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Some Statistical Properties of Linear Volterra Integral Equation solutions

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Abstract

Our aim of this research is to find the results of numerical solution of Volterra linear integral equation of the second kind using numerical methods such that Trapezoidal and Simpson's rule. That is to derive some statistical properties expected value, the variance and the correlation coefficient between the numerical and exact solution□

1. Introduction

The linear integral equation in u(x) can be represented as[1]:

$$h(x)u(x) = f(x) + \int_a^{b(x)} K(x,t)\,u(t)dt \quad \text{.........................(1)}$$

Equation (1) is called a Volterra integral equation when b(x)=x,

$$h(x)u(x) = f(x) + \int_a^{x} K(x,t)\,u(t)dt \quad \text{.........................(2)}$$

Equation (2) is called a Volterra equation of the first kind when h=0, and is called a Volterra equation of the second kind when h=1, K(x,t) is known function and is called the kernel of the integral equation, where a is constant[1].

If X be a continuous type of random variable having a p.d.f. f(x), and u(x) be a function of X such that $\int_{-\infty}^{\infty} u(x)\, f(x)\, dx$, and if X is a discrete type of random variable[2] such that $\sum_x u(x)\; f(x)$. Then the integral, or the sum called the expected value[2,3] of u(x). And the variance of X denoted by σ^2 or V(x), defined [2,3] as:

$V(x) = E(X-\mu)^2 = E(X^2) - \mu^2$ where X is a discrete or continuous of random variable and μ=E(X) □

2. Solving Linear One Dimension Volterra Integral Equation of The Second Kind Using Trapezoidal Rule[1]:

$$u(x) = f(x) + \int_a^{x} K(x,t)\,u(t)dt \quad \text{.........................(3)}$$

By dividing the interval [a,x] into n subintervals $[x_i, x_{i+1}]$, i=0,1,..,n-1, such that $x_i=a+i\Delta t$, i=0,1,..,n, where $\Delta t=(x_n-a)/n$, x_n is the end point for x, then by setting $x=x_i$, i=0,1,..,n, in equation (3) we can have:

$$u(x_i) = f(x_i) + \int_a^{x_i} K(x_i,t)\,u(t)dt \quad \text{.........................(4)}$$

Then by replacing the integral term in the right hand side of equation (4) by the Trapezoidal rule, to get:

$$u(x_0) = f(x_0)$$

$$u(x_i) = f(x_i) + \Delta t[\frac{1}{2}K(x_i,t_0)u_0 + K(x_i,t_1)u_1 + \ldots + \quad \ldots\ldots\ldots(5)$$

$$+ K(x_i,t_{j-1})u_{j-1} + \frac{1}{2}K(x_i,t_j)u_j], \quad i = 1,\ldots, n, j \le i$$

3. Solving Linear One Dimension Volterra Integral Equation of The Second Kind Using Simpson Rule[1].

As above solution, we divide the interval [a,x] into n subintervals $[x_i,x_{i+1}]$, i=0,1,..,n-1, such that $x_i=a+i\Delta t$, i=0,1,..,n, where n is restricted to be an even integer, and $\Delta t=(x_n-a)/n$, x_n is the end point for x. By replacing the integral term in the right hand side of equation (4) by Simpson rule, to get:

$$u(x_0) = f(x_0)$$

$$u(x_i) = f(x_i) + h[K(x_i,t_0)u_0 + 4K(x_i,t_1)u_1 + 2K(x_i,t_2)u_2 +$$

$$+ \ldots + 4K(x_i,t_{j-1})u_{j-1} + K(x_i,t_j)u_j], \quad \ldots(6)$$

$$\text{where } h = \frac{\Delta t}{3}, \ i = 1,\ldots, n, \ j \le i$$

4. The Formulation of Problem

Consider the one-dimensional Volterra linear integral equation of the second kind:

$$h(x)u(x) = f(x) + \int_a^x K(x,t)\,u(t)\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

First step: Solve the integral equation in (7) using two numerical methods Trapezoidal rule, then by Simpson rule which illustrated in section two.

Second step: Interpolate the data obtained in step one for each method respectively by using Newton Forward Formula [4].

Third step: Find the expected value and the variance [2,3] for the result obtained by Trapezoidal rule, then for the result obtained using Simpson rule.

