Comparison of Bayes' Estimators for the Exponential Reliability Function Under Different Prior Functions

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Abstract

In this study, we derived the estimation for Reliability of the Exponential distribution based on the Bayesian approach. In the Bayesian approach, the parameter of the Exponential distribution is assumed to be random variable .We derived posterior distribution the parameter of the Exponential distribution under four types priors distributions for the scale parameter of the Exponential distribution is: Inverse Chi-square distribution, Inverted Gamma distribution, improper distribution, Non-informative distribution. And the estimators for Reliability is obtained using the two proposed loss function in this study which is based on the natural logarithm for Reliability function .We used simulation technique, to compare the resultant estimators in terms of their mean squared errors (MSE).Several cases assumed for the parameter of the exponential distribution for data generating of different samples sizes (small, medium, and large). The results were obtained by using simulation technique, Programs written using MATLAB-R2008a program were used. In general, we obtained a good estimations of reliability of the Exponential distribution under the second proposed loss function according to the smallest values of mean squared errors (MSE) for all samples sizes (n) comparative to the estimated values for MSE under the first proposed loss function.

Key words: The Exponential, Bayes method, the prior distributions: Inverse Chi-square distribution , Inverted Gamma distribution, improper distribution, non-informative distribution, mean squared errors (MSE).

Introduction

The exponential distribution is one of the most important distributions in life-testing and reliability studies. Inference procedures for the exponential distribution and applications in the context of life-testing and reliability have been discussed by many authors. We mention some of them in a brief manner: Chiou (1993) [1] proposed two empirical Bayes shrinkage estimators for the reliability of the exponential distribution and study their properties. Under the uniform prior distribution and the inverted gamma prior distribution these estimators are developed and compared with a preliminary test estimator with a shrinkage estimator in terms of mean squared error. Baklizi (2003) [2] investigated the advantages of incorporating prior information in the reliability function through the shrinkage estimators. His work is an effort to coin unified shrinkage estimators of reliability function for five lifetime distributions commonly used to model lifetime data by biologists, physicists, engineers and statisticians for living and non-living entities. Sarhan (2003) [3] exploits past experiments to approximate a prior information (prior density) into the model, in estimating reliability function and parameter of exponential distribution, by using bayesian approach. Balakrishnan, Lin and Chan(2005) [4] made a comparison of these two prediction intervals based on the expected width of the prediction interval, as well as by means of the probability of the width of one being smaller than the other. Friesl and Hurt (2007) [5] gave some basic ideas of both the construction and investigation of the properties of the Bayesian estimates of certain parametric functions of the exponential distribution under the model of random censorship assuming the Koziol–Green model. Various prior distributions are investigated and the corresponding estimates are derived.Liu and Ren (2013) [6] studied the empirical Bayes estimation of the parameter of the exponential distribution. In the empirical Bayes procedure, they employ the non-parameter polynomial density estimator to the estimation of the unknown marginal probability density function, instead of estimating the unknown prior probability density function of the parameter. They Empirical derived bayes estimators for the parameter of the exponential distribution under squared error and LINEX loss functions. So in this paper, we try to find best estimation for Reliability function ($R(t)$) of exponential distribution which it means the probability of surviving at least till age t. According to the smallest value of Mean Square Errors (MSE) were calculated to compare bayes estimators under four types of prior distributions to get bayes estimation :Inverse Chi-square distribution , Inverted Gamma distribution, Improper distribution ,Non-informative distribution when the Bayesian estimation is based on two proposed loss functions .Several cases from exponential distribution for data generating ,for different sample sizes (small, medium, and large).The results were obtained by using simulation technique, Programs written using MATLAB-R2008a program were used.

Exponential Distribution

We consider $t_1, t_2, \ldots, t_n$ is a random sample of n independent observations from an Exponential distribution having the probability density function (pdf) define as [7]:

$$f(t;\theta) = \theta^{-1} \exp(-\frac{t}{\theta}) \quad , \quad t > 0 \qquad \ldots (1)$$

where $\theta > 0$ is mean, standard deviation, and scale parameter of the distribution, θ is a survival parameter in the sense that if a random variable t is the duration of time that a given biological or mechanical system manages to survive, and $t \sim Exp(\theta)$ then the expected duration of survival of the system is θ units of time .So the cumulative (distribution) function is

$$F(t)=\int_0^t f(u)du = 1-\exp(-\frac{t}{\theta}) \quad , \quad t>0 \qquad \ldots (2)$$

Also, the Reliability function is

$$R(t)=1-F(t)=\int_t^\infty f(u)du=\exp(-\frac{t}{\theta}) \qquad \ldots (3)$$

Where R(t) is probability of surviving at least till age t .And F (t) is the cumulative distribution function.

Bayes Estimation Method

In this section, we used several methods to estimate Reliability function($R(t)$).Let $\underline{t}=(t_1,t_2,\ldots,t_n)$ be a random sample of size n with probability density function given in equation (1) and likelihood function from the Exponential pdf given in (1) will be as follows[7]:

$$L(\underline{t}\backslash\theta)=\prod_{i=1}^{n} f(t;\theta)=\theta^{-n}\exp(-\frac{\sum_{i=1}^{n} t_i}{\theta}) \qquad \ldots (4)$$

In this paper the posterior distributions for the unknown parameter θ are derived using the following four types of priors, and then get bayes estimation [7]:

1. Inverse Chi-square distribution[8].
2. Inverted Gamma distribution[9].
3. Improper distribution.
4. Non-informative distribution.

1- The posterior distribution using different priors

It is assumed that θ follows four types of prior distributions with pdf as given in table below:

The four types of prior distributions ($P(\theta)$) with pdf for θ.

Prior distribution	$P(\theta)$
$\theta\sim$Inverse Chi-square(v)	$P(\theta)\ \alpha\ \frac{1}{2^{\frac{v}{2}}}\ \theta^{-\frac{v}{2}-1}\ \exp(-\frac{1}{2\theta})$ for $v,\ \theta>0$
$\theta\sim$Inverted Gamma(α,β)	$P(\theta)\ \alpha\ \frac{\beta^{\alpha}}{\Gamma\alpha}\ \theta^{-(\alpha+1)}\ \exp(-\frac{\beta}{\theta})$ for $\alpha,\beta,\ \theta>0$
$\theta\sim$Improper(a,b)	$P(\theta)\ \alpha\ \theta^{-(a+1)}\ \exp(-\frac{b}{\theta})$ for $b,\ \theta>0$ and $-\infty<a<\infty$
$\theta\sim$Non-informative(c)	$P(\theta)\ \alpha\ \frac{1}{\theta^{c}}$ for $\theta, c>0$

Then the posterior distribution of given data $\underline{t}=(t_1,t_2,\ldots,t_n)$ is:

$$P(\theta\backslash t)=\frac{L(\underline{t}\backslash\theta)\ P(\theta)}{\int_\theta L(\underline{t}\backslash\theta)\ P(\theta)d\theta} \qquad \ldots (5)$$

Substituting the equation (4) and for each $P(\theta)$ as shown in the table above in equation (5), we get the posterior distributions for the unknown parameter θ are derived using the following four types of priors (for more details see Appendix-A).

The posterior distributions ($P(\theta \backslash t)$) for the unknown parameter (θ) are derived using the following four types of priors.

Prior dist^n.	The posterior distribution ($P(\theta \backslash t)$)
Inverse Chi-square	$P_1(\theta \backslash t) \sim$ Inverted Gamma ($\alpha_{(new)} = (n+\frac{v}{2}), \beta_{(new)} = (\sum_{i=1}^{n} t_i + \frac{1}{2})$) with pdf $P_1(\theta \backslash t) = \frac{(\sum_{i=1}^{n} t_i + \frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2})+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + \frac{1}{2}))$ $n, v, \theta > 0$
Inverted Gamma	$P_2(\theta \backslash t) \sim$ Inverted Gamma ($\alpha_{(new)} = (n+\alpha), \beta_{(new)} = (\sum_{i=1}^{n} t_i + \beta)$) with pdf $P_2(\theta \backslash t) = \frac{(\sum_{i=1}^{n} t_i + \beta)^{(n+\alpha)} \theta^{-[(n+\alpha)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + \beta))}{\Gamma(n+\alpha)}$ $n, \beta, \alpha, \theta > 0$
Improper	$P_3(\theta \backslash t) \sim$ Inverted Gamma ($\alpha_{(new)} = (n+a), \beta_{(new)} = (\sum_{i=1}^{n} t_i + b)$) with pdf $P_3(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i + b)^{(n+a)} \theta^{-[(n+a)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + b))}{\Gamma(n+a)}$ $n, b, \theta > 0$ and $-\infty < a < \infty$
Non-informative	$P_4(\theta \backslash t) \sim$ Inverted Gamma ($\alpha_{(new)} = (n+c-1), \beta_{(new)} = (\sum_{i=1}^{n} t_i)$) with pdf $P_4(\theta \backslash t) = \frac{(\sum_{i=1}^{n} t_i)^{(n+c-1)} \theta^{-[(n+c-1)+1]} \exp(-\frac{1}{\theta}\sum_{i=1}^{n} t_i)}{\Gamma(n+c-1)}$ $n, c, \theta > 0$

2- Bayes' Estimators

Bayes' estimators for Reliability function ($R = R(t)$), was considered with four different priors and under two loss functions proposed:

1. The first proposed loss function $L_1(\hat{R}, R) = (\ln \hat{R} - \ln R)^2$.

2. The second proposed loss function $L_2(\hat{R}, R) = \frac{(\ln \hat{R} - \ln R)^2}{\ln R}$.

Where $\hat{R}$ an estimator for is R, was considered with different four priors, and under two loss functions proposed. The following is the derivation of these estimators:

- The first proposed loss function

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$$L_1(\hat{R}, R) = (\ln \hat{R} - \ln R)^2 \quad \dots (6)$$

After simplified steps, we get Bayes estimator of R(t) denoted by $\hat{R}_{pro.1(i)}(t)$ for the above prior as follows

$$\hat{R}_{pro.1(i)}(t) = \exp \int_0^\infty \ln R(t) P(\theta \backslash t) d\theta \quad \dots (7)$$

So, the following results are the derivations of these estimators under the first proposed loss function with different four priors (for more details see Appendix-B).

The estimators ($\hat{R}_{pro.1(i)}(t)$) under the squared error loss function with different four priors.

Prior distribution	$\hat{R}_{pro.1(i)}(t) = \exp(\int_0^\infty \ln R(t) P(\theta \backslash t) d\theta)$
Inverse Chi-square	$\hat{R}_{pro.1(1)}(t) = \exp[\frac{-t\ \Gamma(n+(v/2)+1)}{\Gamma(n+(v/2))\ (\sum_{i=1}^n t_i + 0.5)}]$, n & v > 0
Inverted Gamma	$\hat{R}_{pro.1(2)}(t) = \exp[\frac{-t\ \Gamma(n+\alpha+1)}{\Gamma(n+\alpha)(\sum_{i=1}^n t_i + \beta)}]$, n, β, α > 0
Improper	$\hat{R}_{pro.1(3)}(t) = \exp[\frac{-t\ \Gamma(n+a+1)}{\Gamma(n+a)(\sum_{i=1}^n t_i + b)}]$, n, b, a > 0
Non-informative	$\hat{R}_{pro.1(4)}(t) = \exp[\frac{-t\ \Gamma(n+c)}{\Gamma(n+c-1)(\sum_{i=1}^n t_i)}]$, n, c > 0

Where $\Gamma(.)$ is a gamma function.

-The second proposed loss function

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$$L_2(\hat{R}, R) = \frac{(\ln \hat{R} - \ln R)^2}{\ln R} \quad \dots (8)$$

After simplified steps, we get Bayes estimator of R(t) denoted by $\hat{R}_{pro2(i)}(t)$ for the above prior as follows

$$\hat{R}_{pro.2(i)}(t) = \exp\left(\frac{1}{\int_0^\infty \frac{1}{\ln R(t)} P(\theta \backslash t) d\theta}\right)^{-1} \quad \dots (9)$$

So, the following results are the derivations of these estimators under the second proposed loss function with different four priors (for more details see Appendix- C).

