t \subseteq 0_{Y(0)}$, which implies $r_k \in$ F-ann y_t.

Therefore $y_t \in T(Y)$, hence $Y \subseteq T(Y)$.

Thus $T(Y) = Y$.

Proposition 3.6 Let f be fuzzy monomorphism from a fuzzy μ-module X into a fuzzy μ-module Y. If Y is torsion fuzzy module then the inverse image of Y is torsion fuzzy μ-module.

Proof. By definitions (3.1) and (3.2), we get that

$f^{-1}(Y) = Y(f(x)) = X(x)$. So to prove X is torsion fuzzy μ-module.

Let $x_t \in X$, then $X(x) \geq t$ and $t \in [0,X(0)]$. Hence $Y(f(x)) \geq t$. Let $f(x) = y$.

Then $y_t = (f(x))_t \in Y = T(Y)$. This implies F-ann $y_t \not\subset 0_1$. So there exists $r_s \in \mu$, $r_s \not\subset 0_1$ such that $r_s\, y_t = 0_{Y(0)}$; That is $r_s\,(f(x))_t = 0_{Y(0)}$. Hence $(r f(x))_\lambda = 0_{Y(0)}$, where $\lambda = \min\{s,t\}$. Thus $f(r\,x) = 0$. It follows that $r\,x = 0$ since f is 1-1, so $r_s\, x_t \subseteq 0_{X(0)}$. On the other hand, $r_s \not\subset 0_1$, hence $x_t \in T(X)$. Thus $X = T(X)$.

Therefore X is torsion fuzzy module.

Proposition 3.7 Let f be fuzzy monomorphism from a fuzzy μ-module X into a fuzzy μ-module Y. Then if Y is torsion free fuzzy module, then the inverse image of Y is a torsion free fuzzy μ-module.

Proof. Since $f^{-1}(Y) = X$, we must prove that X is torsion free.

Suppose there exists $x_t \in X$, $x_t \not\subset 0_{X(0)}$ such that F-ann $x_t \not\subset 0_1$, so there exists $r_s \in \mu$, such that $r_s \not\subset 0_1$ and $r_s\, x_t \subseteq 0_{X(0)}$. Hence $x \neq 0$, $r \neq 0$ and $r\,x = 0$

But $x \neq 0$ implies $y = f(x) \neq 0$ since f is 1-1.

Moreover, $x_t \in X$ implies $X(x) \geq t$ and hence $Y(y) = X(x) \geq t$. Thus $y_t \in Y$.

On the other hand, $r_s\, y_t = (r\,y)_\lambda = (f(rx))_\lambda = 0_\lambda \subseteq 0_{Y(0)}$, where $\lambda = \min\{s,t\}$.

Thus $y_t \in T(Y)$ and so $T(Y) \neq \{0_k : 0 \leq k \leq Y(0)\}$; That is Y is not torsion free, which is a contradiction. Therefore X is torsion free. Notice that we have no examples to explain that the conditions f is an epimorphism in prop. 3.5 and f is monomorphism in prop. 3.6 and 3.7 can not be dropped.

Definition 3.8 [5] If γ, γ' are fuzzy μ-modules and $f : M \to M'$ is homomorphism between them. The fuzzy kernel of f, F-ker f is the fuzzy subset of M defined by:

$$F-\ker f(x) = \begin{cases} \gamma(0) & \text{if } x \in \ker f \\ 0 & \text{if } x \notin \ker f \end{cases}$$

Now, we give the following proposition which we needed later.

Proposition 3.9 Let γ and γ' be fuzzy modules fuzzy rings M and M' respectively, let f be fuzzy homomorphism between them, then

F-ker $f = \{x_t \in \gamma : x = 0 \text{ or } t = 0\} \Leftrightarrow f$ is 1-1.

Proof. ($\Leftarrow$) To prove F-ker $f = \{x_t \in \gamma : x = 0 \text{ or } t = 0\}$.

Suppose $x_t \in$ F-ker f and $x \neq 0$

$x \neq 0$ implies $x \notin \ker f$, because f is 1-1. Hence F-ker $f = 0$ by definition (3.8), But $x_t \in$ F-ker f, then (F-ker f) $(x) \geq t$. Hence $0 \geq t$. This implies $t = 0$ since $t \geq 0$.

$(\Rightarrow)$ To prove f is 1-1. Let $x, y \in$ M such that $f(x) = f(y)$. It's follows that $x - y \in \ker f$. Hence F-ker $f(x - y) = \gamma(0)$ by definition (3.8)

This implies $(x - y)_{\gamma(0)} \in$ F-ker f, then either $x - y = 0$ or $\gamma(0) = 0$

But $\gamma(0) = 0$ is not true, hence $x - y = 0$

Therefore $x = y$ and f is 1-1.

Proposition 3.10 Let X be a fuzzy module over a fuzzy integral domain μ, then f_r is 1-1 iff X is a torsion free fuzzy μ-module. Where f_r is the left multiplication endomorphism of μ by r.

Proof. If f_r is 1-1. Suppose X is not a torsion free fuzzy μ-module.

Suppose there exists $x_t \in$ X such that $x_t \not\subset 0_{X(0)}$ and $x_t \in$ T(X).

Then $x \neq 0$ and F-ann $x_t \not\subset 0_1$.

Hence $\exists\ r_k \in \mu$ such that $r \neq 0$, $0 < k \leq 1$ such that $r_k x_t \subseteq 0_{X(0)}$.

It follows that $r x = 0$ by prop.(1.9)

Hence $x \in \ker f_r$ So that $x = 0$ which is contradiction. Since $X(0) \neq 0$

Thus X is a torsion free fuzzyμ'-module.

Conversely, If X is torsion free, to prove f_r is 1-1, we shall use prop.(3.9)

Suppose there exists $x_t \in$ F-ker f_r such that $x \neq 0$ and $t \neq 0$. Then (F-ker f) $\geq t$, and hence $x \in \ker f_r$ (by definition (3.8).

It follows that $f_r(x) = r x = 0$, and $\forall\ k$, $0 < k \leq 1$,$r_k x_t = (r x)_\lambda = 0_\lambda \subseteq 0_{X(0)}$, where $\lambda = \min\{k,t\}$.

Thus $x_t \in$ T(X) and $x_t \not\subset 0_t$ which is a contradiction since X is torsion free.

Therefore either $x = 0$ or $t = 0$, and hence by prop.(3.9), f_r is 1-1.

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المقاسات الملتوية (طليقة الالتواء) الضبابية على الساحات الضبابية

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الخلاصة

في هذا البحث بينا مفهوم المقاسات الضبابية ملتوية (طليقة الالتواء) على ساحة ضبابية كتعميم لمفهوم المقاسات الملتوية (طليقة الالتواء).