Fourth step: Compute the Correlation Coefficient [2,3] of the solutions obtained in above steps.

Example

$$u(x) = (2x^2 + x + \frac{4}{3})\exp(-2x) + \int_0^x (x-t)\,u(t)\,dt \quad \ldots\ldots\ldots\ldots\ldots(8)$$

$$where \quad 0 \le x \le 1$$

We will solve this integral equation (8) by using two methods.

First, by using trapezoidal rule. To do this, we divide the interval of integration [0,1] into 10 equal subintervals of width $\Delta t = \frac{1-0}{10} = \frac{1}{10}$.

Since we use trapezoidal rule, we apply equation (5) and we can written the equation (5) in the following compact form:

$$u_0 = f_0$$

$$u_i = f_i + \Delta t[\frac{1}{2}k_{i0}u_0 + k_{i1}u_1 + \ldots + k_{ij-1}u_{j-1} + \frac{1}{2}k_{ij}u_j] \qquad \ldots\ldots(9)$$

$$i = 1,2,\ldots,\ n \quad k_{ij} = k(x_i, t_j) \quad j \le i$$

Here we have

$$f(x) = (2x^2 + x + \frac{4}{3})\exp(-2x^2), \qquad \mathrm{k(x,t) = x - t,} \quad \mathrm{t \le x}$$

Then the equation (9) becomes:

$$u_0 = f_0$$

$$u_i = (2x_i^2 + x_i + \frac{4}{3})exp(-2x_i) + \Delta t[\frac{1}{2}k_{i0}u_0 + k_{i1}u_1 + \ldots + k_{ij-1}u_{j-1} + \frac{1}{2}k_{ij}u_j] \qquad \ldots\ldots(10)$$

$$i = 1,2,\ldots,n \quad j \le i$$

By evaluating equation (10) at each i=1,2,…,10 one can get the following values: u_0=1.3333333333, u_1=1.1965553611, u_2=1.1067485612, u_3=1.0501770263, u_4=1.0178222086, u_5=1.0039651300, u_6=1.0051677590, u_7=1.0195489952, u_8=1.0462767757, u_9=1.0852176618, u_{10}=1.1367003013.

By using the following Newton Forward Formula[4]:

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!h^2} + \ldots + (x - x_0)(x - x_1)(x - x_2)\ldots\ldots(x - x_{n-1})\frac{\Delta^n f_0}{n!h^n}$$

We can find the polynomial which is:

f(x)=0.013214064 x^{10} - 0.0766672228 x^9 + 0.2415794909 x^8 - 0.6073832681 x^{7+} 1,3600009821 x^6 - 2.5771430663 x^5 + 3.8357067640 x^4 – 4.0526880921 x^3 + 3.3306605692 x^2 – 1.6639099065 x + 1.3333333333.

Now we can obtain the expected value[2,3] and the variance[2,3],

$$E(x) = \int_0^x x\ f(x)\ dx$$

$$E(x) = \int_0^1 x\, f(x)\ dx = 0.5260811871$$

$$E(x^2) = \int_0^1 x^2 \ f(x) dx = 0.45192848\,88$$

$$V(x) = E(x^2) - (E(x))^2 = 0.0751670733$$

Second, we will solve the integral equation (8) by using Simpson's rule. To do this, we divide the interval of integration [0,1] into equal subintervals of width $h = \frac{\Delta t}{3}$ $where\ \Delta t = \frac{1-0}{10} = \frac{1}{10}$

Then to use Simpson's rule, we apply equation(6) and we can write equation (6) in the following compact form: :

$$u_0 = f_0$$

$$u_i = f_i + h[k_{i0}u_0 + 4k_{i1}u_1 + 2k_{i2}u_2 + \dots + 4k_{ij-1}u_{j-1} + k_{ij}u_j] \quad \dots\dots\dots(11)$$

$$i = 1,2,\dots,\ n \qquad k_{ij} = k(x_i, t_j) \qquad j \le i$$

Then the equation (11) becomes:

$$u_0 = f_0$$

$$u_i = (2x_i^2 + x_i + \frac{4}{3})exp(-2x_i) + h[k_{i0}u_0 + 4k_{i1}u_1 + 2k_{i2}u_2 + \dots + 4k_{ij-1}u_{j-1} + k_{ij}u_j] \ \ i = 1,2,\dots,n \quad j \le i$$

. ………(12)

By evaluating equation (12) at each i=1,2,…,10 one can get the following values: u_0=1.3333333333, u_1=1.1943331389, u_2=1.1062630050, u_3=1.0477357372, u_4=1.0168932262, u_5=1.0011495203, u_6=1.0037745311, u_7=1.0162183017, u_8=1.0443627039, u_9=1.0812198620, u_{10}=1.1341828617.