The estimators ($\hat{R}_{pro.2(i)}(t)$) under the squared error loss function with different four priors.

Prior distribution	$\hat{R}_{pro.2(i)}(t)=\exp\left(\int_0^{\infty}\frac{1}{\ln R(t)}P(\theta\backslash t)\,d\theta\right)^{-1}$
Inverse Chi-square	$\hat{R}_{pro.2(1)}(t)=\exp\left(-\frac{(\sum_{i=1}^{n}t_i+\frac{1}{2})\Gamma(n+\frac{v}{2}-1)}{t\quad\Gamma(n+\frac{v}{2})}\right)^{-1}$, $n\,\&\,v>0$
Inverted Gamma	$\hat{R}_{pro.2(2)}(t)=\exp\left(-\frac{\Gamma(n+\alpha-1)(\sum_{i=1}^{n}t_i+\beta)}{t\quad\Gamma(n+\alpha)}\right)^{-1}$, $n,\beta,\alpha>0$
Improper	$\hat{R}_{pro.2(3)}(t)=\exp\left(-\frac{\Gamma(n+a-1)(\sum_{i=1}^{n}t_i+b)}{t\quad\Gamma(n+a)}\right)^{-1}$, $n,b,a>0$
Non-informative	$\hat{R}_{pro.2(4)}(t)=\exp\left(-\frac{(\sum_{i=1}^{n}t_i)\Gamma(n+c-2)}{t\quad\Gamma(n+c-2)}\right)^{-1}$, $n,c>0$

Where $\Gamma(.)$ is a gamma function,

Simulation Study

In this study, we have generated random samples from Exponential distribution and compared the performance of Bayes estimators based on them. So we have considered several steps to perform simulation study as follows:

1. We have chosen sample size n = 30, 60, 90 and 120 to represent small, moderate and large sample size.

2. We generated data from Exponential distribution for the scale parameter; we have considered randomly several values for the parameter of Exponential distribution $\theta=0.5,\ 1,\ 1.5,\ 2.5$.

3. We choose the values for the parameters of the prior distributions that give the appropriate estimation for Reliability function ($R(t)$), as shown belows:

- We used two values for the parameter of the Inverse Chi-square distribution (v= 4) as prior distribution for θ.

- We used two values for the parameters of the Inverted Gamma distribution $(\alpha,\beta)=(5,2)$ as prior distribution for θ.

- We used two values for the parameters of the improper distribution $(a,b)=(9,3)$ as prior distribution for θ.

- We used randomly three values for the function of the non-informative prior distribution c =1.

4. The number of replication used was ($L = 1000$) for each sample size (n).
5. The true $R(t)$ is computed according to the formula (3) with $\theta = 0.5,\ 1,\ 1.5, 2.5$ and the true t is $t = 0.5,\ 1.5,\ 2.5, 3.5$.
6. We obtained estimators for Reliability function ($R(t)$), the estimators in the table in section (3.2.1), it means the estimators $\hat{R}_{pro.1(i)}(t)$ under the first proposed loss function with four different priors .And the estimators in the table in section (3.2.2) , it means the estimators $\hat{R}_{pro2(i)}(t)$ under the second proposed loss function with different four priors.

The simulation program was written by using MATLAB-R2008a program. After the Reliability function ($R(t)$), was estimated, Mean Square Errors (MSE) was calculated to compare between the bayes estimators, So we have the following criterion:

$$MSE(\hat{R}(t)) = \frac{1}{L}\sum_{L=1}^{1000}(\hat{R}_L(t) - R(t))^2 \qquad ...(10)$$

See appendix-D, for the programs algorithm. The results of the simulation study are summarized and tabulated in tables (1 to 4) see appendix-E.In each row of tables (1 to 4) ,we have four estimated values for R(t) $\hat{R}_{pro.1(i)}(t)$, with MSE for all sample size (n) and values (v, (α,β), (a , b), c) respectively.
Also the results of the simulation study are summarized and tabulated in tables (5 to 8) see appendix-E . In each row of tables (5 to 8) ,we have four estimated values for R(t) $\hat{R}_{pro.2(i)}(t)$ with MSE for all sample size (n) and values (v, (α,β), (a, b), c) respectively .The Bayes estimators under four types of prior distribution .So our criteria is the best estimator that gives the smallest value of MSE. We list the results in the tables (1 to 8) in appendix-E.

Discussion

In general, as we see in the tables (1 to 8) by using different estimation methods, See appendix-E. We find the Mean Square Errors (MSE) was decreased when sample size increased in all cases. That means the estimation of $(\hat{R}(t))$ get better for the large sample sizes. We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values for MSE under the first proposed loss function.
As we see <u>in table -1,</u> when the true value of θ ($\theta = 0.5$) and the prior distribution for θ is

- Inverted Gamma distribution with ($\alpha = 5, \beta = 2$) for t=0.5.
- Improper distribution with (a=9, b=3) for t=1.5, 2.5, 3.5.

And we see <u>in table -2,</u> when the true value of θ ($\theta = 1$) and the prior distribution for θ is

- Non-informative distribution with c=1 for t=0.5, 1.5.
- Inverse Chi-square distribution with v=4 for t=2.5.

- Inverted Gamma distribution with $(\alpha = 5, \beta = 2)$ for t=3.5.

And we see in table -3, when the true value of $\theta(\theta = 1.5)$ and the prior distribution for θ is

- Non-informative distribution with c=1 for t=0.5, 1.5, 2.5.
- Inverse Chi-square distribution with v=4 for t=3.5.

And we see in table -4, when the true value of $\theta(\theta = 2.5)$ and the prior distribution for θ is Non-informative distribution with c=1 for all. See appendix-E.
We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values for MSE under the second proposed loss function.
As we see in table -5, when the true value of $\theta(\theta = 0.5)$ and the prior distribution for θ is

- Inverted Gamma distribution with $(\alpha = 5, \beta = 2)$ for t=0.5 .
- Improper distribution with (a=9, b=3) for t=1.5, 2.5, 3.5.

And we see in table -6, when the true value of $\theta(\theta = 1)$ and the prior distribution for θ is

- Non-informative distribution with c=1 for t=0.5.
- Inverted Gamma distribution with $(\alpha = 5, \beta = 2)$ for t=1.5, 2.5.
- Improper distribution with (a=9, b=3) for t=3.5.

And we see in table -7, when the true value of $\theta(\theta = 1.5)$ and the prior distribution for θ is

- Non-informative distribution with c=1 for t=0.5, 1.5.
- Inverse Chi-square distribution with v=4 for t=2.5.
- Inverted Gamma distribution with $(\alpha = 5, \beta = 2)$ for t=3.5.

And we see in table -8, when the true value of $\theta(\theta = 2.5)$ and the prior distribution for θ is

- Non-informative distribution with c=1 for t=0.5, 1.5, 2.5.
- Inverse Chi-square distribution with v=4 for t=3.5.

See the summary of discussion for $MSE(\hat{R}(t))$ in tables (9) and (10) in Appendix-E.

Conclusion

When we compared the estimated values for Reliability(R(t)) of the Exponential distribution by using Bayes with respect to Mean Square Errors (MSE) of estimated exponential reliability function under the two proposed loss function in this study .We find that MSE is decreasing when sample size is increasing in all cases. The estimated values for Reliability ($\hat{R}_{pro.2}(t)$) under the second proposed loss function is the best of the estimated values for ($\hat{R}_{pro.1}(t)$) under the first proposed loss function, according to the smallest values of MSE for all sample sizes (n), for the same prior distribution for θ for some t , when the true value of $\theta(\theta = 0.5 \& \theta = 2.5)$.See tables (9) and (10) in Appendix-E. Also ,we obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values for MSE under the second proposed loss function, when the prior distribution for θ is

- Non-informative distribution with c=1 for t=0.5, and inverted Gamma distribution with $(\alpha=5, \beta=2)$ for t= 1.5, 2.5 when the true value of $\theta\,(\theta=1)$.

- Non-informative distribution with c=1 for t=0.5 and inverse Chi-square distribution with v=4 for t=2.5, and inverted Gamma distribution with $(\alpha=5, \beta=2)$ for t= 3.5, when the true value of $\theta\,(\theta=1.5)$. See tables (9) and (10) in Appendix-E.

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<u>Appendix-A</u>: The posterior distribution using different Priors.

1. <u>**The posterior distribution using Inverse Chi-square distribution as prior:**</u>

It is assumed that θ follows the Inverse Chi-square distribution with pdf as given below:

$$P(\theta)\ \alpha\ \frac{1}{2^{\frac{v}{2}}}\ \theta^{-\frac{v}{2}-1}\ \exp\left(-\frac{1}{2\theta}\right) \quad \text{for} \quad v,\ \theta > 0 \qquad \ldots(A.1)$$

Then the posterior distribution of given the data $\underline{t} = (t_1, t_2, \ldots, t_n)$ is:

$$P(\theta \setminus t) = \frac{L(t \setminus \theta)\ \ P(\theta)}{\int_{\theta} L(\underline{t} \setminus \theta)\ \ P(\theta)\, d\theta} \qquad \ldots(A.2)$$

Substituting the equation (4) and the equation (A.1) in equation (A.2), we get:

$$P_i(\theta \setminus t) = \frac{\theta^{-n}\ \exp\left(-\frac{\Sigma_{i=1}^{n} t_i}{\theta}\right)\left[\frac{1}{2^{\frac{v}{2}}}\ \theta^{-\frac{v}{2}-1}\ \exp\left(-\frac{1}{2\theta}\right)\right]}{\int_0^{\infty} \theta^{-n}\ \exp\left(-\frac{\Sigma_{i=1}^{n} t_i}{\theta}\right)\left[\frac{1}{2^{\frac{v}{2}}}\ \theta^{-\frac{v}{2}-1}\ \exp\left(-\frac{1}{2\theta}\right)\right] d\theta} \qquad \ldots(A.3)$$

$$P_i(\theta \setminus t) = \frac{\theta^{-[(n+\frac{v}{2})+1]}\ e^{-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \frac{1}{2})}}{\int_0^{\infty} \theta^{-[(n+\frac{v}{2})+1]}\ e^{-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \frac{1}{2})}\, d\theta} \qquad \ldots(A.4)$$

By multiplying the integral in equation (A.4) by the quantity which equals to

$$\left(\frac{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})}\right)\left(\frac{\Gamma(n+\frac{v}{2})}{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})^{(n+\frac{v}{2})}}\right)$$

, where $\Gamma(.)$ is a gamma function .Then we get,

$$P_i(\theta \setminus t) = \frac{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})^{(n+\frac{v}{2})}\ \theta^{-[(n-\frac{v}{2})+1]}\ \exp\left(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \frac{1}{2})\right)}{\Gamma(n+\frac{v}{2})\quad A(t;\theta)} \qquad \ldots(A.5)$$

Where $A(t;\theta)$ equals to

$$A(t;\theta) = \int_0^{\infty} \frac{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})^{(n+\frac{v}{2})}\ \theta^{-[(n+\frac{v}{2})+1]} \exp\left(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \frac{1}{2})\right)}{\Gamma(n+\frac{v}{2})}\, d\theta = 1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{t} = (t_1, t_2, \ldots, t_n)$ is

$$P_1(\theta \backslash t) = \frac{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2})+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \frac{1}{2})) \qquad \ldots (A.6)$$

It means that $P_1(\theta \backslash t)) \sim$ Inverted Gamma distribution with new parameters $(\alpha_{(new)} = (n + \frac{v}{2}), \beta_{(new)} = (\Sigma_{i=1}^{n} t_i + \frac{1}{2}))$.

2. The posterior distribution using Inverted Gamma distribution as prior:

It is assumed that θ follows the Inverted Gamma distribution with pdf as given below:

$$P(\theta)\ \alpha\ \frac{\beta^{\alpha}}{\Gamma\alpha} \theta^{-(\alpha+1)} \exp(-\frac{\beta}{\theta}) \quad \text{for} \quad \alpha, \beta, \theta > 0 \qquad \ldots (A.7)$$

Then the posterior distribution of given the data $\underline{t} = (t_1, t_2, \ldots, t_n)$ according to the equation (A.2), we get it by substituting the equation (4) and the equation (A.7) in equation (A.2), so we have

$$P_2(\theta \backslash t) = \frac{\theta^{-n} \exp(-\frac{\Sigma_{i=1}^{n} t_i}{\theta}) [\frac{\beta^{\alpha}}{\Gamma\alpha} \theta^{-(\alpha+1)} \exp(-\frac{\beta}{\theta})]}{\int_0^{\infty} \theta^{-n} \exp(-\frac{\Sigma_{i=1}^{n} t_i}{\theta}) [\frac{\beta^{\alpha}}{\Gamma\alpha} \theta^{-(\alpha+1)} \exp(-\frac{\beta}{\theta})] d\theta} \qquad \ldots (A.8)$$

$$P_2(\theta \backslash t) = \frac{\theta^{-[(n+\alpha)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \beta))}{\int_0^{\infty} \theta^{-[(n+\alpha)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \beta) d\theta} \qquad \ldots (A.9)$$

By multiplying the integral in equation (A.9) by the quantity which equals to

$$\left(\frac{(\Sigma_{i=1}^{n} t_i + \beta)^{(n+\alpha)}}{\Gamma(n+\alpha)}\right) \left(\frac{\Gamma(n+\alpha)}{(\Sigma_{i=1}^{n} t_i + \beta)^{(n+\alpha)}}\right)$$

, where $\Gamma(.)$ is a gamma function .Then we get,

$$P_2(\theta \backslash t) = \frac{(\Sigma_{i=1}^{n} t_i + \beta)^{(n+\alpha)} \theta^{-[(n+\alpha)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \beta)}{\Gamma(n+\alpha) \quad B(t;\theta)} \qquad \ldots (A.10)$$

Where $B(t;\theta)$ equals to

$$P_2(\theta \backslash t) = \frac{(\Sigma_{i=1}^{n} t_i + \beta)^{(n+\alpha)}}{\Gamma(n+\alpha)} \theta^{-[(n+\alpha)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \beta)) \qquad \ldots (A.11)$$

$B(t;\theta) = \int_0^{\infty} \left(\frac{(\Sigma_{i=1}^{n} t_i + \beta)^{(n+\alpha)}}{\Gamma(n+\alpha)}\right) \theta^{-[(n+\alpha)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \beta))\, d\theta = 1$. Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{t} = (t_1, t_2, \ldots, t_n)$ is

It means that $P_2(\theta \backslash t) \sim$ Inverted Gamma distribution with new parameters ($\alpha_{(new)} = (n+\alpha)$, $\beta_{(new)} = (\sum_{i=1}^{n} t_i + \beta)$).

3. The posterior distribution using improper distribution as prior:

It is assumed that θ follows the improper distribution with pdf as given below:

$$P(\theta) \propto \theta^{-(a+1)} \exp(-\frac{b}{\theta}) \quad \text{for } b, \theta > 0 \text{ and } -\infty < a < \infty \qquad \dots (A.12)$$

Then the posterior distribution of given the data $\underline{t} = (t_1, t_2, \dots, t_n)$ according to the equation (A.2), we get it by substituting the equation (4) and the equation (A.12) in equation (A.2), so we have

$$P_3(\theta \backslash t) = \frac{\theta^{-n} \exp(-\frac{\sum_{i=1}^{n} t_i}{\theta})[\theta^{-(a+1)} \exp(-\frac{b}{\theta})]}{\int_0^\infty \theta^{-n} \exp(-\frac{\sum_{i=1}^{n} t_i}{\theta})[\theta^{-(a+1)} \exp(-\frac{b}{\theta})]d\theta} \qquad \dots (A.13)$$

$$P_3(\theta \backslash t) = \frac{\theta^{-[(n+a)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + b))}{\int_0^\infty \theta^{-[(n+a)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + b))d\theta} \qquad \dots (A.14)$$

By multiplying the integral in equation (A.14) by the quantity which equals to

$(\frac{(\sum_{i=1}^{n} t_i + b)^{(n+a)}}{\Gamma(n+a)})(\frac{\Gamma(n+a)}{(\sum_{i=1}^{n} t_i + b)^{(n+a)}})$, where $\Gamma(.)$ is a gamma function .Then we get,

$$P_3(\theta \backslash t) = \frac{(\sum_{i=1}^{n} t_i + b)^{(n+a)} \theta^{-[(n+a)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + b))}{\Gamma(n+a) \quad C(t;\theta)} \qquad \dots (A.15)$$

Where $C(x;\theta)$ equals to

$C(t;\theta) = \int_0^\infty \frac{(\sum_{i=1}^{n} t_i + b)^{(n+a)} \theta^{-[(n+a)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + b))}{\Gamma(n+a)} d\theta = 1$. Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{t} = (t_1, t_2, \dots, t_n)$ is

$$P_3(\theta \backslash t) = \frac{(\sum_{i=1}^{n} t_i + b)^{(n+a)}}{\Gamma(n+a)} \theta^{-[(n+a)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + b)) \qquad \dots (A.16)$$

It means that $P_3(\theta \backslash t) \sim$ Inverted Gamma distribution with new parameters ($\alpha_{(new)} = (n+a)$, $\beta_{(new)} = (\sum_{i=1}^{n} t_i + b)$).

4. The posterior distribution using Non-informative distribution as prior:

It is assumed that θ follows the non-informative distribution with pdf as given below:

$$P(\theta)\ \alpha\ \frac{1}{\theta^{c}} \quad \text{for} \quad \theta, c > 0 \qquad \ldots(A.17)$$

Then the posterior distribution of given the data $\underline{t}=(t_1,t_2,\ldots,t_n)$ according to the equation (A.2), we get it by substituting the equation (4) and the equation (A.17) in equation (A.2), so we have

$$P_s(\theta \backslash t) = \frac{\theta^{-n}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)[\ \theta^{-c}\ ]}{\int_0^{\infty} \theta^{-n}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)[\ \theta^{-c}\ ]\,d\theta} \qquad \ldots (A.18)$$

$$P_s(\theta \backslash t) = \frac{\theta^{-(n+c)}\ \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} t_i\right)}{\int_0^{\infty} \theta^{-(n+c)}\ \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} t_i\right)d\theta} \qquad \ldots (A.19)$$

We can write $\theta^{-(n+c)}$ as $\theta^{-[(n+c-1)+1]}$, and by multiplying the integral in equation (A.19), by the quantity which equals to

$\left(\frac{(\sum_{i=1}^{n} t_i)^{(n+c-1)}}{\Gamma(n+c-1)}\right)\left(\frac{\Gamma(n+c-1)}{(\sum_{i=1}^{n} t_i)^{(n+c-1)}}\right)$, where $\Gamma(.)$ is a gamma function. Then we get

$$P_s(\theta \backslash t) = \frac{(\sum_{i=1}^{n} t_i)^{(n+c-1)}\ \theta^{-[(n+c-1)+1]}\exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} t_i\right)}{\Gamma(n+c-1)\quad D(t;\theta)} \qquad \ldots (A.20)$$

Where $D(x;\theta)$ equals to

$D(t;\theta) = \int_0^{\infty} \frac{(\sum_{i=1}^{n} t_i)^{(n+c-1)}\ \theta^{-[(n+c-1)+1]}\exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} t_i\right)}{\Gamma(n+c-1)}\,d\theta = 1$. Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{t}=(t_1,t_2,\ldots,t_n)$ is

$$P_s(\theta \backslash t) = \frac{(\sum_{i=1}^{n} t_i)^{(n+c-1)}}{\Gamma(n+c-1)}\theta^{-[(n+c-1)+1]}\exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} t_i\right) \qquad \ldots (A.21)$$

It means that $P_s(\theta \backslash t) \sim$ Inverted Gamma distribution with new parameters $(\alpha_{(new)} = (n+c-1),\ \beta_{(new)} = (\sum_{i=1}^{n} t_i))$.

Appendix-B: **The following is the derivation of these estimators under the first proposed loss function.**

1. **The first proposed loss function**

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$L_1(\hat{R}, R) = (\ln\hat{R} - \ln R)^2$, where $R = R(t)$ & $\hat{R} = \hat{R}(t)$, the risk function is:

$$Risk = E[L_1(\hat{R}, R)] \quad \ldots(B.1)$$

$$Risk = E[(\ln\hat{R} - \ln R)^2]$$

$$Risk = \int_\theta (\ln\hat{R} - \ln R)^2 \; P(\theta \setminus t)\, d\theta$$

$$Risk = \int_\theta (\ln\hat{R}^2 - 2\ln R \ln\hat{R} + \ln R^2)\; P(\theta \setminus t)\, d\theta$$

$$Risk = \ln\hat{R}^2 \int_0^\infty P(\theta \setminus t)\, d\theta - 2\ln\hat{R}\int_0^\infty \ln R\, P(\theta \setminus t)\, d\theta + \int_0^\infty \ln R^2\, P(\theta \setminus t)\, d\theta \Rightarrow$$

$$Risk = \ln\hat{R}^2 - 2\ln\hat{R}E(\ln R \setminus t) + E(\ln R^2 \setminus t) \quad \ldots(B.2)$$

Let $\frac{\partial}{\partial \ln\hat{R}} Risk = 0$, we get Bayes estimator of R denoted by $\hat{R}_{pra.1}(t)$ for the above prior as follows

$$\ln\hat{R}(t) = E(\ln R \setminus t) = \int_0^\infty \ln R(t)\, P(\theta \setminus t)\, d\theta \quad \ldots(B.3)$$

$$\hat{R}_{pra.1(i)}(t) = \exp\left(\int_0^\infty \ln R(t)\, P_i(\theta \setminus t)\, d\theta\right) \quad ; i = 1,2,3,4 \quad \ldots(B.4)$$

1.1 **Bayes estimation using Inverse chi-squared distribution as prior:**

To obtain the Bayes' estimator under inverse chi-squared distribution as prior. Substituting the equation (A. 6) in equation (B.4), we get:

$$\hat{R}_{pra.1(i)}(t) = \exp\left(\int_0^\infty \ln R(t)\, P_i(\theta \setminus t)\, d\theta\right) \quad ; i = 1 \quad \ldots(B.4)$$

$$\hat{R}_{pra.1(1)}(t) = \exp\left(\int_0^\infty \ln(\exp(-\frac{t}{\theta}))\frac{(\Sigma_{i=1}^n t_i + (1/2))^{(n+\frac{v}{2})}}{\Gamma(n+(v/2))}\theta^{-[(n+\frac{v}{2})+1]}\exp(-\frac{1}{\theta}(\Sigma_{i=1}^n t_i + \frac{1}{2}))\, d\theta\right) \ldots(B.5)$$

$$\hat{R}_{pra.1(1)}(t)\exp\left(\int_0^\infty (-\frac{t}{\theta})\frac{(\Sigma_{i=1}^n t_i + (1/2))^{(n+\frac{v}{2})}}{\Gamma(n+(v/2))}\theta^{-[(n+\frac{v}{2})+1]}\exp(-\frac{1}{\theta}(\Sigma_{i=1}^n t_i + \frac{1}{2}))\, d\theta\right) \quad \ldots(B.6)$$

By multiplying the integral in equation (B.6) by the quantity which equals to $K_1 = \frac{\Gamma(n+(v/2)+1)}{\Gamma(n+(v/2)+1)}$,where $\Gamma(.)$ is a gamma function. So we have

$$\hat{R}_{gsa.1(1)}(t) = \exp(-t\,\kappa_1 \int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})^{(n+(v/2)-1+1}}{\Gamma(n+(v/2))} \theta^{-[(n+\frac{v}{2}+1)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \frac{1}{2}) d\theta\ ) \quad \ldots (B.7)$$