By using Newton Forward Formula[4], we can find the polynomial which is: f(x)= 2387.0379995 x^{10} – 11940.289032 x^9 + 25626.121937 x^8 – 30845.226184x^7 + 22833.549096 x^6 – 10715.042130 x^5 + 3164.5486860 x^4 – 563.76739209 x^3 + 56.57019699 x^2 – 3.7023280574 x +1.333333333.

Now we can obtain the expected value[2,3] and the variance[2,3],

$$E(x) = \int_0^1 x \ f(x) \ dx = 0.52359400\,67$$

$$E(x^2) = \int_0^1 x^2 \ f(x) \ dx = 0.34985003\,85$$

$$V(x) = E(x^2) - (E(x))^2 = 0.07569935\,46$$

To find the exact solution we must evaluate integral equation (8) at each x_i, i=0,1,…,10, then one can get following values:

u_0=1.3333333333, u_1=1.1965553611, u_2=1.1053807815, u_3=1.0449205686, u_4=1.0062157851, u_5=0.9841623359, u_6=0.9763178627, u_7=0.9822767277, u_8=1.0034686676, u_9=1.0433565263, u_{10}=1.1081311572.

By using Newton Forward Formula[4], we can find the polynomial which is:
$f(x) = -0.0078939518\ x^{10} - 0.1017527499\ x^9 + 0.8462405483\ x^8 - 2.4760805677\ x^7 + 4.4612226593\ x^6 - 5.7691603786\ x^5 + 6.0710939102\ x^4 - 5.0167100778\ x^3 + 3.4343687856\ x^2 - 1.6665858973\ x + 1.3333333333.$

Now we can obtain the expected value[2,3] and the variance[2,3],

$$E(x) = \int_0^1 x\ f(x)\,dx = 0.5113201\ 765$$

$$E(x^2) = \int_0^1 x^2\ f(x)\,dx = 0.34063514\ 95$$

$$V(x) = E(x^2) - (E(x))^2 = 0.07918682\ 65$$

We will denote the solutions of Trapezoidal, Simpson's rule and exact by X,Y and Z, respectively and there mean by $\overline{X}, \overline{Y}$ *and* $\overline{Z}$ respectively. To compute the correlation coefficient of these solutions by using[2,3]:

$$r_{XY} = \frac{\sum_{i=0}^{10}(X_i - \overline{X})(Y_i - \overline{Y})}{\sqrt{\sum_{i=0}^{10}(X_i - \overline{X})^2 \sum_{i=0}^{10}(Y_i - \overline{Y})^2}} = \frac{0.10182923384578}{0.10183523150808} = 0.99994110425038$$

r_{XY} represents the correlation coefficient of solution of Trapezoidal and Simpson's rule.

$$r_{XZ} = \frac{\sum_{i=0}^{10}(X_i - \overline{X})(Z_i - \overline{Z})}{\sqrt{\sum_{i=0}^{10}(X_i - \overline{X})^2 \sum_{i=0}^{10}(Z_i - \overline{Z})^2}} = \frac{0.1100728803\ 3002}{0.1110460085\ 3648} = 0.9912367115\ 2806$$

r_{xz} represents the correlation coefficient of solution of Trapezoidal and exact.

$$r_{YZ} = \frac{\sum_{i=0}^{10}(Y_i - \overline{Y})(Z - \overline{Z})}{\sqrt{\sum_{i=0}^{10}(Y_i - \overline{Y})^2 \sum_{i=0}^{10}(Z_i - \overline{Z})^2}} = \frac{0.11072549919335}{0.11162558906690} = 0.99193657447\ 99$$

r_{YZ} represent the correlation coefficient of solution of Simpson's rule and exact.