Then we have

$$\hat{R}_{gsa.1(1)}(t) = \exp(\frac{-t(\Gamma(n+(v/2)+1)}{\Gamma(n+(v/2))(\Sigma_{i=1}^{n} t_i + \frac{1}{2})} (\kappa_2(t;\theta)) \quad \ldots (B.8)$$

Where $\kappa_2(t;\theta)$ equals to

$$\kappa_2(t;\theta) = \int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})^{(n+\frac{v}{2}+1)}}{\Gamma(n+\frac{v}{2}+1)} \theta^{-[(n+\frac{v}{2}+1)+1]} \exp(-\frac{1}{\theta}((\Sigma_{i=1}^{n} t_i + \frac{1}{2}))\,d\theta = 1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of R as the following formula:

$$\hat{R}_{gsa.1(1)}(t) = \exp(\frac{-t(\Gamma(n+(v/2)+1)}{\Gamma(n+(v/2))(\Sigma_{i=1}^{n} t_i + \frac{1}{2})}) \quad ; n \,\&\, v > 0 \quad \ldots (B.9)$$

1.2 <u>Bays estimation using Inverted gamma distribution as prior:</u>

To obtain the Bayes' estimator under the inverted gamma distribution as prior. Substituting the equation (A.11) in equation (B.4), we get:

$$\hat{R}_{gsa.1(i)}(t) = \exp(\int_0^\infty \ln R(t)\, P_i(\theta \setminus t)\, d\theta) \quad ; i = 2 \quad \ldots (B.4)$$

$$\hat{R}_{gsa.1(2)}(t) = \exp(\int_0^\infty (\ln(\exp(-\frac{t}{\theta})) \frac{(\Sigma_{i=1}^{n} t_i + \beta)^{(n+\alpha)}}{\Gamma(n+\alpha)} \theta^{-[(n+\alpha)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \beta))d\theta \quad \ldots (B.10)$$

$$\hat{R}_{gsa.1(2)}(t) = \exp(-t\int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i + \beta)^{(n+\alpha+1-1)}}{\Gamma(n+\alpha)} \theta^{-[(n+\alpha-1)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + +\beta))d\theta) \quad \ldots (B.11)$$

By multiplying the integral in equation (B.11) by the quantity which equals to $h_1 = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+1)}$, where $\Gamma(.)$ is a gamma function. So we have

$$\hat{R}_{gsa.1(2)}(t) = \exp(-t\ h_1 \int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i + \beta)^{(n+\alpha+1-1)}}{\Gamma(n+\alpha)} \theta^{-[(n+\alpha-1)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \beta))d\theta) \quad \ldots (B.11)$$

Then we have

$$\hat{R}_{gsa.1(2)}(t) = \exp(\frac{-t\ \Gamma(n+\alpha+1)}{\Gamma(n+\alpha)\,(\Sigma_{i=1}^{n} t_i + \beta)} (h_2(t;\theta))) \quad \ldots (B.12)$$

Where $h2(t;\theta)$ equals to

$$h2(t;\theta) = \int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i + \beta)^{(n+\alpha+1)}}{\Gamma(n+\alpha+1)} \theta^{-[(n+\alpha+1)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \beta))d\theta = 1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of R as the following formula:

$$\hat{R}_{gm.1(2)}(t) = \exp\left(\frac{-t\quad \Gamma(n+\alpha+1)}{\Gamma(n+\alpha)\,(\Sigma_{i=1}^{n} t_i+\beta)}\right) \quad ;n,\beta,\alpha>0 \qquad \ldots(B.13)$$

1.3 Bayes estimation using improper distribution as prior:

To obtain the Bayes' estimator under improper distribution as prior. Substituting the equation (A.16) in equation (B.4), we get:

$$\hat{R}_{gm.1(3)}(t) = \exp\left(\int_0^\infty \ln R(t)\, P_i(\theta \backslash t)\, d\theta\right) \quad ;i=3 \qquad \ldots(B.4)$$

$$\hat{R}_{gm.1(3)}(t) = \exp\left(\int_0^\infty \ln(\exp(-\frac{t}{\theta}))\, \frac{(\Sigma_{i=1}^{n} t_i+b)^{(n+a)}}{\Gamma(n+a)}\, \theta^{-[(n+a)+1]}\, \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i+b))\, d\theta\right) \ldots(B.12)$$

$$\hat{R}_{gm.1(3)}(t) = \exp\left(-t\int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i+b)^{(n+a+1-1)}}{\Gamma(n+a)}\, \theta^{-[(n+a+1)+1]}\exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i+b))\, d\theta\right) \quad \ldots(B.13)$$

By multiplying the integral in equation (B.13) by the quantity which equals to $E_1 = \frac{\Gamma(n+a+1)}{\Gamma(n+a+1)}$,where $\Gamma(.)$ is a gamma function. So we have

$$\hat{R}_{gm.1(3)}(t) = \exp\left(-t\,E_1\int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i+b)^{(n+a+1-1)}}{\Gamma(n+a)}\, \theta^{-[(n+a)+1]}\, \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i+t+b))\, d\theta\right) \ldots(B.14)$$

Then we have

$$\hat{R}_{gm.1(3)}(t) = \exp\left(\frac{-t\quad \Gamma(n+a+1)}{\Gamma(n+a)(\Sigma_{i=1}^{n} t_i+b)}\,(E_2(t;\theta))\right) \qquad \ldots(B.15)$$

Where $E_2(t;\theta)$ equals to

$$E_2(t;\theta) = \int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i+b)^{(n+a+1)}}{\Gamma(n+a+1)}\, \theta^{-[(n+a+1)+1]}\, \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i+b))\, d\theta = 1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of R as the following formula:

$$\hat{R}_{gm.1(3)}(t) = \exp\left(\frac{-t\quad \Gamma(n+a+1)}{\Gamma(n+a)(\Sigma_{i=1}^{n} t_i+b)}\right) \qquad n,b,a>0 \qquad \ldots(B.16)$$

1.4 Bayes estimation using non-informative distribution as prior:

To obtain the Bayes' estimator under non informative distribution as prior. Substituting the equation (A.21) in equation (B.4), we get:

$$\hat{R}_{gm.1(4)}(t) = \exp\left(\int_0^\infty \ln R(t)\, P_i(\theta \backslash t)\, d\theta\right) \quad ;i=4 \qquad \ldots(B.4)$$

$$\hat{R}_{gm.1(4)}(t) = \exp\left(\int_0^\infty \ln(\exp(-\frac{t}{\theta}))\, \frac{(\Sigma_{i=1}^{n} t_i)^{(n+c-1)}}{\Gamma(n+c-1)}\, \theta^{-[(n+c-1)+1]}\, \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i))\, d\theta\right) \ldots(B.17)$$

$$\hat{R}_{pm.1(4)}(t) = \exp\left(-t\int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i)^{(n+c-l)+l-1}}{\Gamma(n+c-1)}\,\theta^{-[(n+c-l+1)+1]} \exp\left(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i)\right) d\theta\right) \quad \ldots (B.18)$$

By multiplying the integral in equation (B.18) by the quantity which equals to $F_1 = \frac{\Gamma(n+c)}{\Gamma(n+c)}$, ,where $\Gamma(.)$ is a gamma function. So we have

$$\hat{R}_{pm.1(4)}(t) = \exp\left(F_1\int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i)^{(n+c-l)+l-1}}{\Gamma(n+c-1)}\,\theta^{-[(n+c-l+1)+1]} \exp\left(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + t)\right) d\theta\right) \quad \ldots (B.19)$$

Then we have

$$\hat{R}_{pm.1(4)}(t) = \exp\left(\frac{-t\ \Gamma(n+c)}{\Gamma(n+c-1)(\Sigma_{i=1}^{n} t_i)}\ (F_2(t;\theta))\right) \quad \ldots (B.20)$$

Where $F_2(t;\theta)$ equals to

$$F_2(t;\theta) = \int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i + t)^{(n+c)}}{\Gamma(n+c)}\,\theta^{-[(n+c)+1]} \exp\left(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i)\right) d\theta = 1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of R as the following formula:

$$\hat{R}_{pm.1(4)}(t) = \exp\left(\frac{-t\ \Gamma(n+c)}{\Gamma(n+c-1)(\Sigma_{i=1}^{n} t_i)}\right) \qquad n,c > 0 \quad \ldots (B.21)$$

Appendix-C: The following is the derivation of these estimators under the second loss function.

2. The second proposed loss function

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$L_2(\hat{R}, R) = \frac{(\ln \hat{R} - \ln R)^2}{\ln R}$, where $R = R(t)$ & $\hat{R} = \hat{R}(t)$, the risk function is:

$$Risk = E[L_2(\hat{R}, R)] \qquad \ldots(C.1)$$

$$Risk = E[\frac{(\ln \hat{R} - \ln R)^2}{\ln R}]$$

$$Risk = \int_\theta \frac{(\ln \hat{R} - \ln R)^2}{\ln R} P(\theta \backslash t)\, d\theta$$

$$Risk = \int_\theta \frac{1}{\ln R}((\ln \hat{R})^2 - 2\ln R \ln \hat{R} + (\ln R)^2)\ P(\theta \backslash t)\, d\theta$$

$$Risk = (\ln \hat{R})^2 \int_0^\infty \frac{1}{\ln R} P(\theta \backslash t)\, d\theta - 2(\ln \hat{R}) \int_0^\infty \frac{1}{\ln R} \ln R\, P(\theta \backslash t)\, d\theta + \int_0^\infty \frac{1}{\ln R} (\ln R)^2\, P(\theta \backslash t)\, d\theta \Rightarrow$$

$$Risk = (\ln \hat{R})^2 \int_0^\infty \frac{1}{\ln R} P(\theta \backslash t)\, d\theta - 2(\ln \hat{R})(1) + E(\ln R \backslash t) \qquad \ldots (C.2)$$

Let $\frac{\partial}{\partial \ln \hat{R}} Risk = 0$, we get Bayes estimator of R denoted by $\hat{R}_{pro.2}(t)$ for the above prior as follows

$$(\ln \hat{R}) = \frac{1}{\int_0^\infty \frac{1}{\ln R} P(\theta \backslash t)\, d\theta} \qquad \ldots (C.3)$$

$$\hat{R}_{pro.2(i)}(t) = \exp(\int_0^\infty \frac{1}{\ln R(t)} P_i(\theta \backslash t)\, d\theta\ )^{-1} \qquad ; i = 1,2,3,4 \qquad \ldots (C.4)$$

2.1 Bayes estimation using Inverse chi-squared distribution as prior:

To obtain the Bayes' estimator under inverse chi-squared distribution as prior.. Substituting the equation (A.6) in the integral in equation (C.4), we get:

$$\hat{R}_{pro.2(1)}(t) = \exp(\int_0^\infty \frac{1}{\ln R(t)} P_1(\theta \backslash t)\, d\theta\ )^{-1} \qquad \text{for } i = 1 \qquad \ldots (C.4)$$

$$\hat{R}_{pro.2(1)}(t) = \exp(\int_0^\infty \frac{1}{\ln(\exp(-\frac{t}{\theta}))} \frac{(\Sigma_{i=1}^n t_i + \frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2})+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^n t_i + \frac{1}{2}))\, d\theta\ )^{-1} \ldots (C.5)$$

$$\hat{R}_{pro.2(1)}(t) = \exp(-\frac{1}{t}\int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2}-1)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \frac{1}{2}))\, d\theta)^{-1} \quad \ldots (C.6)$$

By multiplying the integral in equation (C.6) by the quantity which equals to $\kappa_1 = \frac{\Gamma(n+\frac{v}{2}-1)}{\Gamma(n+\frac{v}{2}-1)}$,

where $\Gamma(.)$ is a gamma function. So we have

$$\hat{R}_{pro.2(1)}(t) = \exp(-\frac{1}{t}\kappa_1\int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})^{(n+\frac{v}{2})-1+1}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2})+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i - t + \frac{1}{2}))\, d\theta)^{-1} \ldots (C.7)$$

Then we have

$$\hat{R}_{pro.2(1)}(t) = \exp(\frac{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})\Gamma(n+\frac{v}{2}-1)}{-t \quad \Gamma(n+\frac{v}{2})} (\kappa_2(t;\theta)))^{-1} \quad \ldots (C.8)$$

Where $\kappa_2(t;\theta)$ equals to

$\kappa_2(t;\theta) = \int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})^{(n+\frac{v}{2}-1)}}{\Gamma(n+\frac{v}{2}-1)} \theta^{-[(n+\frac{v}{2}-1)+1]} \exp(-\frac{1}{\theta}((\Sigma_{i=1}^{n} t_i + \frac{1}{2}))d\theta = 1$. Be the integral of the pdf of the Inverted Gamma distribution. So we get the Bayes estimator of R as the following formula:

$$\hat{R}_{pro.2(1)}(t) = \exp(\frac{(\Sigma_{i=1}^{n} t_i + \frac{1}{2})\Gamma(n+\frac{v}{2}-1)}{-t \quad \Gamma(n+\frac{v}{2})})^{-1} \quad , n \,\&\, v > 0 \quad \ldots (C.9)$$

2.2 Bays estimation using Inverted gamma distribution as prior:

To obtain the Bayes' estimator under the inverted gamma distribution as prior. Substituting the equation (A.11) in the integral in equation (C.4), we get:

$$\hat{R}_{pro.2(i)}(t) = \exp(\int_0^\infty \frac{1}{\ln R(t)} P_i(\theta \setminus t)\, d\theta)^{-1} \quad \text{for } i = 2 \quad \ldots (C.4)$$

$$\hat{R}_{pro.2(2)}(t) = \exp(\int_0^\infty \frac{1}{\ln(\exp(\frac{t}{\theta}))} \frac{(\Sigma_{i=1}^{n} t_i + \beta)^{(n+\alpha)}}{\Gamma(n+\alpha)} \theta^{-[(n+\alpha)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \beta))d\theta)^{-1} \ldots (C.10)$$

$$\hat{R}_{pro.2(2)}(t) = \exp(-\frac{1}{t}\int_0^\infty \frac{(\Sigma_{i=1}^{n} t_i + \beta)^{(n+\alpha)}}{\Gamma(n+\alpha)} \theta^{-[(n+\alpha-1)+1]} \exp(-\frac{1}{\theta}(\Sigma_{i=1}^{n} t_i + \beta))d\theta)^{-1} \quad \ldots (C.11)$$

By multiplying the integral in equation (C.11) by the quantity which equals to $h_1 = \frac{\Gamma(n+\alpha-1)}{\Gamma(n+\alpha-1)}$,

where $\Gamma(.)$ is a gamma function. So we have

$$\hat{R}_{gsa.2(2)}(t) = \exp\left(-\frac{1}{t} h_1 \int_0^{\infty} \frac{(\sum_{i=1}^{n} t_i + \beta)^{(n+\alpha)-1+1}}{\Gamma(n+\alpha)} \theta^{-[(n+\alpha-1)+1]} \exp\left(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + \beta)\right) d\theta\right)^{-1} \quad \dots (C.12)$$

Then we have

$$\hat{R}_{gsa.2(2)}(t) = \exp\left(-\frac{\Gamma(n+\alpha-1)(\sum_{i=1}^{n} t_i + \beta)}{t \;\; \Gamma(n+\alpha)} (h_2(t;\theta))\right)^{-1} \quad \dots (C.13)$$

Where $h2(t;\theta)$ equals to

$$h2(t;\theta) = \int_0^{\infty} \frac{(\sum_{i=1}^{n} t_i + \beta)^{(n+\alpha-1)}}{\Gamma(n+\alpha-1)} \theta^{-[(n+\alpha)+1]} \exp\left(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + \beta)\right) d\theta = 1.$$ Be the integral of the pdf of the Inverted Gamma distribution. So we get the Bayes estimator of R as the following formula:

$$\hat{R}_{gsa.2(2)}(t) = \exp\left(-\frac{\Gamma(n+\alpha-1)(\sum_{i=1}^{n} t_i + \beta)}{t \;\; \Gamma(n+\alpha)}\right)^{-1} \quad , n, \beta, \alpha > 0 \quad \dots (C.14)$$

2.3 <u>Bayes estimation using improper distribution as prior:</u>

To obtain the Bayes' estimator under improper distribution as prior. Substituting the equation (A.16) in the integral in equation (C.4), we get:

$$\hat{R}_{gsa.2(i)}(t) = \exp\left(\int_0^{\infty} \frac{1}{\ln R(t)} P_i(\theta \setminus t)\, d\theta\right)^{-1} \quad \text{for } i = 3 \quad \dots (C.4)$$

$$\hat{R}_{gsa.2(3)}(t) = \exp\left(\int_0^{\infty} \frac{1}{\ln(\exp(\frac{t}{\theta}))} \frac{(\sum_{i=1}^{n} t_i + b)^{(n+a)}}{\Gamma(n+a)} \theta^{-[(n+a)+1]} \exp\left(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + b)\right) d\theta\right)^{-1} \dots (C.15)$$

$$\hat{R}_{gsa.2(3)}(t) = \exp\left(-\frac{1}{t}\int_0^{\infty} \frac{(\sum_{i=1}^{n} t_i + b)^{(n+a)}}{\Gamma(n+a)} \theta^{-[(n+a-1)+1]} \exp\left(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + b)\right) d\theta\right)^{-1} \quad \dots (C.16)$$

By multiplying the integral in equation (C.16) by the quantity which equals to $E_1 = \frac{\Gamma(n+a-1)}{\Gamma(n+a-1)}$, where $\Gamma(.)$ is a gamma function. So we have

$$\hat{R}_{gsa.2(3)}(t) = \exp\left(-\frac{1}{t} E_1 \int_0^{\infty} \frac{(\sum_{i=1}^{n} t_i + b)^{(n+a)-1+1}}{\Gamma(n+a)} \theta^{-[(n+a-1)+1]} \exp\left(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + b)\right) d\theta\right)^{-1} \dots (C.17)$$

Then we have

$$\hat{R}_{gsa.2(3)}(t) = \exp\left(-\frac{\Gamma(n+a-1)(\sum_{i=1}^{n} t_i + b)}{t \;\; \Gamma(n+a)} (E_2(t;\theta))\right)^{-1} \quad \dots (C.18)$$

Where $E2(t;\theta)$ equals to

$$E2(t;\theta) = \int_0^{\infty} \frac{(\sum_{i=1}^{n} t_i + b)^{(n+a-1)}}{\Gamma(n+a-1)} \theta^{-[(n+a-1)+1]} \exp\left(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i + b)\right) d\theta = 1.$$ Be the integral of the pdf of the Inverted Gamma distribution. So we get the Bayes estimator of R as the following formula:

$$\hat{R}_{pra.2(3)}(t) = \exp\left(-\frac{\Gamma(n+a-1)(\sum_{i=1}^{n} t_i + b)}{t\ \ \Gamma(n+a)}\right)^{-1} \qquad n, b, a > 0 \qquad \ldots (C.19)$$

2.4 Bayes estimation using non-informative distribution as prior:

To obtain the Bayes' estimator under non informative distribution as prior. Substituting the equation (A.21) in the integral in equation (C.4), we get:

$$\hat{R}_{pra.2(i)}(t) = \exp\left(\int_0^{\infty} \frac{1}{\ln R(t)} P_i(\theta \setminus t)\, d\theta\right)^{-1} \qquad \text{for } i = 4 \qquad \ldots (C.4)$$

$$\hat{R}_{pra.2(4)}(t) = \exp\left(\int_0^{\infty} \frac{1}{\ln(\exp(\frac{t}{\theta}))} \frac{(\sum_{i=1}^{n} t_i)^{(n+c-1)}}{\Gamma(n+c-1)} \theta^{-[(n+c-1)+1]} \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} t_i\right) d\theta\right)^{-1} \qquad \ldots (C.20)$$

$$\hat{R}_{pra.2(4)}(t) = \exp\left(-\frac{1}{t}\int_0^{\infty} \frac{(\sum_{i=1}^{n} t_i)^{(n+c-1)}}{\Gamma(n+c-1)} \theta^{-[(n+c-2)+1]} \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} t_i\right) d\theta\right)^{-1} \qquad \ldots (C.21)$$

By multiplying the integral in equation (C.21) by the quantity which equals to $F_1 = \frac{\Gamma(n+c-2)}{\Gamma(n+c-2)}$, where $\Gamma(.)$ is a gamma function. So we have

$$\hat{R}_{pra.2(4)}(t) = \exp\left(-\frac{1}{t}F_1\int_0^{\infty} \frac{(\sum_{i=1}^{n} t_i)^{(n+c-1)-1+1}}{\Gamma(n+c-1)} \theta^{-[(n+c-2)+1]} \exp\left(-\frac{1}{\theta}(\sum_{i=1}^{n} t_i)\right) d\theta\right)^{-1} \quad \ldots (C.22)$$

Then we have

$$\hat{R}_{pra.2(4)}(t) = \exp\left(-\frac{(\sum_{i=1}^{n} t_i)\Gamma(n+c-2)}{t\ \ \Gamma(n+c-2)} (F_2(t;\theta))\right)^{-1} \qquad \ldots (C.23)$$

Where $F_2(t;\theta)$ equals to

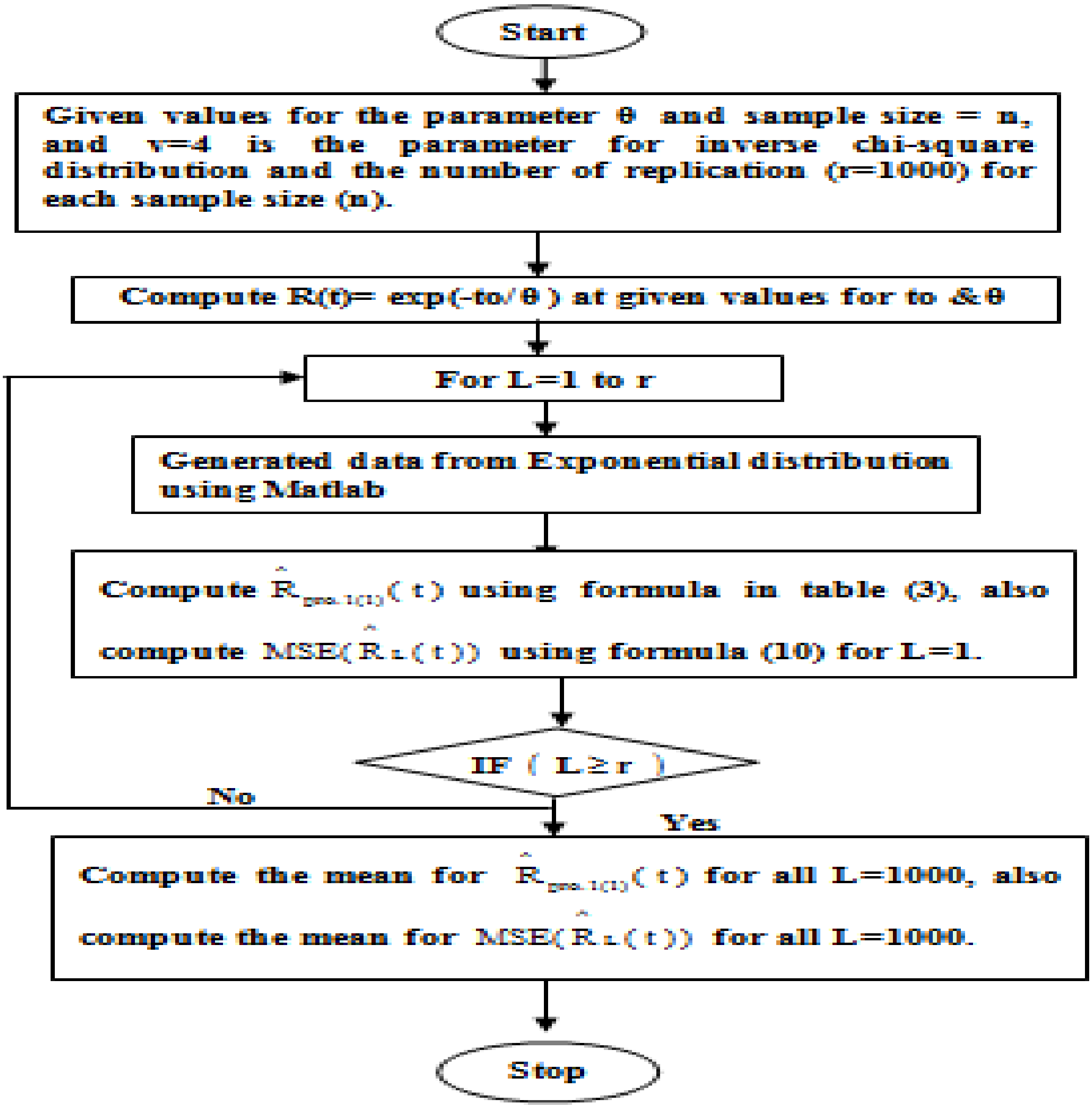
$$F_2(t;\theta) = \int_0^{\infty} \frac{(\sum_{i=1}^{n} t_i)^{(n+c-2)}}{\Gamma(n+c-2)} \theta^{-[(n+c-2)+1]} \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} t_i\right) d\theta = 1.$$