References

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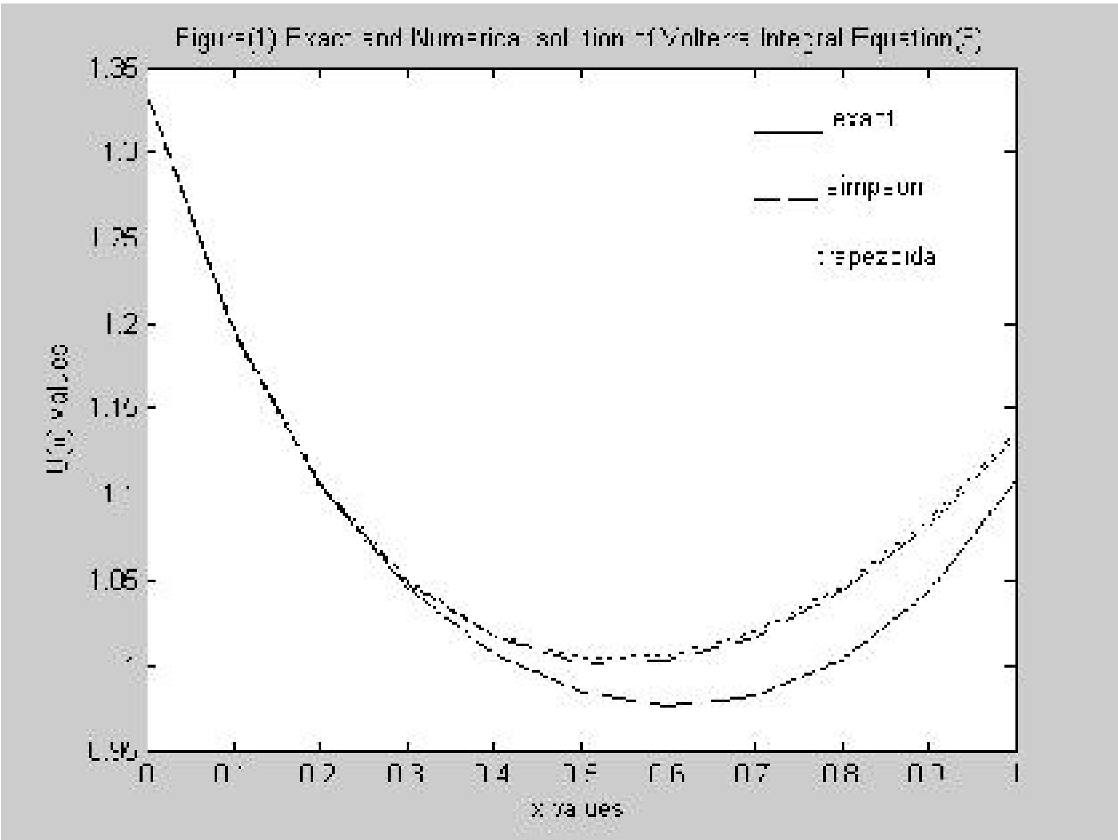
Table :(1) represents the exact and the numerical solution of Volterra integral equation(3)

i	x_i	U_i(Trapezoidal)	U_i(Simpson)	Exact solution
0	0	1.3333333333	1.3333333333	1.3333333333
1	0.1	1.1965553611	1.1943331389	1.1965553611
2	0.2	1.1067485612	1.1062630050	1.1053807815
3	0.3	1.0501770263	1.0477357372	1.0449205686
4	0.4	1.0178222086	1.0168932262	1.0062157851
5	0.5	1.0039651300	1.0011495203	0.9841623359
6	0.6	1.0051677590	1.0037745311	0.9763178627
7	0.7	1.0195489952	1.0162183017	0.9822767277
8	0.8	1.0462767757	1.0443627039	1.0034686676
9	0.9	1.0852176618	1.0812198620	1.0433565263
10	1	1.1367003013	1.1341828617	1.1081311572



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استخدام بعض الاحصاءات لحلول معادلة فولتيرا الخطية التكاملية

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الخلاصة

الهدف الرئيس للبحث هو إيجاد الحل الصحيح لمعادلة فولتيرا الخطية التكاملية من الدرجة الثانية والحل العددي لهذه المعادلة بطريقة شبه المنحرف وطريقة سيمبسون. ثم إيجاد القيمة المتوقعة والتباين لكل الحلول ومعرفة العلاقة بين الحل المضبوط والحلول العددية باستخدام معامل الارتباط.