Be the integral of the pdf of the Inverted Gamma distribution. So we get the Bayes estimator of R as the following formula:

$$\hat{R}_{pra.2(4)}(t) = \exp\left(-\frac{(\sum_{i=1}^{n} t_i)\Gamma(n+c-2)}{t\ \ \Gamma(n+c-2)}\right)^{-1}, \qquad n, c > 0 \qquad \ldots (C.24)$$

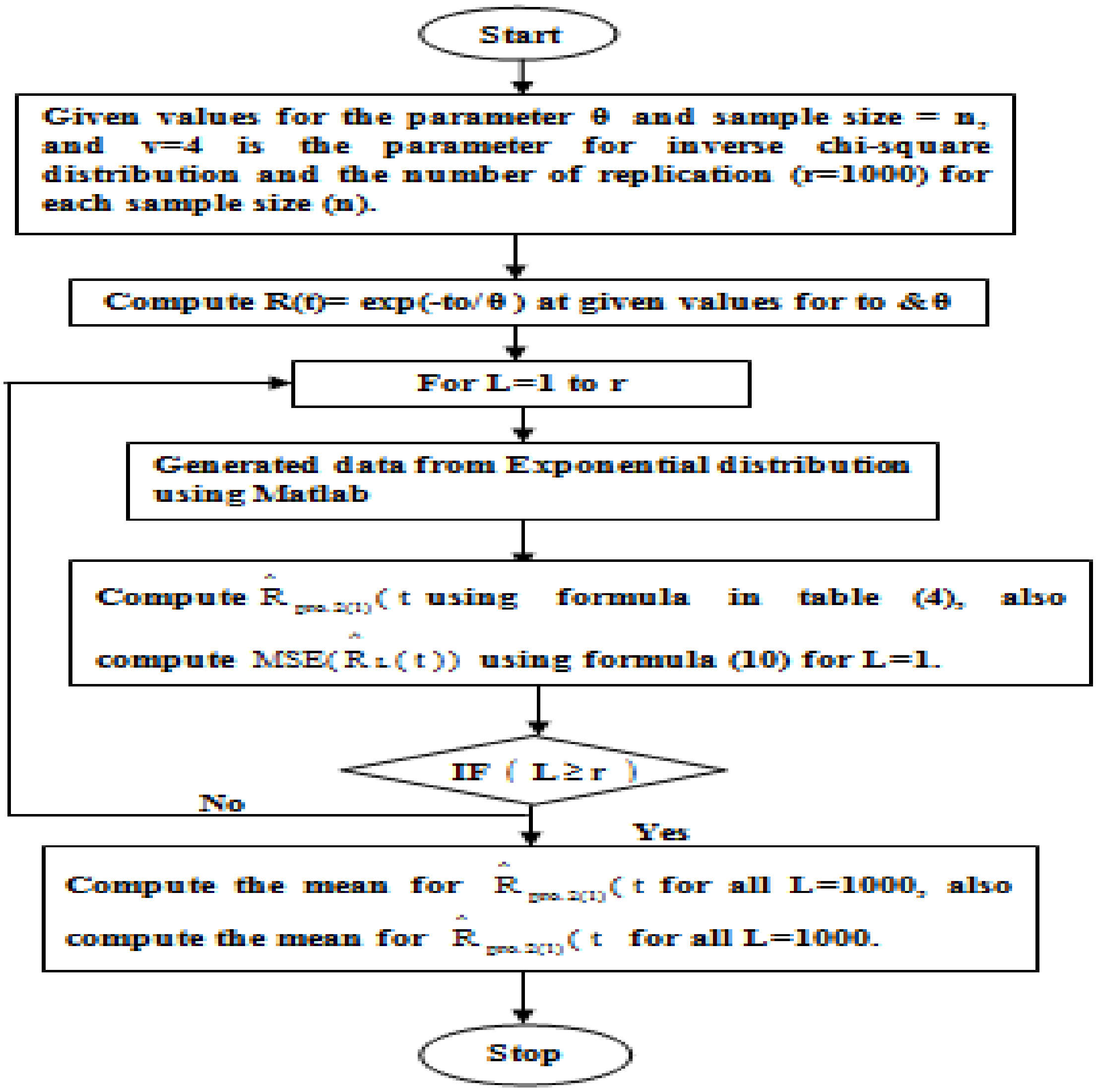
Appendix-D **The following is the programs algorithm.**

Algorithm (1): To compute Bayes estimators ($\hat{R}_{pro.1(1)}(t)$) using inverse chi-square as prior distribution for θ with MSE for $\hat{R}_{pro.1(1)}(t)$.



Note (1): we can reformulate the Algorithm (1) to compute Bayes estimators for $\hat{R}_{pro.1(i)}(t), i = 2,3,4$ under using other distributions as prior distribution for θ with MSE for $\hat{R}_{pro.1(i)}(t), i = 2,3,4$.

<u>Algorithm (2):</u> To compute Bayes estimators ($\hat{R}_{pro.2(1)}(t)$) using inverse chi-square as prior distribution for θ with MSE for $\hat{R}_{pro.2(1)}(t)$.



Note (2): we can reformulate the Algorithm (2) to compute Bayes estimators for $\hat{R}_{pro.2(i)}(t), i = 2,3,4$ under using other distributions as prior distribution for θ with MSE for $\hat{R}_{pro.2(i)}(t), i = 2,3,4$.

Table(1): MSE of estimated exponential reliability function under the first proposed loss function.

parameters				$R(t)$	$(\hat{R}(t))$				$MSE(\hat{R}(t))$			
					Sample Size(n)				Sample Size(n)			
					30	60	90	120	30	60	90	120
θ	t	v	-		Inverse Chi-square distribution $(P_1(\theta \backslash x))$							
0.5	0.5	4		0.36788	0.35325	0.36045	0.36197	0.3635	0.00402	0.00216	0.00139	0.00113
	1.5			0.04978	0.04809	0.04909	0.04889	0.04924	0.00057	0.00033	0.00021	0.00018
	2.5			0.00673	0.00722	0.00707	0.00686	0.00688	$3.62e^{-5}$	$1.8959e^{-5}$	$1.1699e^{-5}$	$9.878e^{-6}$
	3.5			0.00091	0.00117	0.00107	0.00099	0.00099	$2.0846e^{-6}$	$8.9858e^{-7}$	$5.034e^{-7}$	$4.2149e^{-7}$
θ	t	α	β		Inverted Gamma distribution $(P_2(\theta \backslash x))$							
0.5	0.5	5	2	0.36788	0.3549	0.36092	0.36221	0.36364	0.00333	0.00196	0.00130	0.00108
	1.5			0.04978	0.04806	0.04907	0.04889	0.04924	0.00048	0.00030	0.00020	0.00017
	2.5			0.00673	0.00708	0.00703	0.00684	0.00687	$2.9611e^{-5}$	$1.7119e^{-5}$	$1.0938e^{-5}$	$9.386e^{-6}$
	3.5			0.00091	0.00112	0.00105	0.00099	0.00098	$1.6067e^{-6}$	$7.9432e^{-7}$	$4.6573e^{-7}$	$3.9755e^{-7}$
θ	t	a	b		Improper distribution $(P_3(\theta \backslash x))$							
0.5	0.5	9	3	0.36788	0.33671	0.35046	0.35489	0.35801	0.00377	0.00209	0.00138	0.00112
	1.5			0.04978	0.04100	0.04492	0.04599	0.04699	0.00043	0.00027	0.00019	0.00016
	2.5			0.00673	0.00543	0.00606	0.00617	0.00635	$1.9237e^{-5}$	$1.3175e^{-5}$	$9.2343e^{-6}$	$8.1662e^{-6}$
	3.5			0.00091	0.00077	0.00085	0.00085	0.00088	$7.812e^{-7}$	$5.1849e^{-7}$	$3.4928e^{-7}$	$3.1632e^{-7}$
θ	t	c	-		Non- informative distribution $(P_4(\theta \backslash x))$							
0.5	0.5	1		0.36788	0.36448	0.36631	0.36593	0.36649	0.00409	0.00218	0.00138	0.00113
	1.5			0.04978	0.05284	0.05153	0.05051	0.05047	0.00070	0.00036	0.00022	0.00018
	2.5			0.00673	0.00844	0.00767	0.00724	0.00717	$5.1505e^{-5}$	$2.2942e^{-5}$	$1.3264e^{-5}$	$1.0877e^{-5}$
	3.5			0.00091	0.00146	0.00120	0.00107	0.00105	$3.3531e^{-6}$	$1.1709e^{-6}$	$6.0318e^{-7}$	$4.8395e^{-7}$

Table(2): MSE of estimated exponential reliability function under the first proposed loss function.

parameters				$R(t)$	$(\hat{R}(t))$				$MSE(\hat{R}(t))$			
					Sample Size(n)				Sample Size(n)			
					30	60	90	120	30	60	90	120
θ	t	v	-		Inverse Chi-square distribution $(P_1(\theta \backslash x))$							
1	0.5	4		0.60653	0.58678	0.59656	0.59937	0.60099	0.00332	0.00162	0.00110	0.00081
	1.5			0.22313	0.20714	0.21502	0.2172	0.21848	0.00328	0.00176	0.00123	0.00093
	2.5			0.08208	0.07537	0.07875	0.07958	0.08009	0.00110	0.00062	0.00044	0.00034
	3.5			0.03019	0.02817	0.02928	0.02947	0.02960	0.00028	0.00016	0.00011	$9.0365e^{-5}$
θ	t	α	β		Inverted Gamma distribution $(P_2(\theta \backslash x))$							
1	0.5	5	2	0.60653	0.5743	0.58968	0.59464	0.59739	0.00378	0.00176	0.00117	0.00085
	1.5			0.22313	0.1941	0.20766	0.21209	0.21457	0.00346	0.00182	0.00126	0.00095
	2.5			0.08208	0.06759	0.07430	0.07649	0.07772	0.00105	0.00060	0.00043	0.00033
	3.5			0.03019	0.02418	0.02699	0.02788	0.02838	0.00024	0.00014	0.00010	$8.6031e^{-5}$
θ	t	a	b		Improper distribution $(P_3(\theta \backslash x))$							
1	0.5	9	3	0.60653	0.54973	0.57608	0.58524	0.59021	0.00596	0.00242	0.00149	0.00104
	1.5			0.22313	0.17062	0.19375	0.20226	0.20697	0.00497	0.00231	0.00151	0.00110
	2.5			0.08208	0.05470	0.06627	0.07071	0.07321	0.00135	0.00070	0.00048	0.00036
	3.5			0.03019	0.01806	0.02303	0.02499	0.02611	0.00027	0.00015	0.00011	$8.8913e^{-5}$
θ	t	c	-		Non- informative distribution $(P_4(\theta \backslash x))$							
1	0.5	1		0.60653	0.60131	0.60398	0.60435	0.60474	0.00293	0.00152	0.00105	0.00078
	1.5			0.22313	0.2226	0.22306	0.22262	0.22258	0.00329	0.00177	0.00123	0.00093
	2.5			0.08208	0.08479	0.08367	0.08290	0.08260	0.00126	0.00067	0.00046	0.00035
	3.5			0.03019	0.03313	0.03185	0.03119	0.03090	0.00037	0.00019	0.00012	$9.694e^{-5}$

Table(3): MSE of estimated exponential reliability function under the first proposed loss function.

parameters				$R(t)$	$(\hat{R}(t))$				$MSE(\hat{R}(t))$			
					Sample Size(n)				Sample Size(n)			
					30	60	90	120	30	60	90	120
θ	t	v	-		Inverse Chi-square distribution $(P_1(\theta \backslash x))$							
1.5	0.5	4		0.71653	0.69711	0.7061	0.71019	0.71165	0.00255	0.00113	0.00070	0.00051
	1.5			0.36788	0.34327	0.3542	0.35961	0.36146	0.00499	0.00243	0.00156	0.00116
	2.5			0.18888	0.1718	0.17909	0.18303	0.1843	0.00316	0.00165	0.00108	0.00082
	3.5			0.09697	0.08725	0.09123	0.09361	0.09432	0.00148	0.00081	0.00054	0.00041
θ	t	α	β		Inverted Gamma distribution $(P_2(\theta \backslash x))$							
1.5	0.5	5	2	0.71653	0.683	0.69859	0.70509	0.70779	0.00329	0.00135	0.00079	0.00057
	1.5			0.36788	0.32301	0.34309	0.35195	0.35562	0.00606	0.00277	0.00171	0.00125
	2.5			0.18888	0.15537	0.16986	0.17659	0.17937	0.00357	0.00179	0.00114	0.00086
	3.5			0.09697	0.07589	0.08475	0.08905	0.09082	0.00154	0.00084	0.00055	0.00042
θ	t	a	b		Improper distribution $(P_3(\theta \backslash x))$							
1.5	0.5	9	3	0.71653	0.66013	0.68626	0.69668	0.70139	0.00547	0.00198	0.00108	0.00073
	1.5			0.36788	0.29215	0.32539	0.33957	0.34612	0.00947	0.00390	0.00223	0.00156
	2.5			0.18888	0.13179	0.15564	0.16642	0.17149	0.00526	0.00240	0.00143	0.00104
	3.5			0.09697	0.06048	0.07506	0.08199	0.08530	0.00212	0.00106	0.00066	0.00049
θ	t	c	-		Non- informative distribution $(P_4(\theta \backslash x))$							
1.5	0.5	1		0.71653	0.71011	0.71267	0.71459	0.71495	0.00212	0.00101	0.00065	0.00048
	1.5			0.36788	0.36246	0.3641	0.36629	0.36649	0.00455	0.00228	0.00150	0.00113
	2.5			0.18888	0.18781	0.18743	0.18869	0.18857	0.00317	0.00163	0.00108	0.00082
	3.5			0.09697	0.09865	0.09718	0.09766	0.09738	0.00163	0.00084	0.00056	0.00042

Table(4): MSE of estimated exponential reliability function under the first proposed loss function.

parameters				$R(t)$	$(\hat{R}(t))$				$MSE(\hat{R}(t))$			
					Sample Size(n)				Sample Size(n)			
					30	60	90	120	30	60	90	120
θ	t	v	-		Inverse Chi-square distribution $(P_1(\theta \backslash x))$							
2.5	0.5	4		0.81873	0.80348	0.81095	0.814	0.8147	0.00122	0.00058	0.00034	0.00026
	1.5			0.54881	0.5211	0.53458	0.54012	0.54135	0.00436	0.00219	0.00131	0.00103
	2.5			0.36788	0.33993	0.35349	0.35907	0.36025	0.00488	0.00257	0.00157	0.00124
	3.5			0.2466	0.22299	0.23445	0.23915	0.24008	0.00392	0.00215	0.00133	0.00106
θ	t	α	β		Inverted Gamma distribution $(P_2(\theta \backslash x))$							
2.5	0.5	5	2	0.81873	0.79112	0.80458	0.80971	0.81147	0.00181	0.00074	0.00040	0.00030
	1.5			0.54881	0.49762	0.52215	0.53167	0.53495	0.00622	0.00271	0.00153	0.00117
	2.5			0.36788	0.31502	0.33997	0.34979	0.3532	0.00666	0.00309	0.00180	0.00139
	3.5			0.2466	0.20065	0.22206	0.23057	0.23355	0.00510	0.00249	0.00149	0.00116
θ	t	a	b		Improper distribution $(P_3(\theta \backslash x))$							
2.5	0.5	9	3	0.81873	0.77301	0.79515	0.80335	0.80665	0.00327	0.00113	0.00057	0.00041
	1.5			0.54881	0.46464	0.50412	0.51928	0.52551	0.01075	0.00404	0.00213	0.00153
	2.5			0.36788	0.28138	0.32075	0.33636	0.3429	0.01109	0.00447	0.00244	0.00178
	3.5			0.2466	0.17162	0.20479	0.21832	0.2241	0.00818	0.00351	0.00198	0.00147
θ	t	c	-		Non- informative distribution $(P_4(\theta \backslash x))$							
2.5	0.5	1		0.81873	0.81333	0.81587	0.81727	0.81716	0.00094	0.00051	0.00031	0.00024
	1.5			0.54881	0.54026	0.54431	0.54664	0.54625	0.00357	0.00198	0.00123	0.00097
	2.5			0.36788	0.36077	0.3642	0.36629	0.36569	0.00424	0.00240	0.00150	0.00120
	3.5			0.2466	0.24214	0.24439	0.24588	0.24516	0.00361	0.00207	0.00131	0.00104

Table(5): MSE of estimated exponential reliability function under the second proposed loss function.

parameters				$R(t)$	$\hat{R}(t)$ Sample Size(n)				$MSE(\hat{R}(t))$ Sample Size(n)			
					30	60	90	120	30	60	90	120
θ	t	v	-		Inverse Chi-square distribution $(P_1(\theta \backslash x))$							
0.5	0.5	4		0.36788	0.36476	0.36638	0.36597	0.36652	0.00382	0.00210	0.00135	0.00111
	1.5			0.04978	0.05268	0.05148	0.0505	0.05046	0.00065	0.00035	0.00022	0.00018
	2.5			0.00673	0.00835	0.00764	0.00723	0.00716	$4.7587e^{-5}$	$2.2097e^{-5}$	$1.2948e^{-5}$	$1.0684e^{-5}$
	3.5			0.00091	0.00143	0.00119	0.00107	0.00104	$3.0286e^{-6}$	$1.1188e^{-6}$	$5.8647e^{-7}$	$4.7402e^{-7}$
θ	t	α	β		Inverted Gamma distribution $(P_2(\theta \backslash x))$							
0.5	0.5	5	2	0.36788	0.36543	0.36658	0.36609	0.36659	0.00318	0.00191	0.00127	0.00106
	1.5			0.04978	0.05226	0.05135	0.05045	0.05043	0.00054	0.00032	0.00021	0.00017
	2.5			0.00673	0.00810	0.00757	0.00720	0.00714	$3.8159e^{-5}$	$1.9826e^{-5}$	$1.2067e^{-5}$	$1.0133e^{-5}$
	3.5			0.00091	0.00134	0.001168	0.00106	0.00104	$2.2851e^{-6}$	$9.814e^{-7}$	$5.4032e^{-7}$	$4.4601e^{-7}$
θ	t	a	b		Improper distribution $(P_3(\theta \backslash x))$							
0.5	0.5	9	3	0.36788	0.34613	0.35579	0.35861	0.36086	0.00329	0.00194	0.00130	0.00107
	1.5			0.04978	0.04439	0.04694	0.04742	0.04810	0.00042	0.00027	0.00019	0.00016
	2.5			0.00673	0.00617	0.00651	0.00649	0.00660	$2.1723e^{-5}$	$1.4306e^{-5}$	$9.7204e^{-6}$	$8.5434e^{-6}$
	3.5			0.00091	0.00092	0.00094	0.00091	0.00093	$1.0106e^{-6}$	$6.0933e^{-7}$	$3.8938e^{-7}$	$3.4563e^{-7}$
θ	t	c	-		Non- informative distribution $(P_4(\theta \backslash x))$							
0.5	0.5	1		0.36788	0.37676	0.37244	0.37002	0.36956	0.00416	0.00219	0.00139	0.00113
	1.5			0.04978	0.05806	0.05408	0.05219	0.05173	0.00085	0.00040	0.00024	0.00019
	2.5			0.00673	0.00981	0.00829	0.00764	0.00747	$7.0534e^{-5}$	$2.7385e^{-5}$	$1.4969e^{-5}$	$1.1924e^{-5}$
	3.5			0.00091	0.00179	0.00133	0.00115	0.00111	$4.9801e^{-6}$	$1.4795e^{-6}$	$7.1255e^{-7}$	$5.497e^{-7}$

Table(6): MSE of estimated exponential reliability function under the second proposed loss function.

parameters				$R(t)$	$\hat{R}(t)$ Sample Size(n)				$MSE(\hat{R}(t))$ Sample Size(n)			
					30	60	90	120	30	60	90	120
θ	t	v	-		Inverse Chi-square distribution $(P_1(\theta \backslash x))$							
1	0.5	4		0.60653	0.59656	0.60153	0.6027	0.6035	0.00294	0.00153	0.00105	0.00078
	1.5			0.22313	0.21735	0.22036	0.2208	0.22121	0.00317	0.00174	0.00121	0.00092
	2.5			0.08208	0.08148	0.08199	0.08177	0.08175	0.00116	0.00064	0.00044	0.00034
	3.5			0.03019	0.03134	0.03095	0.03060	0.03046	0.00033	0.00017	0.00012	$9.3801e^{-5}$
θ	t	α	β		Inverted Gamma distribution $(P_2(\theta \backslash x))$							
1	0.5	5	2	0.60653	0.5834	0.59448	0.59789	0.59985	0.00321	0.00160	0.00109	0.00081
	1.5			0.22313	0.20321	0.21268	0.21555	0.21721	0.00311	0.00172	0.00120	0.00092
	2.5			0.08208	0.07281	0.07728	0.07856	0.07931	0.00101	0.00059	0.00042	0.00033
	3.5			0.03019	0.02676	0.02849	0.02893	0.02919	0.00025	0.00015	0.00010	$8.7141e^{-5}$
θ	t	a	b		Improper distribution $(P_3(\theta \backslash x))$							
1	0.5	9	3	0.60653	0.55817	0.58068	0.58841	0.59262	0.00502	0.00214	0.00135	0.00096
	1.5			0.22313	0.17836	0.19836	0.20553	0.2095	0.00430	0.00209	0.00139	0.00103
	2.5			0.08208	0.05879	0.06887	0.07260	0.07469	0.00121	0.00065	0.00045	0.00035
	3.5			0.03019	0.01993	0.02429	0.02592	0.02685	0.00025	0.00015	0.00010	$8.6394e^{-5}$
θ	t	c	-		Non- informative distribution $(P_4(\theta \backslash x))$							
1	0.5	1		0.60653	0.61151	0.60906	0.60773	0.60728	0.00283	0.00150	0.00103	0.00077
	1.5			0.22313	0.23378	0.22864	0.22634	0.22536	0.00352	0.00183	0.00125	0.00094
	2.5			0.08208	0.09180	0.08713	0.08519	0.08431	0.00147	0.00073	0.00048	0.00036
	3.5			0.03019	0.03693	0.03368	0.03239	0.03179	0.00047	0.00021	0.00013	0.00010

Table(7): MSE of estimated exponential reliability function under the second proposed loss function.

parameters				R(t)	$(\hat{R}(t))$				MSE$(\hat{R}(t))$			
					Sample Size(n)				Sample Size(n)			
					30	60	90	120	30	60	90	120
θ	t	v	-		Inverse Chi-square distribution $(P_1(\theta \backslash x))$							
1.5	0.5	4		0.71653	0.70497	0.71006	0.71283	0.71364	0.00222	0.00104	0.00066	0.00049
	1.5			0.36788	0.35473	0.36013	0.36361	0.36448	0.00458	0.00230	0.00151	0.00114
	2.5			0.18888	0.18124	0.18405	0.1864	0.18685	0.00306	0.00161	0.00107	0.00081
	3.5			0.09697	0.09390	0.09475	0.09602	0.09614	0.00152	0.00081	0.00054	0.00042
θ	t	α	β		Inverted Gamma distribution $(P_2(\theta \backslash x))$							
1.5	0.5	5	2	0.71653	0.69044	0.70245	0.70768	0.70974	0.00277	0.00121	0.00073	0.00053
	1.5			0.36788	0.33343	0.34873	0.35581	0.35856	0.00526	0.00253	0.00160	0.00119
	2.5			0.18888	0.16362	0.17449	0.17981	0.18184	0.00321	0.00167	0.00109	0.00083
	3.5			0.09697	0.08147	0.08797	0.09131	0.09256	0.00144	0.00080	0.00053	0.00041
θ	t	a	b		Improper distribution $(P_3(\theta \backslash x))$							
1.5	0.5	9	3	0.71653	0.66715	0.69001	0.69922	0.70332	0.00466	0.00175	0.00097	0.00067
	1.5			0.36788	0.30135	0.33068	0.34327	0.34896	0.00821	0.00348	0.00203	0.00145
	2.5			0.18888	0.13861	0.15983	0.16943	0.17383	0.00463	0.00217	0.00132	0.00097
	3.5			0.09697	0.06482	0.07787	0.08406	0.08693	0.00190	0.00097	0.00061	0.00046
θ	t	c	-		Non-informative distribution $(P_4(\theta \backslash x))$							
1.5	0.5	1		0.71653	0.71821	0.71669	0.71725	0.71695	0.00199	0.00098	0.00064	0.00048
	1.5			0.36788	0.37472	0.37023	0.37038	0.36956	0.00457	0.00228	0.00151	0.00113
	2.5			0.18888	0.19828	0.19266	0.19218	0.1912	0.00340	0.00168	0.00111	0.00083
	3.5			0.09697	0.10628	0.10096	0.10019	0.09927	0.00187	0.00089	0.00058	0.00044

Table(8): MSE of estimated exponential reliability function under the second proposed loss function.

parameters				R(t)	$(\hat{R}(t))$				MSE$(\hat{R}(t))$			
					Sample Size(n)				Sample Size(n)			
					30	60	90	120	30	60	90	120
θ	t	v	-		Inverse Chi-square distribution $(P_1(\theta \backslash x))$							
2.5	0.5	4		0.81873	0.80898	0.81369	0.81582	0.81607	0.00103	0.00053	0.00032	0.00025
	1.5			0.54881	0.53171	0.53998	0.54374	0.54407	0.00381	0.00205	0.00125	0.00099
	2.5			0.36788	0.3514	0.35942	0.36307	0.36326	0.00439	0.00244	0.00152	0.00121
	3.5			0.2466	0.23346	0.23993	0.24287	0.24289	0.00364	0.00208	0.00131	0.00104
θ	t	α	β		Inverted Gamma distribution $(P_2(\theta \backslash x))$							
2.5	0.5	5	2	0.81873	0.79641	0.80727	0.81151	0.81282	0.00150	0.00066	0.00037	0.00028
	1.5			0.54881	0.50754	0.52737	0.53521	0.53763	0.00524	0.00244	0.00142	0.00110
	2.5			0.36788	0.32541	0.34561	0.35365	0.35614	0.00571	0.00281	0.00167	0.00131
	3.5			0.2466	0.20985	0.22719	0.23413	0.23626	0.00444	0.00230	0.00140	0.00111
θ	t	a	b		Improper distribution $(P_3(\theta \backslash x))$							
2.5	0.5	9	3	0.81873	0.77811	0.79779	0.80512	0.80799	0.00278	0.00100	0.00052	0.00037
	1.5			0.54881	0.47376	0.50912	0.52271	0.52813	0.00926	0.00360	0.00193	0.00141
	2.5			0.36788	0.29051	0.32603	0.34006	0.34575	0.00965	0.00401	0.00222	0.00165
	3.5			0.2466	0.17936	0.20948	0.22167	0.2267	0.00718	0.00317	0.00181	0.00136
θ	t	c	-		Non-informative distribution $(P_4(\theta \backslash x))$							
2.5	0.5	1		0.81873	0.81893	0.81864	0.8191	0.81853	0.00087	0.00048	0.00030	0.00024
	1.5			0.54881	0.55135	0.54982	0.5503	0.549	0.00342	0.00194	0.00121	0.00096
	2.5			0.36788	0.37305	0.37033	0.37038	0.36875	0.00422	0.00240	0.00151	0.00119
	3.5			0.2466	0.25361	0.25013	0.24971	0.24803	0.00374	0.00211	0.00133	0.00105

Table(9): Best estimation according to the smallest value for MSE($\hat{R}(t)$) under the first proposed loss function.

parameters		The distribution	MSE($\hat{R}(t)$)			
		Sample Size(n) 30 60 90 120	Sample Size(n) 30	60	90	120
θ	t					
0.5	0.5	Inverted Gamma distribution with (α=5, β=2)	0.00333	0.00196	0.00130	0.00108
	1.5	Improper distribution with (a=9 ,b=3)	0.00043	0.00027	0.00019	0.00016
	2.5	Improper distribution with (a=9 ,b=3)	$1.9237e^{-5}$	$1.3175e^{-5}$	$9.2343e^{-6}$	$8.1662e^{-6}$
	3.5	Improper distribution with (a=9 ,b=3)	$7.812e^{-7}$	$5.1849e^{-7}$	$3.4928e^{-7}$	$3.1632e^{-7}$
θ	t					
1	0.5	Non- informative distribution with (c=1)	0.00293	0.00152	0.00105	0.00078
	1.5	Non- informative distribution with (c=1)	0.00329	0.00177	0.00123	0.00093
	2.5	Inverse Chi-square distribution with (v=4)	0.00110	0.00062	0.00044	0.00034
	3.5	Inverted Gamma distribution with (α=5, β=2)	0.00024	0.00014	0.00010	$8.6031e^{-5}$
θ	t					
1.5	0.5	Non- informative distribution with (c=1)	0.00212	0.00101	0.00065	0.00048
	1.5	Non- informative distribution with (c=1)	0.00455	0.00228	0.00150	0.00113
	2.5	Non- informative distribution with (c=1)	0.00317	0.00163	0.00108	0.00082
	3.5	Inverse Chi-square distribution with (v=4)	0.00148	0.00081	0.00054	0.00041
θ	t					
2.5	0.5	Non- informative distribution with (c=1)	0.00094	0.00051	0.00031	0.00024
	1.5	Non- informative distribution with (c=1)	0.00357	0.00198	0.00123	0.00097
	2.5	Non- informative distribution with (c=1)	0.00424	0.00240	0.00150	0.00120
	3.5	Non- informative distribution with (c=1)	0.00361	0.00207	0.00131	0.00104

Table(10): Best estimation according to the smallest value for MSE($\hat{R}(t)$) under the second proposed loss function.

parameters		The distribution	MSE($\hat{R}(t)$)			
		Sample Size(n) 30 60 90 120	Sample Size(n) 30	60	90	120
θ	t					
0.5	0.5	Inverted Gamma distribution with (α=5, β=2)	0.00318	0.00191	0.00127	0.00106
	1.5	Improper distribution with (a=9 ,b=3)	0.00042	0.00027	0.00019	0.00016
	2.5	Improper distribution with (a=9 ,b=3)	$2.1723e^{-5}$	$1.4306e^{-5}$	$9.7204e^{-6}$	$8.5434e^{-6}$
	3.5	Improper distribution with (a=9 ,b=3)	$1.0106e^{-6}$	$6.0933e^{-7}$	$3.8938e^{-7}$	$3.4563e^{-7}$
θ	t					
1	0.5	Non- informative distribution with (c=1)	0.00283	0.00150	0.00103	0.00077
	1.5	Inverted Gamma distribution with (α=5, β=2)	0.00311	0.00172	0.00120	0.00092
	2.5	Inverted Gamma distribution with (α=5, β=2)	0.00101	0.00059	0.00042	0.00033
	3.5	Improper distribution with (a=9 ,b=3)	0.00025	0.00015	0.00010	$8.6394e^{-5}$
θ	t					
1.5	0.5	Non- informative distribution with (c=1)	0.00199	0.00098	0.00064	0.00048
	1.5	Non- informative distribution with (c=1)	0.00457	0.00228	0.00151	0.00113
	2.5	Inverse Chi-square distribution with (v=4)	0.00306	0.00161	0.00107	0.00081
	3.5	Inverted Gamma distribution with (α=5, β=2)	0.00144	0.00080	0.00053	0.00041
θ	t					
2.5	0.5	Non- informative distribution with (c=1)	0.00087	0.00048	0.00030	0.00024
	1.5	Non- informative distribution with (c=1)	0.00342	0.00194	0.00121	0.00096
	2.5	Non- informative distribution with (c=1)	0.00422	0.00240	0.00151	0.00119
	3.5	Inverse Chi-square distribution with (v=4)	0.00364	0.00208	0.00131	0.00104

مقارنة مقدرات بيز لدالة المعولية الاسية باستعمال دوال اولية مختلفة

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الكلية التقنية الادارية /جامعة بغداد

استلم في :4/تشرين الأول/ 2016 قبل في: 7تشرين الثاني 2016

الخلاصة

في هذا البحث نشتق تقدير معولية التوزيع الاسي الاعتماد على اسلوب بيز .تفترض معلمة التوزيع الاسي لتكون متغيرا عشوائيا في اسلوب بيز.نشق التوزيع اللاحق لمعلمة التوزيع الاسي باستعمال اربعة توزيعات اولية لمعلمة القياس للتوزيع الاسي هي:توزيع معكوس مربع كاي وتوزيع معكوس كاما وتوزيع غير الملائم (Improper) وتوزيع Non-informative.وتستحصل مقدرات المعولية باستعمال دالتين خسارة مقترحه في هذا البحث التي تعتمد على اللوغاريتم الطبيعي لدالة المعولية . استعملنا اسلوب المحاكاة لمقارنة المقدرات الناتجه بدلالة متوسط مربعات الاخطاءها .افترضت عدة حالات لمعلمة التوزيع الاسي لتوليد البيانات ولاحجام عينات مختلفة (الصغيرة , المتوسطة ,و الكبيرة).الستحصلت النتائج باستعمال أسلوب المحاكاة,بكتابة برامج باستخدام MATLAB-R2008a. عموما , حصلنا على تقديرات جيدة لمعولية التوزيع الاسي باستعمال دالة الخسارة المقترحة الثانية , وفقا لاصغر قيمة لمتوسط مربعات الاخطاء (MSE) ولكل احجام العينات مقارنة بالمقيم المقدرة لمتوسط مربعات الاخطاء باستعمال دالة الخسارة المقترحة الاولى .

الكلمات المفتاحية : التوزيع الاسي , طريقة بيز ,التوزيعات الأولية : توزيع معكوس مربع كاي ,توزيع معكوس كاما , توزيع غير الملائم (Improper), توزيع Non-informative, متوسط مربعات الخطاء (MSE).